The Use of Computer Algebra Systems in Calculus Teaching: Principles and Sample Applications

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1. Introduction

After computers have come into our lives, we have started to use them to teach mathematics also. Many math educators have studied on how to use computers more efficiently as a teaching material. There are lots of studies, which debate faciliatating efficiency of computers in teaching any concept. This chapter focuses on presenting some ways of using specific software rather than advocating computers against more traditional educational tools.

As a result of research studies in the literature, it can be said that although no one can guarantee that a computer will be beneficial as a vehicle for delivering instruction, they increase the probability of success. High quality and creative instructional design coupled with careful evaluation and revision are also necessary (Alessi & Trollip, 2001).

In this chapter, we will try to draw an outline on using special software to teach and learn some specific concepts on mathematics, which are included in calculus courses.

2. Computer Algebra Systems (CAS)

After 1990's, computers have become an electronic math tutor, who can interact with learner. For this purpose, some software has been designed for especially teaching math like dynamic algebra and geometry programs. Beside this, Computer Algebra Systems, which are originally created for the use of engineers and applied mathematician, has started to be used for educational purposes.

SAC, MACSYMA, REDUCE, MAGMA, DERIVE, MAPLE, MATHEMATICA, COCOA are some of the computer algebra systems.

One of the first studies on theoretical construction using CAS in math education had been in 1994 (Kutzler, 1994). The use of CAS in math education has started formally, with a declaration on International Conference of Mathematics Education in Seville in 1996 (ICME-8). In this meeting, an international conference, named as Computer Algebra in Mathematics Education (CAME), has been decided to be organized. CAME's principle activity is a twoyearly Symposium: the first was held in Israel, 1999; the second in The Netherlands, 2001; the third in France, 2003; the fourth in the USA, 2005; and the fifth in Hungary, 2007. CAME6 will be held in Belgrade, Serbia, 16 and 17 July 2009, coinciding with the 33rd Psychology of Mathematics Education Conference in Thessaloniki, Greece, 19-24 July.

2.1 CAS and Math Education

Principles of the constructivist theory are also needed to be dominant for an affective use of computers in teaching math. That is, it is needed to make our students interact with the computer. By this way, constructivist learning theory, whose pioneers may be said to be Piaget and Vygotsky, has opened a new era for using computers in teaching math.

According to the constructivist theory, if someone searches and discovers a concept in its natural environment like scientist, this is something more than learning. We may use the word "acquiring" for this process. Most of the mathematical concepts had been discovered through a long period. Constructivist theory says that, think your students as a mathematician and facilitate their work to discover the concept. Special computer software may help us to implement this process.

Computer Algebra Systems may be thought as a good platform for applying constructivist principles in Math education, especially on Calculus concepts. When integrated with the constructivism, CAS has brought a new aspect to the Math education.

Conventional math education is "algorithm" centered, that is students are responsible for performing standard algorithms in general. But, math education has to be focused on "understanding math and earning mathematical thinking capability" (Kokol-Voljc, 2000). According to Kokol-Vlojc, students, who can only perform operations, may be called as a craftsman, even if that operation is the most difficult one.

The aim of mathematics education should be making someone mathematically literate. Mathematical literacy means "understanding the role of mathematics in the world and ability of making well-structured mathematical judgments" (Brown, 2001).

In this context, researcher has been realized that Computer Algebra Systems may be a good tool to create constructivist math learning environment and lots of research studies have been conducted.

Ruthven, Rousham and Chaplin (1997) had the following findings at the end of their research;

- CAS has a positive role on reorganizing the thinking system as a cognitive tool.
- CAS may give us a chance for struggling with non-routine problems.
- CAS may provide an interactive learning environment.
- CAS has a capacity of enlarging the border of the mind.

Aspestberger (1998) has suggested using CAS as a solution for the following problems, which are stated in his research;

- When the teachers are asked to choose a phrase for the concept of integration, most of them had chosen the phrase "inverse of derivative" instead of "Riemann Sum"
- Teachers have spent lots of time to assign the rules of finding the inverse function of derived function.
- Difficulties of paper and pencil operations require choosing simple problems.

Hannah (1998) has reported that,

- graphic calculator has presented a rich environment to discover math concepts,

- CAS makes students to think deeper by helping them encounter a new mathematical situation.

Vlachos and Kehagias' (2000) empirical study showed that the success of students, who have been trained by CAS is higher than others, who have not been trained by CAS. Furthermore, CAS caused to increase students' attitude towards mathematics. After their research, which has positive results in favor of CAS, they decided to use CAS in their every math courses.

Kahng (2005) conducted a project in The University of Minnesota between 2005 and 2007. He designed a course environment based on interactive Mathematica worksheets for the courses Calculus-1, 2 and 3.

Leinbach, Pountney and Etchells (2002) emphasize that when CAS is used in math teaching, we need to reorganize what we teach, besides how we teach. Leinbach and others advocate that, thanks to CAS, students can spend much more time on problem solving activities. When they encounter a complicated and difficult operation in the solution procedure, CAS may help them. This situation provides more time and opportunity for students' cognitive activities.

Cnop (1997) used CAS applications to teach inequalities, limit, continuity concepts, Cnop pointed out that CAS applications had a great potential of visualization. On the other hand, we must check that mathematics concept is dominant rather than CAS applications in the course. Cnop (2001) also found that the courses, which were designed by using CAS applications providing opportunities of making experiments, were very successful on developing students' understanding (2001).

Kendal conducted an investigation searching on how to teach differential calculus to 11th graders with CAS (2001). He concluded that graphical and symbolic representation was one of the most effective methods.

3. Constructing a Learning Environment through CAS

Based on the literature background about using CAS in teaching math , we can say that CAS assisted learning environments have a meaningful effect on the students' math success, especially on calculus concepts. On the other hand, a carefully designed instructional process is needed. Some principles, required for a well-structured teaching with CAS, may be counted as following;

- Skills about using a CAS should not be the dominant during the course. To satisfy this condition, we have to be sure how much our student know the selected CAS.
- During our CAS applications, students must use their previous mathematical knowledge. We need to pay attention to this point, while designing the teaching environment.
- It should not be forgotten that most of the computer applications is an only visualization and never means a formal mathematical proof. Students must be aware of the have to support their theories, which are reached after a CAS application, by a formal mathematical proof.
- Any CAS application must be suitable for easy update to let students make their own trials.

3.1 Sample Applications

In this study, we have just aimed at presenting some examples of using a CAS to teach and learn Calculus concepts. We have chosen the Maple as CAS and our selected concepts do not belong to a specific subject. You can see our 6 applications as

- an innovative view to the concept
- a constructivist teaching and learning environment
- some advanced usage examples of the Maple

We advice readers to study below applications with the computer assist. Every application has been constructed as a unique and complete structure. This means, unless you construct a unique and complete Maple worksheet for every application, some commands may not work or give error because of the unassigned or miss-assigned variable.

3.1.1 Difference between being convergent of a sequence and having limit or its accumulation point

Limit or accumulation point is first mentioned for a set in Calculus courses. Besides this, when a learner encounter the context of a number sequence, it may be confusing that a sequence is convergent and it has an accumulation point. Let's look at the formal definitions of both concepts first;

- Let $A \subseteq IR$ and $a \in IR$. If every ε neighborhood of the number a consists of at least one member of the set A except the number a itself, a is called "accumulation point" of the set A. It may be called as "limit point" also.
- Let (a_n) be a real number sequence, $a \in IR$ and N is a member of index set n. For every $\varepsilon>0$, if there exists at least one N such that for every $n \geq N$, a_n is a member of the neighborhood of the number a, the sequence (a_n) is said to be convergent and a is called as limit of the sequence.

Both definition mentions an accumulation around a specific number's neighborhood. Remember that a sequence is also a set and let's observe the following sequences.

```
> with(plots):
> a:=n->(n^2+1)/(n^2-3);
```

$$a := n \to \frac{n^2+1}{n^2-3} \tag{1}$$

Let's evaluate the limit value, named as L, of the sequence a_n, which is defined as maple function a(n).

```
> L:=limit(a(n),n=infinity);
```

$$L := 1 \tag{2}$$

To make a graphical observation, let's plot some points of a(n) and the line y=L, which obtained in the equation (2).

```
> B:=seq([n,a(n)],n=100..2000):
> A:=pointplot([B]):
> C:=plot(L,x=0..2000,color=blue,thickness=2):
> display(A,C);
```

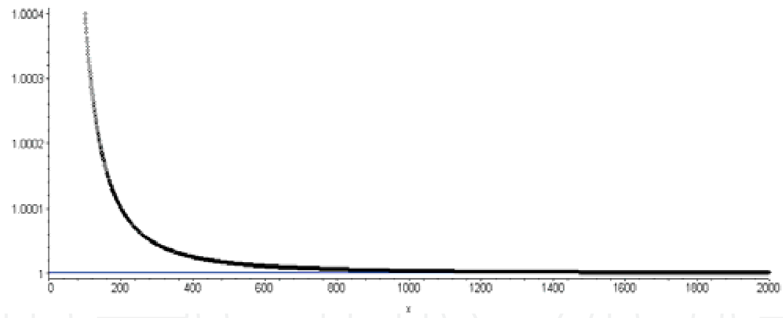


Fig. 1. Members of the sequence a_n between 100 and 2000

In the above figure, you can see how the points of a(n) accumulating around the real value L=1. You can try to display points, closer to infinity, by changing the interval of n. Before making your last judgment, watch the following animation.

```
> animate(pointplot,[[n,a(n)]],n=100..2000,frames=1901,background=C);
```

You will obtain an animation for a(n) sequence, when you run the above command.

For this example, convergence and accumulation point definitions are equivalent. But, when you observe following sequence b_n, you may recognize the difference;

```
> b:=n->(-1)^n*(n^2+1)/(n^2-3);
```

$$b := n \rightarrow \frac{(-1)^n (n^2+1)}{n^2-3} \tag{3}$$

Again, let's evaluate the limit value first;

```
> L2:=limit(b(n),n=infinity);
```

$$L2 := -1 \,..\, 1 \tag{4}$$

Let's assume that, the maple output (4) is not clear for a while and observe the following visualization.

```
> E:=seq([n,b(n)],n=1..200):
> F:=pointplot([E]):
> K:=plot([op(1,L2),op(2,L2)],x=0..200,color=blue,thickness=2):
> display(F,K);
```

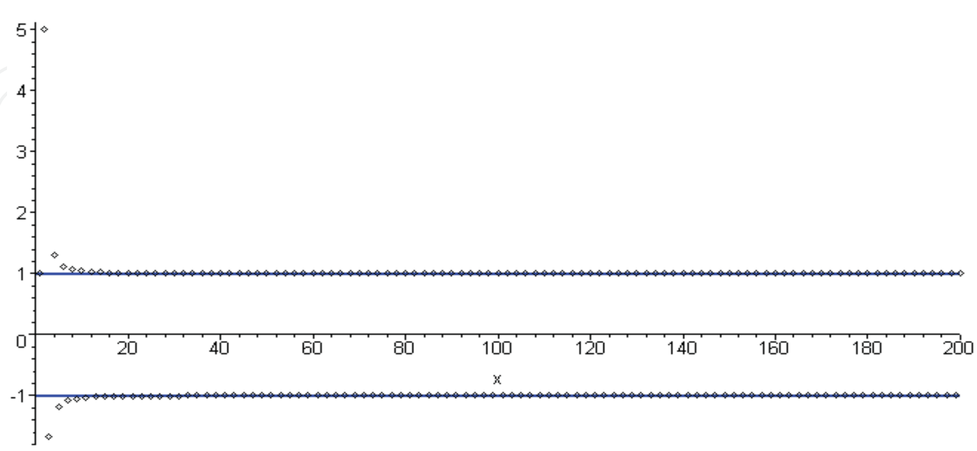


Fig. 2. Members of the sequence b_n between 1 and 200

Now, we can make more accurate decision for the maple output (4) such that, members of b_n sequence are accumulated around two different values, which are 1 and -1. For deeper investigation, following animation should be watched.

```
> animate(pointplot,[[n,b(n)]],n=1..200,frames=200,background=K);
```

You will obtain an animation for b(n) sequence, when you run the above command.

In this animation, it can be easily seen that every successive members of the sequence b_n are being accumulated around 1 and -1 respectively. That is, if 201st member is in the neighborhood of -1, 202nd member is in the neighborhood of 1. While this situation satisfies the accumulation point definition for the numbers 1 and -1, it does not satisfy convergence definition.

Since, every member of the sequence is not accumulated around any number, the real number sequence b_n is not convergent, but it has two different accumulation points.

Consequently we have reached a well known calculus theorem, which is stated as "A convergent sequence has a unique limit point".

3.1.2 Visualizing the Derivative concept

The derivative is related to the idea of a tangent line from geometry. A line tangent to a curve at a point on the curve is the line that passes through that point and has a slope equal to the slope of the curve at that point. The derivative of a function $f(x)$ is another function, $f'(x)$, that gives the slope of the tangent line to $y=f(x)$ at any point.

In this application we try to answer the following questions:

- What should we understand from the concept of derivative?
- What is the relation between "the derivative of a function $f(x)$" and "the slope of the tangent line to $y=f(x)$ at any point"?

Suppose that while examining the contracts of the last 6 years, an officer wants to determine the rate of change in 2 years. He prepared a table as follows:

Year	Cost (million $)
1	1.1
2	4.2
3	8.9
4	16.1
5	25
6	36.2

Table. 1. Cost Per Year

Using the Maple program we can translate these data into graphical form as below. We can sketch a graph using these data by plotting these points by Maple.

```
> plot([[1,1.1],[2,4.2],[3,8.9],[4,16.1],[5,25],[6,36.2]],style=point);
```

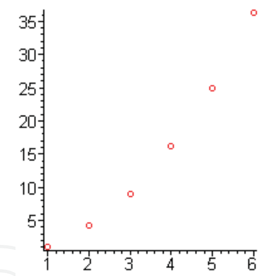


Fig. 3. Cost Per Year

It is the function of $y=x^2$ that fits well into this graph. As the officer wants to determine the rate of change at $x = 2$ we have to find the slope of the line which is tangent to the curve of the function at $x = 2$.

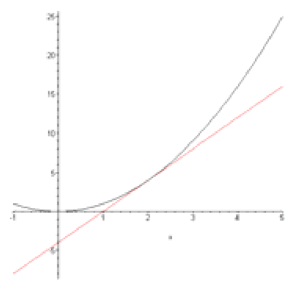


Fig. 4. Tangent Line at $x = 2$

In order to find the slope of this line we can use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{5}$$

Yet, we know only one point (2,4) of this line. Let the point (2,4) be P. We need another point to calculate the slope of the line. We can choose this point on the curve which is closer to P. Let's Q be the point at x=4. Then the coordinates of the point Q is (4,16). The line passing through the points P and Q is a secant line which intersects the graph at two points.

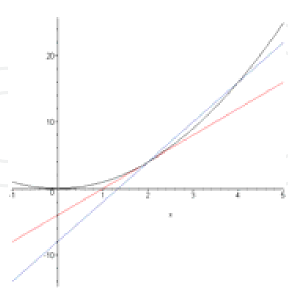


Fig. 5. Tangent and Secant Line

Now we can calculate the slope of this secant line using the formula below:

$$m_s = \frac{y_2 - y_1}{x_2 - x_1} = \frac{16 - 4}{4 - 2} = \frac{12}{2} = 6 \tag{6}$$

By using the Maple function below, we can make essays giving different values to *h*. Here *a* is the abscissa value of the point P and *h* is the difference of the abscissa values of the points P and Q.

$$P(2,4) \text{ and } Q(4,16) \;\Rightarrow\; h = x_2 - x_1 = 4 - 2 = 2 \tag{7}$$

```
> f:=x->x^2;
```

$$f := x \rightarrow x^2 \tag{8}$$

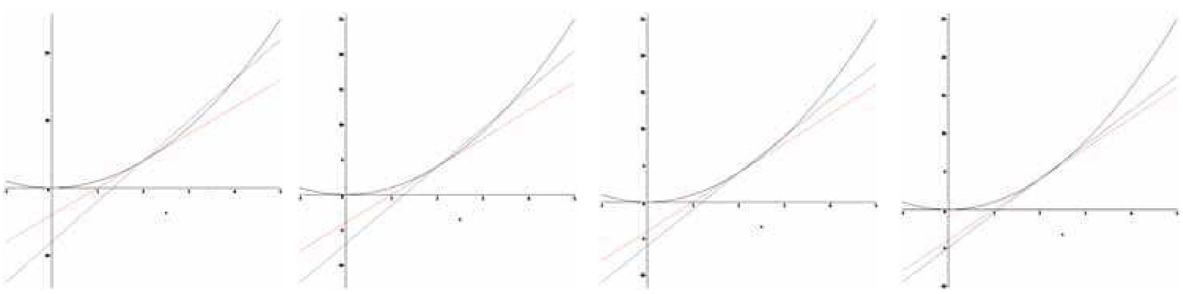
```
> a:=2;
```

$$a := 2 \tag{9}$$

```
> h:=2;
```

$$h := 2 \tag{10}$$

```
>   plot([f(x),4*x-4,((f(a+h)-f(a))/h)*(x-a)+f(a)],x=-1..5,color=[black,red,
blue]);
```



for h=2 for h=1.5 for h=1 for h=0.5

Fig. 6. The Graph of $y=x^2$, Tangent Line and Secant Line For Different *h* Values

By using the animation below, we can see the movement of the secant line as *h* value decreases.

```
> with(plots):
> n:=20;
```

$$n := 20 \tag{11}$$

```
>   Background:=display(pointplot({[a,f(a)],[a,0]},symbol=circle),plot(f(x),
x=(a-(h+signum(h)*1))..(a+(h+signum(h)*1)))):

>      Mover:=display(seq(pointplot({[a+(n-i)/(n/h),f(a+(n-i)/(n/h))],[a+(n-
i)/(n/h),0]},symbol=circle,color=blue),i=0..n-1),insequence=true):

>          Secants:=display(seq(plot((f(a+(n-i)/(n/h))-f(a))/(n-i)*(n/h)*(x-
a)+f(a),x=(a-(h+signum(h)*1))..(a+(h+signum(h)*1)),color=blue),i=0..n-
1),insequence=true):

>          Slopes:=display(seq(textplot([a+.25,f(a)+14,cat("slope           =
",convert(evalf((f(a+(n-i)/(n/h))-f(a))/(n-i)*(n/h),5),string))],
align=ABOVE,color=BLUE,font=[TIMES,ROMAN,18]),i=0..n-1),insequence=true):
```

```
> HValues:=display(seq(textplot([a+.28,f(a)+18,cat("h  =  ",convert(evalf((n-i)/(n/h),4),string))],color=BLUE,font=[TIMES,ROMAN,18]),i=0..n-1),insequence=true):

> display(Secants,Slopes,Background,Mover,HValues);
```

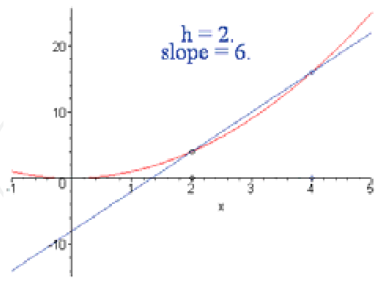


Fig. 7. Secant Line Becomes Tangent Line As *h* Value Decrease

As the *h* value decreases the point Q gets closer to the point P and the secant line becomes tangent line. Instead of giving different values to *h* each time we can redefine the coordinates of the point Q in terms of *h*. Then the new coordinates of the point Q will be $(2+h,(2+h)^2)$. If *h* has negative values, the point Q approaches from right to the point P. If it has positive values, the point Q approaches from left to the point P.
Now the slope of the secant line passing through the points P and Q will be

$$m_s = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(2+h)^2 - 2^2}{(\not{2}+h) - \not{2}} = \frac{\not{4} + 4h + h^2 - \not{4}}{h} = \frac{4h + h^2}{h} = \frac{\not{h}(4+h)}{\not{h}} = 4 + h \qquad (12)$$

In order to find the slope of the tangent line we have to make *h* closer and closer to 0. In this case the point $(a+h, f(a+h))$ gets closer and closer to the original point $(a, f(a))$ and the secant line looks more and more like a tangent line.

When we make *h* closer and closer to 0, the value of $m_t = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(2+h)^2 - 2^2}{(2+h) - 2}$ can be calculated by the help of limit operation:

$$\lim_{h \to 0} \frac{(2+h)^2 - 2^2}{(2+h) - 2} = \lim_{h \to 0} 4 + h = 4 \qquad (13)$$

So the slope of the tangent line at the point (2,4) is 4.
This value gives us also the value of the derivative of the function for $x=2$. We show this as

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{(2+h) - 2} = 4 \qquad (14)$$

This means we can compute the slope of the tangent line by finding

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} \qquad (15)$$

This limit is also the derivative of the function $f(x)$ at the point x=2. We can conclude that the derivative of a function at a point is the slope of the tangent line at that point.

3.1.3 Visualizing the meaning of convergence interval of the function series

Convergence of any number series means that, an infinite sum has a finite result. This situation is easy to understand. A number sequence can be classified as either convergent and divergent. On the other hand, the context is a bit different in the function series. The elements of a function series are functions, while the elements of a number series are numbers. So, sum of the numbers has an easily understandable meaning whereas sum of the functions does not.

In this application we will try to find out an answer for the following questions;

- What should we understand from the partial sum of a function series goes to infinity?
- What does "a function series is convergent" mean?
- What does convergence interval mean?

Let's define a function series as;

$$\sum_{n=1}^{\infty} \frac{n+1}{x^n} \tag{16}$$

First, we need to define this series as a Maple function;

```
> A:=k->sum((n+1)/x^n,n=1..k);
```

$$A := k \rightarrow \sum_{n=1}^{k} \frac{n+1}{x^n} \tag{17}$$

By this Maple function, we have an opportunity of calling the kth element of the series as A(k) in anywhere of the dynamic Maple worksheet. We can evaluate the limit value of A(k) as k goes to infinity.

```
> limit(A(k),k=infinity);
```

$$\lim_{k \to \infty} -\frac{\left(\frac{1}{x}\right)^{(k+1)} x\,(x-k-1+(k+1)\,x)}{(-1+x)^2} + \frac{2\,x-1}{(-1+x)^2} \tag{18}$$

We need to think Maple as an electronic source of mathematics. While it can perform the most complex operations, it can not make a decision like a human mathematician. When we obtain a result as in the function (3), we need to make a decision as following;

As we focus on the $\left(\frac{1}{x}\right)^{k+1}$ part of the function (3), if x goes to infinity we can easily decide the limit will be

$$\frac{2x-1}{(x-1)^2}\text{ , if } |x|>1 \tag{19}$$

Now, it is time to show a graphical approach. We have a common knowledge from real number series that infinitely many elements of a partial sum sequence of a convergent real number series are collected around the finite sum value of the series.

By this view, let's examine the elements of partial sum sequence, which is the Maple function A(k), of our sample function series.

```
> with(plots):
> B:=seq(A(k),k=1..10):
> E:=plot([B],x=-6..6,y=-20..20,color=black):
> display(E);
```

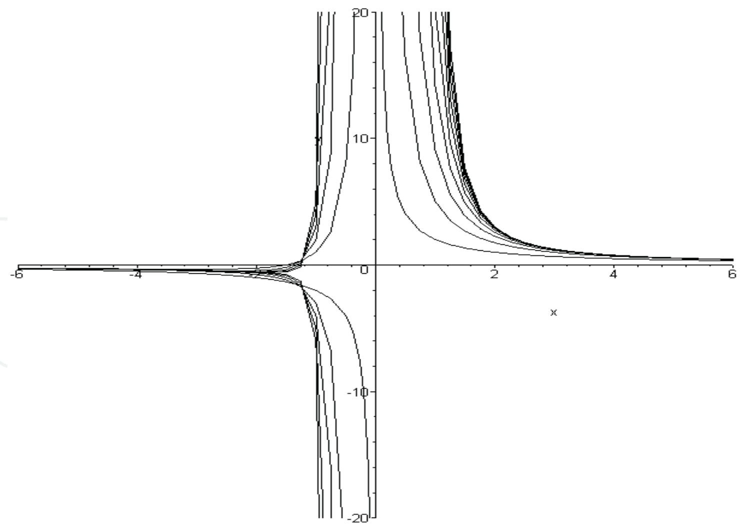


Fig. 8. First 10 elements of partial sum sequence

In the figure-1 first 10 elements have been visualized. Let's increase the number of the elements.

```
> B:=seq(A(k),k=1..150):
> E:=plot([B],x=-6..6,y=-20..20,color=black):
> display(E);
```

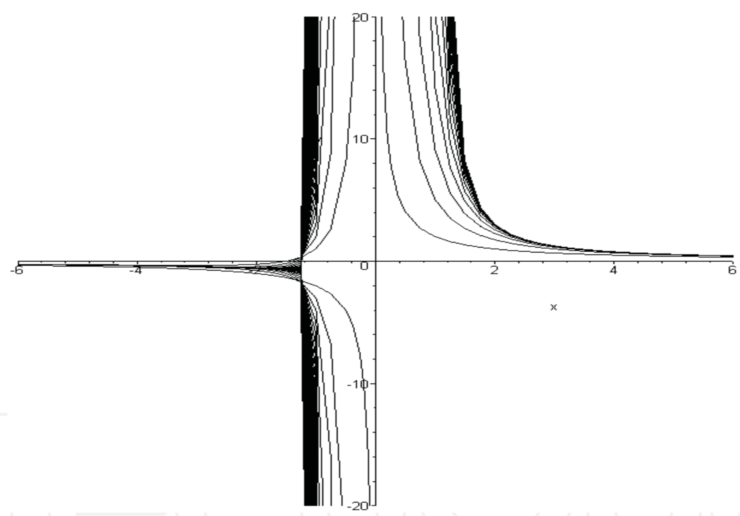


Fig. 9. First 150 elements of partial sum sequence

As we see the first 150 elements, it can be easily seen that the functions, which are the elements of partial sum sequence, are collected on a specific region. We also know that for the concept of convergence, we must neglect the first finite elements.

```
> B:=seq(A(k),k=120..150):
> E:=plot([B],x=-6..6,y=-20..20,color=black):
> display(E);
```

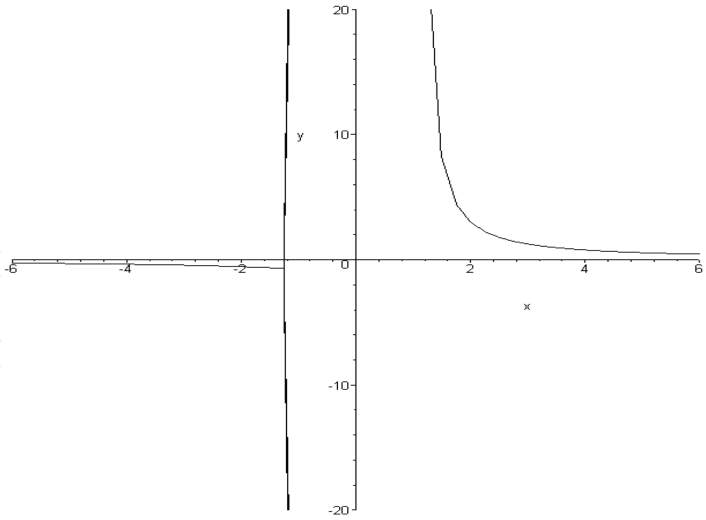


Fig. 10. the elements between 120 and 150 of partial sum sequence

Of course, these elements are meaningless as compared with remaining infinite elements. But this is a simple visualization and let's plot the graph of the function (4), which is the limit of A(k) as k goes to infinity.

```
> F:=plot((2*x-1)/(-1+x)^2,x=-6..6,y=-20..20,color=red,thickness=5):
> display(E,F);
```

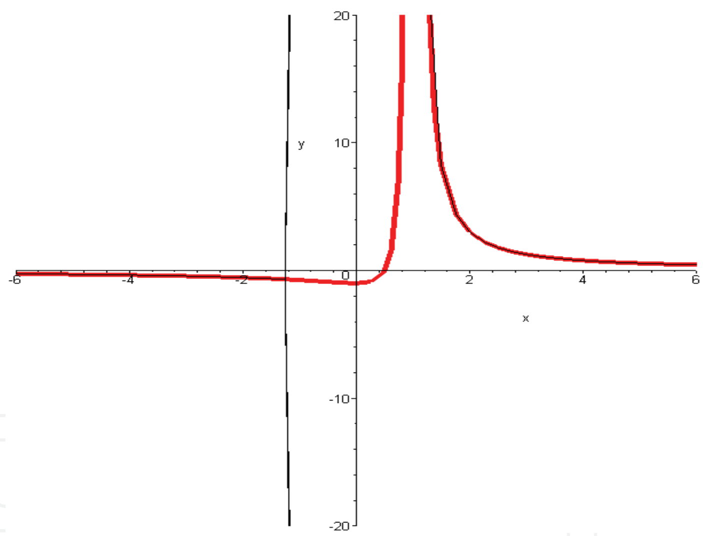


Fig. 11. Comparing the limit function and elements between 120 and 150

As you can see, the graph of the function, which is the limit of A(k), approximately fitted on the graphs, which belong to the elements of A(k) for k from 120 to 150. As we remember, red graph is the limit function for the $|x| > 1$, it will be better to examine one more thing.

```
> G:=implicitplot([x=-1,x=1],x=-2..2,y=-20..20,color=blue,thickness=2):
> display(E,F,G);
```

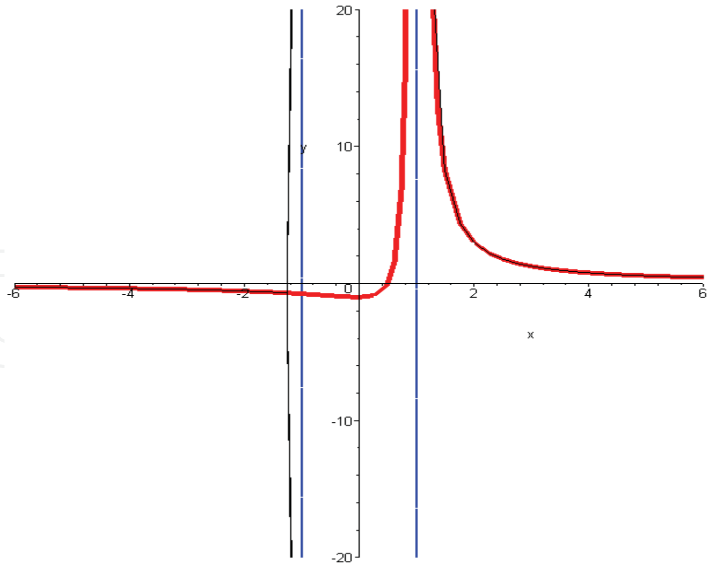


Fig. 12. Visualization of Convergence interval

At last, we have seen that the function of the limit of the A(k) as k goes to infinity has a potential of coinciding with all elements of function series, except first finite element in a specific region, which can be called as convergence interval, which is IR – [-1,1] in this example.
Readers are advised to apply this visualization for any other function series and to compare with their formal knowledge of finding the convergence interval of a function series.

3.1.4 Understanding the construction of a surface

Maple can plot a two variable function's graph using a special command directly. But this method of plotting only provides a nice three dimensional picture. If we want to use computer as an electronic tutor, we should use an indirect method rather than obtaining a perfect picture. Let's choose a specific function as an example to clarify what we mean exactly.

You can plot the graph of function $z = f(x, y) = \frac{y}{x^2}$ by using "plot3d" maple command as following,

```
> plot3d(y/x^2,x=-5..5,y=-5..5,view=-4..4,axes=normal);
```

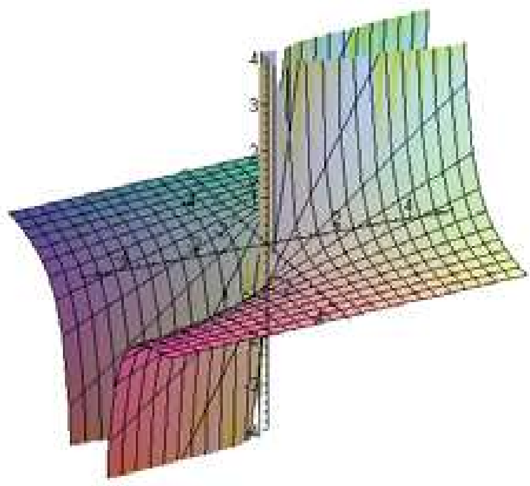


Fig. 13. Obtaining the graph directly

Now, let's try to construct the surface above step by step. In a two variable function, for every constant value of z, we obtain a relation between x and y. This relation represents a curve which is called "level curve" or "contour line" with its more common name.
Let's define the function first;

```
> z:=y/x^2;
```

$$z := \frac{y}{x^2} \tag{20}$$

Now, let's draw some contour lines of the surface.

```
> with(plots):
> A:=seq([z=c],c=-10..10):
> implicitplot([A],x=-2 ..2,y=-10..10,numpoints=5000,color=black);
```

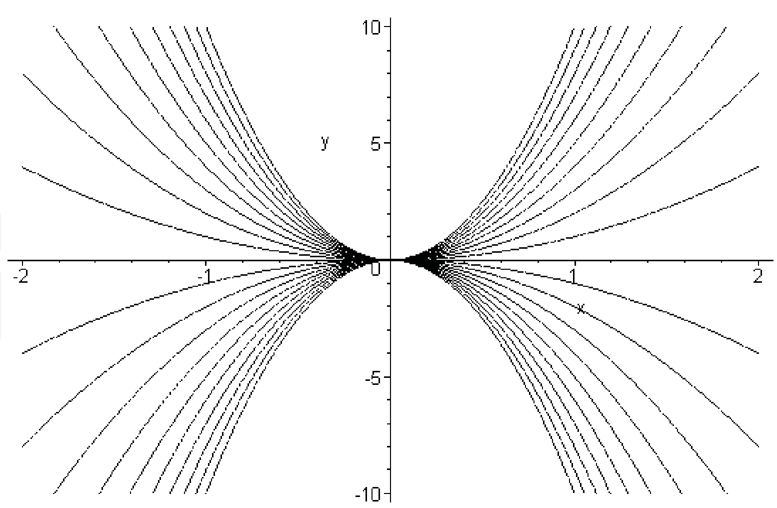


Fig. 14. Some contour lines of the surface

We should look at the figure above as a topographic map. We may advice our students to specify every contour line's height. It may be written on the figure, after obtaining the lines gradually. An original topographic map, which shows the relation between contour lines and surface, is presented as an example in the figure.15.

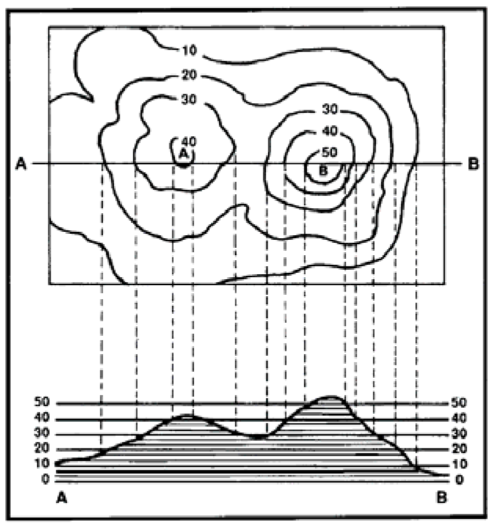


Fig. 15. A topographic Map (www.uml.edu/tsongas/activities/read_map.htm)

We may also locate every contour line on their original height in 3D environment.

```
> B:=seq([t,c*t^2,c],c=-10..10):
> spacecurve({B},t=-2..2,color=black,numpoints=5000, axes=normal);
```

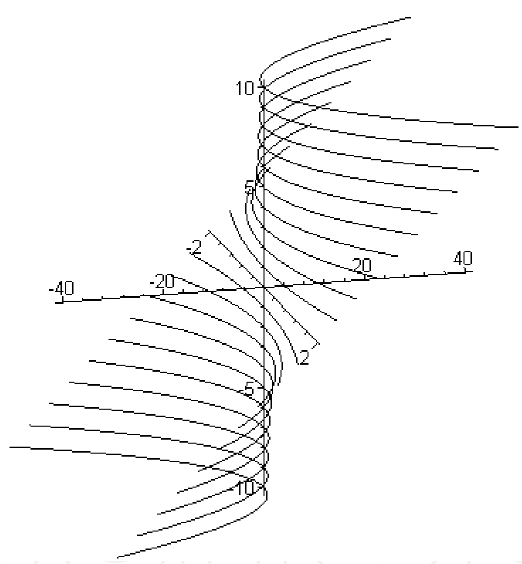


Fig. 16. Skeleton of the surface

As a final result we have obtained a figure, which may be called surface skeleton. If you monitor more contour lines, figure.5 may make more sense for you. Try to construct another 3D surface by using the same procedure.

3.1.5 A Real Context Problem: Donkey in a circle bounded field

Most of the mathematical problems, in a math class, are well constructed. We mean, the results of the problems are generally a rational number or solving procedure is similar to previous ones or easily estimated. On the other hand, mathematical concepts are originally developed on the way to the solution of a real problem. A real problem is not set to be easily solved. Solution procedure may be difficult to construct. In addition, operation, which is necessary for the solution, may be very difficult or impossible to operate by hand. Thanks to a Computer Algebra System like Maple, we have an opportunity to expose our students to a real (not well-constructed) problem. They need only to construct a solution algorithm. Afterwards, they can use Maple to terminate the solution. Here is an example;

Problem: A donkey is being fed in a circle shaped field, which has a radius of 10 meter. Donkey is tied on the boundary of the field. How many meters must be the length that we allow the donkey to reach at most half of the field? (Sertöz, 1996)

Solution: We need to produce a mathematical model for the problem as in the following illustration.

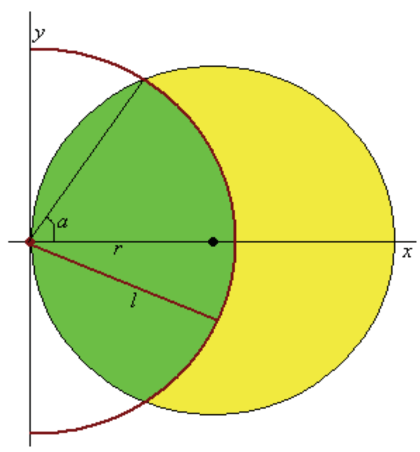


Fig. 17. Graphical model for the problem

In the above figure, let the circle, with radius ***r***, be the field and the dark region be the area, which the donkey can reach. Now, dark region must be half of the complete circle. Complete circle's area can be easily calculated as

$$A_1 = \pi r^2 \tag{21}$$

Dark region is also a part of the circle, with radius ***l***. But it can not be easily calculated like A_1. Let this region call as A_2. A_2 can be thought as a double integration domain. Let's calculate the upper part of the x-axis first and multiply by 2.

We need to find an acute angle ″a ″ on the figure. By this way, the region can be separated into two Polar Regions. One's boundaries will be $0 \le \theta \le a$ and $0 \le \sigma \le l$. The other one's boundaries will be $a \le \theta \le \pi/2$ and $0 \le \sigma \le 2r\cos(\theta)$. Remember that complete circle's polar equation is $\sigma = 2r\cos(\theta)$.

Now the question is ″what is the value of the acute angle a? ″

$$2r\cos\theta = l \Rightarrow \cos\theta = \frac{l}{2r} \Rightarrow \theta = \arccos\left(\frac{l}{2r}\right) \Rightarrow a = \arccos\left(\frac{l}{2r}\right) \tag{22}$$

The area of A2 region can be evaluated as follows;

$$A_2 = 2\left(\int_0^{\arccos\left(\frac{l}{2r}\right)} \int_0^{l} \sigma d\sigma d\theta + \int_{\arccos\left(\frac{l}{2r}\right)}^{\frac{\pi}{2}} \int_0^{2r\cos(\theta)} \sigma d\sigma d\theta \right) \quad (23)$$

Since, we want the donkey reach half of the whole field, to find the final solution following equation is needed to be solved;

$$A_1 = 2A_2 \quad (24)$$

So far, we redesigned the real problem as a formal mathematical problem. Now we have following mathematical problem;

- evaluate the integral numbered (23)
- solve the equation numbered (24) for r =10

While a student struggle on a real life problem, redesigning of problem in terms of mathematical language can be seen sufficient. In this step, using a CAS like maple can encourage our students to struggle on new problem. Let's see how the problem above is solved by using Maple;

```
> A2:=2*(Int(Int(ro,ro=0..l),theta=0..arccos(l/(2*r)))+
Int(Int(ro,ro=0..2*r*cos(theta)),theta=arccos(l/(2*r))..Pi/2));
```

$$A2 := 2\int_0^{\arccos\left(\frac{l}{2\,r}\right)} \int_0^{l} ro\ dro\ d\theta + 2\int_{\arccos\left(\frac{l}{2\,r}\right)}^{\frac{\pi}{2}} \int_0^{2\,r\cos(\theta)} ro\ dro\ d\theta \quad (25)$$

Let's evaluate the integral above;

$$A2 := l^2 \arccos\left(\frac{l}{2\,r}\right) - \frac{r\,l\sqrt{\frac{4\,r^2 - l^2}{r^2}}}{2} - 2\,r^2 \arccos\left(\frac{l}{2\,r}\right) + r^2\,\pi \quad (26)$$

Now, let's solve the equation (4) for r =10.

```
> r:=10:
> fsolve(2*A2=Pi*r^2,l);
```

$$11.58728473 \quad (27)$$

This is the final result. Our donkey is needed to be tied with a rope, with the length of about 11,58 meter.

The maple command "fsolve" operates a numerical method to solve equation. When you try to use "solve" command, which can operate analytical methods, Maple can not solve this equation. And the result is an irrational number. That is, if you tried to solve this equation by hand, you would not managed. Double integration by polar coordinates is used here. You can try to use single integration and rectangular coordinates.

3.1.6 Finding the area the city of Antalya by using Riemann Sum idea

This CAS application is again related to a real context project. We will try to find out the area of Antalya City, which is one of the most popular cities of Turkey on the coast of the Mediterranean Sea. We will use the Riemann Sum idea, which is the fundamental of Integral concept, as estimating the area of Virginia (Clark et.all, 2003).

In this application, we have used auxiliary software, called swish application, to get the coordinates of the Antalya's border. We have obtained the Antalya map from Turkish General Directorate of Forestry internet site. In the below figure, every click, on the map, determines the coordinates of the cursor. After collecting the coordinates, you can easily transfer them to Maple worksheet by copy and paste.

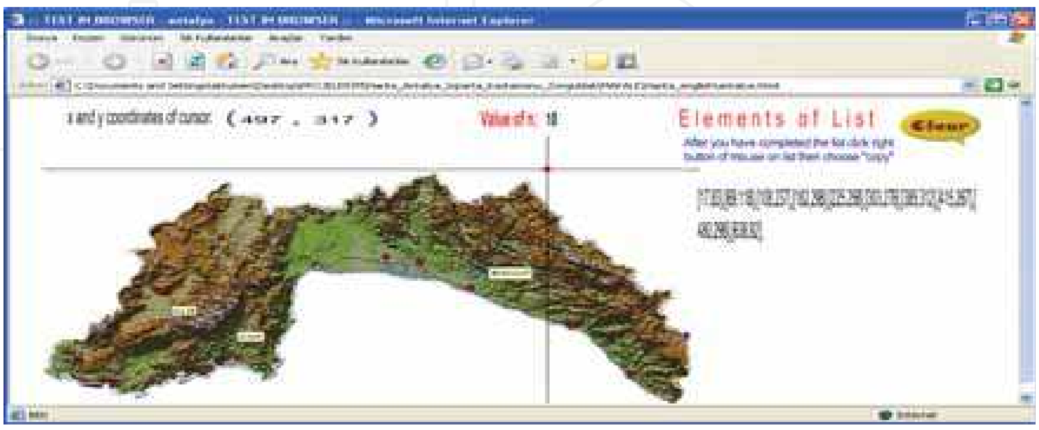


Fig. 18. Swish Application, determining the Antalya's Border

As every modeling problem, first we need to define the problem in terms of formal mathematical terminology. As we calculate the area between the two curves, we have to see the Antalya map as two curves, which need to satisfy the properties of being a function. That is, according to these curves every point in the x-axis must have only one image in the y-axis. In the figure.19, blue curve and red curve can satisfy this rule. Let's call these curves as north border and south border respectively.

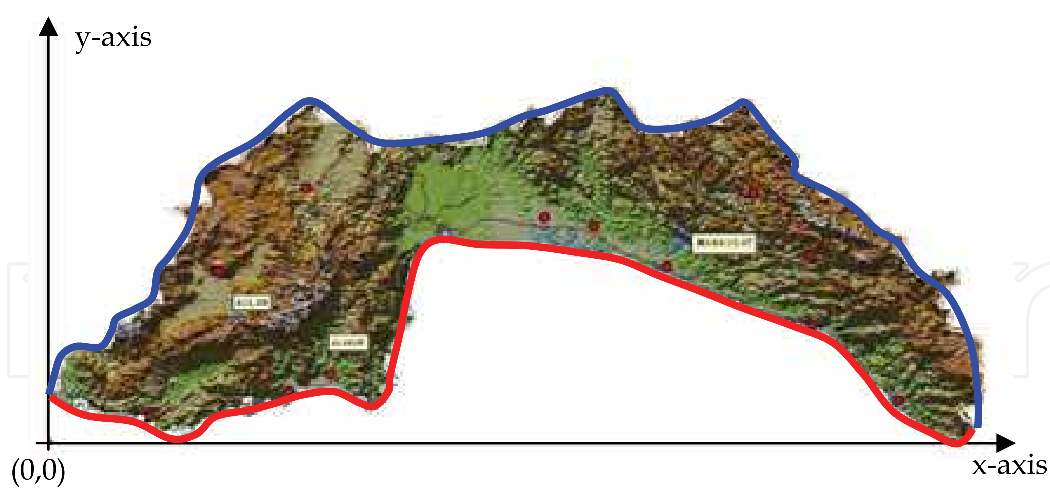


Fig. 19. Determining the Antalya's Border by North and South Curves.

Now we need to find Riemann sum of the possible function of the north curve and south curve, and then subtract the area under the south curve from area under the north curve. To implement this procedure, we need to take some point on the curves by using Swish application.

Finding the area under the north border:

Let's define the coordinates, which we obtain from Swish application;

```
> n:=50:
northcoordinates:=[4,44],[11,65],[19,84],[40,87],[48,105],[64,117],[73,129],
[75,146],[86,157],[96,177],[97,194],[101,210],[105,233],[122,246],[143,264],
[160,279],[178,290],[201,284],[218,274],[228,265],[247,264],[268,264],[285,2
63],[299,270],[317,280],[333,287],[352,293],[373,301],[388,305],[398,292],[4
07,277],[427,275],[452,283],[472,295],[486,281],[495,269],[507,253],[514,233
],[528,219],[544,217],[554,204],[571,185],[584,174],[587,159],[604,145],[612
,124],[618,104],[629,90],[634,76],[639,57]:
```

Now, let's visualize the north border by using the coordinates;

```
> with(plots):
northborderdata:=PLOT(POINTS(northcoordinates)):
display(northborderdata,view=[0..750,0..375]);
```

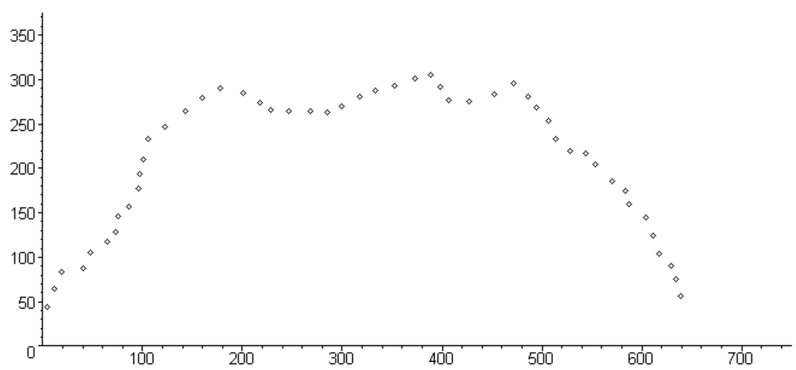


Fig. 20. North border's coordinates

To implement the Riemann Sum approach, we need to construct left or right rectangles as following;

Following loop transfers the first and the second components of the north coordinates to xborder[i] and yborder[i] variables respectively.

```
> for i from 1 to n do
xborder[i]:=northcoordinates[i,1];
yborder[i]:=northcoordinates[i,2];
od:
```

And, width of every left rectangle has been constructed by the following loop;

```
> for i from 1 to n-1 do
pplotred[i]:=plot(yborder[i],x=xborder[i]..xborder[i+1],color=red,thickness=
2):
od:
```

Following Maple command group visualizes the left rectangles for the north border.

```
> partition[1]:=PLOT(CURVES([[xborder[1],0],[xborder[1],yborder[1]]])):
partition[n]:=PLOT(CURVES([[xborder[n],0],[xborder[n],yborder[n-1]]])):
  for i from 2 to n-1 do
partition[i]:=PLOT(CURVES([[xborder[i],0],[xborder[i],max(yborder[i-1],
yborder[i])]])):
od:
  partitionlines:=display(partition[k] $k=1..n):
  display(northborderdata,partitionlines,pplotred[t]          $t=1..n-1,
view=[0..670,0..375]);
```

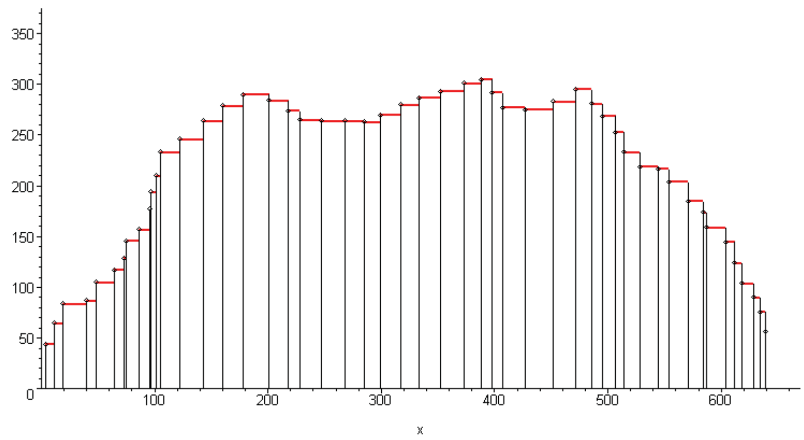


Fig. 21. Left rectangles of North border

Finally, we need to find out the sum of the all rectangles;

```
> for m from 1 to n-1 do
base[m]:=xborder[m+1]-xborder[m];
od:
> sum(base[j]*yborder[j],j=1..n-1): area:=evalf(%);
```

$$area := 145656. \tag{28}$$

The result, which we have found, is in terms of pixel square. According to our map, 1 pixel = 0,52 km. So, 0,2704 km^2 = 1 pixel^2. Therefore, we can find the result as;

```
> area_km_square:=area*(0.2704);
```

$$area\_km\_square := 39385.3824 \tag{29}$$

We have reached the area of the region between north border and x-axis as 39.385 km^2. After finding the area of the region between south border and x-axis by applying the same procedure, we can reach the final result. Here, a complete visualization has been presented below.

```
> n:=44:
southcoordinates:=[2,43],[19,32],[38,27],[60,21],[79,13],[97,11],[113,24],[1
28,28],[146,32],[164,36],[177,48],[196,48],[208,40],[221,34],[230,47],[234,6
3],[236,83],[243,104],[246,129],[250,155],[255,169],[265,177],[287,176],[307
,175],[327,176],[346,171],[365,168],[383,166],[403,159],[421,149],[438,138],
[455,128],[475,123],[497,110],[515,103],[532,92],[544,79],[555,65],[565,51],
[576,33],[590,18],[610,6],[621,2],[632,9]:
> southborderdata:=PLOT(POINTS(southcoordinates)):
  northborderdata:=PLOT(POINTS(northcoordinates)):
  display(southborderdata,northborderdata,view=[0..750,0..375]);
```

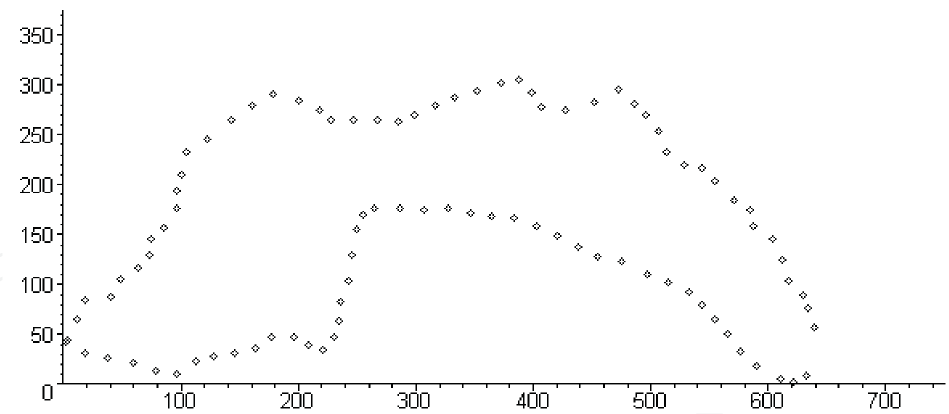


Fig. 22. Complete map of Antalya, obtained by the coordinates.

As seen in the figure.22, the area between north and south borders will be Antalya's area. Finding the area, under the south border, is left for the readers. You will apply the same procedure for the calculation. You only need to change the coordinates. Of course, you need to pay attention to changing variable names as "south" instead of "north".

You will obtain the area under the south border as 15.097 km^2. To find the area of Antalya, we have to subtract 15.097 from 39.385, and the final result will be 24.288 km^2. The original area of Antalya is 20.815 km^2 (http://tr.wikipedia.org/wiki/Antalya). This means an error of 0,17. Although, this error may be thought as unacceptable for a formal knowledge, it may be accepted as a successful result for this kind of calculation. Obtaining a more precise value is possible related to the following potential items;

- Much more coordinate points should be taken from the borders.
- Taking more points may not be sufficient, because we need to make the border curve is suitable for a formal function. This means, we have to neglect some bays in the map. To prevent these missing areas, map must be divided into little parts.

This application should be seen as an introduction to the concept of integration. In a real calculus course environment, students may have a chance to realize the fundamental idea of a formal calculus concept.

4. Conclusion

Every math concept has an exploration and elaboration story in the history of math. There are a lot of interesting and easy views of every concept. Of course, this exploration and elaboration process may take hundreds of years. A carefully designed CAS based learning and research environment has a potential of consisting this process in itself and it only takes a couple of class period.

Working through experiences is the accepted method of learning mathematics. With the help of CAS, student can do a lot of exercises than work with the classical way, not only the number of exercises but also the variety of the problem will increase.

With the help of CAS, the complicated computational processes can be reduced to enabling students to focus more on the analysis of the problem. Moreover, students will be able to do a lot of experiments, especially with the visualization problems. In this case, we may spare them the usual boringness in understanding the abstraction process, so in the long run they will be able to understand this process better.

5. References

Alessi, S.& Trollip, S. (2001). *Multimedia for Learning Methods and Development,* Allyn and Bacon, ISBN 9780205276912, USA

Aspestberger, K. (1998), Teaching Integrals with TI-92: A Chance of Making a Complex Mathematical Concept Elementary, *International Conference on Teaching of Mathematics*, 3-6 July, 1998, pp.29-31, Samos, Greece.

Clark, J., Diefenderfer, C., Hammer, S., Hammer, T., (2003). Estimating the Area of Virginia. *Journal of Online Mathematics and its Applications*, from http://mathdl.maa.org/ mathDL/4/?pa=content&sa=viewDocument&nodeId=507

Cnop, I., (1997), Implementing Abstract Mathematical Concepts Using Computer Algebra Packages, *International Conference on Technology in Mathematics Teaching*, ICTMT-3.

Cnop, I. (2001), New Insight in Mathematics by Live CAS Documents, Annual Conference on Applications of Computer Algebra, 12th. Albuquerque, NM, May 31 – June 3.

Hannah, J. (1998), Student Use of Graphic Calculator: Tool or Crutch?, International *Conference on Teaching of Mathematics*, 3-6 July, 1998, Samos, Geeece.

Kahng, B., (2005), Computer Assisted Calculus Education Project, Division of Science and Mathematics, a project in University of Minnesota, Morris, MN 56267.

Kendal, M. (2001), Teaching And Learning Introductry Differential Calculus With A Computer Algebra System, *PhD Thesis*, Department of Science and Mathematics Education, The University of Melbourne

Kokol-Voljc, V. (2000), Examination questions when Using CAS for School Mathematics Teaching, *The International Journal of Computer Algebra in Mathematics Education*, 7(1), 63–75.

Kutzler, B. (1994). The future of teaching mathematics, *International Derive Journal* 1(1):37–48.

Leinbach, C., Pountney D. C. ve Etchells, T. (2002), Appropriate use of a CAS in the teaching and learning of mathematics, *International Journal of Mathematical Education in Science and Technology*, vol. 33, no. 1, 1 – 14.

Ruthven, K., Rousham, L. and Chaplin, D., (1997), The Long-Term Influence of a "Calculator-Aware" Number Curriculum on Pupils' Mathematical Attainments and Attitudes in the Primary Phase., *Research Papers in Education: Policy and Practice*, v12 n3 p249-81

Sertöz, S., (1996), *Matematik Dünyası*, Turkish Mathematical Society, Vol.6, 13-17.

Vlachos, P. ve Kehagias, A., (2000), A Computer Algebra System and a New Approach for Teaching Business Calculus, *The International Journal of Computer Algebra in Mathematics Education*, Vol. 7, No.2.

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The widespread deployment and use of Information Technologies (IT) has paved the way for change in many fields of our societies. The Internet, mobile computing, social networks and many other advances in human communications have become essential to promote and boost education, technology and industry. On the education side, the new challenges related with the integration of IT technologies into all aspects of learning require revising the traditional educational paradigms that have prevailed for the last centuries. Additionally, the globalization of education and student mobility requirements are favoring a fluid interchange of tools, methodologies and evaluation strategies, which promote innovation at an accelerated pace. Curricular revisions are also taking place to achieved a more specialized education that is able to responds to the society's requirements in terms of professional training. In this process, guaranteeing quality has also become a critical issue. On the industrial and technological side, the focus on ecological developments is essential to achieve a sustainable degree of prosperity, and all efforts to promote greener societies are welcome. In this book we gather knowledge and experiences of different authors on all these topics, hoping to offer the reader a wider view of the revolution taking place within and without our educational centers. In summary, we believe that this book makes an important contribution to the fields of education and technology in these times of great change, offering a mean for experts in the different areas to share valuable experiences and points of view that we hope are enriching to the reader. Enjoy the book!

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