Chapter

Generalized and Fundamental Solutions of Motion Equations of Two-Component Biot's Medium

Lyudmila Alexeyeva and Yergali Kurmanov

Abstract

Here processes of wave propagation in a two-component Biot's medium are considered, which are generated by arbitrary forces actions. By using Fourier transformation of generalized functions, a fundamental solution, Green tensor, of motion equations of this medium has been constructed in a non-stationary case and in the case of stationary harmonic oscillation. These tensors describe the processes of wave propagation (in spaces of dimensions 1, 2, 3) under an action of power sources concentrated at coordinates origin, which are described by a singular delta-function. Based on them, generalized solutions of these equations are constructed under the action of various sources of periodic and non-stationary perturbations, which are described by both regular and singular generalized functions. For regular acting forces, integral representations of solutions are given that can be used to calculate the stress-strain state of a porous water-saturated medium.

Keywords: Biot's medium, solid and liquid components, Green tensor, Fourier transformation, regularization

1. Introduction

Various mathematical models of deformable solid mechanics are used to study the seismic processes of earth's crust. The processes of wave propagation are most studied in elastic media. But these models do not take into account many real properties of an ambient array. These are, for example, the presence of groundwater, which affects the magnitude and distribution of stresses. Models, which take into account the water saturation of earth's crust structures, presence of gas bubbles, etc., are multicomponent media. A variety of multicomponent media, complexity of processes associated with their deformation, lead to a large difference in methods of analysis and modeling used in solving such problems.

Porous medium saturated with liquid or gas, from the point of view of continuum mechanics, is essentially a two-phase continuous medium, one phase of which is particles of liquid (gas) and other solid particles are its elastic skeleton. There are various mathematical models of such media, developed by various authors. The most famous of them are the models of Biot, Nikolaevsky, and Horoshun [1–5]. However, the class of solved tasks to them is very limited and mainly associated with the construction and study of particular solutions of these equations based on methods of full and partial separation of variables and theory of special functions in the works of Rakhmatullin, Saatov, Filippov, Artykov [6, 7], Erzhanov, Ataliev, Alexeyeva, Shershnev [8, 9], etc. In this regard, it is important to develop effective methods of solution of boundary value problems for such media with the use of modern mathematical methods.

Periodic on time processes are very widespread in practice. By this cause, here we consider also processes of wave propagation in Biot's medium, posed by the periodic forces of different types. Based on the Fourier transformation of generalized functions, we constructed fundamental solutions of oscillation equations of Biot's medium. It is Green tensor, which describes the process of propagation of harmonic waves at a fixed frequency in the space–time of dimension N = 1, 2, 3, under the action concentrated at the coordinates origin. By using this tensor, we construct generalized solutions of these equations for arbitrary sources of periodic disturbances, which can be described as both regular and singular distributions. They can be used to calculate the stress-strain state of a porous water-saturated medium by seismic wave propagation.

2. The parameters and motion equations of a two-component Biot's medium

The equations of motion of a homogeneous isotropic two-component Biot's medium are described by the following system of second-order hyperbolic equations [1–3]:

$$\begin{gathered}(\lambda+\mu)\operatorname{grad}\operatorname{div}u_s+\mu\Delta u_s+Q\operatorname{grad}\operatorname{div}u_f+F^s(x,t)=\rho_{11}\ddot{u}_s+\rho_{12}\ddot{u}_f\\ Q\operatorname{grad}\operatorname{div}u_s+R\operatorname{grad}\operatorname{div}u_f+F^f(x,t)=\rho_{12}\ddot{u}_s+\rho_{22}\ddot{u}_f\\ (x,t)\in R^N\times[0,\infty).\end{gathered}\tag{1}$$

Here N is the dimension of the space. At a plane deformation N = 2, the total spatial deformation corresponds to N = 3, at N = 1 the equations describe the dynamics of a porous liquid-saturated rod.

We denote $u_s=u_{sj}(x,t)e_j$ is a displacement vector of an elastic skeleton, $u_f=u_{fj}(x,t)e_j$ is a displacement vector of a liquid, and $e_j(\ j=1,\ \dots,N)$ are basic orts of Lagrangian Cartesian coordinate system (everywhere by repeating indices, there is summation from 1 to N).

Constants ρ_{11}, ρ_{12}, ρ_{22} have the dimension of mass density, and they are associated with densities of masses of particles, composing a skeleton $\rho_s$and a fluid ρ_f, by relationships:

$$\rho_{11}=(1-m)\rho_s-\rho_{12},\quad \rho_{22}=m\rho_f-\rho_{12},$$

where m is a porosity of the medium. The constant of attached density ρ_{12} is related to a dispersion of deviation of micro-velocities of fluid particles in pores from average velocity of fluid flow and depends on pores geometry. Elastic constants λ,μ are Lama's parameters of an isotropic elastic skeleton, and Q, R characterize an interaction of a skeleton with a liquid on the basis of.

2.1 Biot's law for stresses

$$\begin{aligned}\sigma_{ij}&=\left(\lambda\partial_k u_{sk}+Q\partial_k u_{fk}\right)\delta_{ij}+\mu\left(\partial_i u_{sj}+\partial_j u_{si}\right)\\ \sigma&=-mp=R\partial_k u_{fk}+Q\partial_k u_{sk}\end{aligned}\tag{2}$$

Here $\sigma_{ij}(x,t)$ are a stress tensor in a skeleton, and $p(x,t)$ is a pressure in a fluid component. External mass forces acting on a skeleton $F^s = F^s_j(x,t)e_j$ and on a liquid component $F^f = F^f_j(x,t)e_j$.

Further we use the next notations for partial derivatives: $\partial_k = \frac{\partial}{\partial x_k}$, $u_{j,k} = \partial_k u_j$, $\Delta = \partial_k \partial_k$ is Laplace operator.

There are three sound speeds in this medium:

$$\begin{aligned} c_1^2 &= \frac{\alpha_1 + \sqrt{\alpha_1{}^2 - 4\alpha_2\alpha_3}}{2\alpha_2}, \\ c_2^2 &= \frac{\alpha_1 - \sqrt{\alpha_1{}^2 - 4\alpha_2\alpha_3}}{2\alpha_2}, \\ c_3^2 &= \sqrt{\frac{\rho_{22}\mu}{\alpha_2}} \end{aligned} \tag{3}$$

where the next constants were introduced as:

$$\begin{aligned} \alpha_1 &= (\lambda + 2\mu)\rho_{22} + R\rho_{11} - 2Q\rho_{12}, \\ \alpha_2 &= \rho_{11}\rho_{22} - (\rho_{12})^2, \\ \alpha_3 &= (\lambda + 2\mu)R - Q^2. \end{aligned}$$

The first two speeds $c_1, c_2 (c_1 > c_2)$ describe the velocity of propagation of two types of *dilatational waves*. The second slower dilatation wave is called *repackaging wave*. A third velocity c_3 corresponds to *shear waves* and at $\rho_{12} = 0$ coincides with velocity of shear wave propagation in an elastic skeleton ($c_3 < c_1$).

We introduce also two velocities of propagation of dilatational waves in corresponding elastic body and in an ideal compressible fluid:

$$c_s = \sqrt{\frac{\lambda + 2\mu}{\rho_{11}}}, \qquad c_f = \sqrt{\frac{R}{\rho_{22}}}$$

3. Problems of periodic oscillations of Biot's medium

Construction of motion equation solutions by periodic oscillations is very important for practice since existing power sources of disturbances are often periodic in time and therefore can be decomposed into a finite or infinite Fourier series in the form:

$$\begin{aligned} F^s(x,t) &= \sum_n F^s_n(x) e^{-i\omega_n t}, \\ F^f(x,t) &= \sum_n F^f_n(x) e^{-i\omega_n t} \end{aligned} \tag{4}$$

where periods of oscillation of each harmonic $T_n = 2\pi/\omega_n$ are multiple to the general period of oscillation T. Therefore, it is enough to consider the case of stationary oscillations, when the acting forces are periodic on time with an oscillation frequency ω:

$$\begin{aligned} F^s(x,t) &= F^s(x) e^{-i\omega t}, \\ F^f(x,t) &= F^f(x) e^{-i\omega t} \end{aligned} \tag{5}$$

The solution of Eq. (1) can be represented in the similar form:

$$u_s(x,t) = u_s(x)e^{-i\omega t}, \quad u_f(x) = u_f(x)e^{-i\omega t} \tag{6}$$

where complex amplitudes of displacements $u_s(x)$, $u_f(x)$ must be determined. If the solution has been known for any frequency ω, then we get similar decomposition for displacements of a medium:

$$\begin{aligned} u_s(x,t) &= \sum_n u_{sn}(x)e^{-i\omega_n t}, \\ u_f(x,t) &= \sum_n u_{fn}(x)e^{-i\omega_n t} \end{aligned} \tag{7}$$

which give us the solution of problem for forces (4).

We get equations for complex amplitudes by stationary oscillations, substituting (6) into the system (1):

$$\begin{gathered} (\lambda+\mu)\operatorname{grad}\operatorname{div} u_s + \mu\Delta u_s + Q\operatorname{grad}\operatorname{div} u_f + \rho_{11}\omega^2 u_s + \rho_{12}\omega^2 u_f + F^s(x) = 0 \\ Q\operatorname{grad}\operatorname{div} u_s + R\operatorname{grad}\operatorname{div} u_f + \rho_{12}\omega^2 u_s + \rho_{22}\omega^2 u_f + F^f(x) = 0 \end{gathered} \tag{8}$$

To construct the solutions of this system for different forces, we define Green tensor of it.

4. Green tensor of Biot's equations by stationary oscillations

Let us construct $U_m^j(x,\omega)e^{-i\omega t} \;\; (j,m = 1, \dots, 2N)$ fundamental solutions of the system (1) for the forces in the form:

$$F(x,t) = \begin{pmatrix} F^s \\ F^f \end{pmatrix} = \begin{pmatrix} \delta_k^{[j]} e_k \\ \delta_{k+N}^{[j]} e_k \end{pmatrix} \delta(x)e^{-i\omega t}, \tag{9}$$
$$k = 1, \dots, N, j = 1, \dots, 2N.$$

Here $\delta_k^j = \delta_{jk}$ is the Kronecker symbol, and $\delta(x)$ is the singular delta-function. They describe a motion of Biot's medium at an action of sources of stationary oscillations, concentrated in the point $x = 0$. The upper index of this tensor (... $^{[k]}$) fixes the current concentrated force and its direction. The lower index corresponds to component of movement of a skeleton and a fluid, respectively, $k = 1, \dots, N$ and $k = N+1, \dots, 2N$.

Their complex amplitudes $U_m^j(x,\omega) \;\; (j,m = 1, \dots, 2N)$ satisfy the next system of equation:

$$\begin{aligned} &(\lambda+\mu)U^k_{j,ji} + \mu U^k_{i,jj} + \omega^2\rho_{11}U_i^k + QU^{k+N}_{j,ji} - \omega^2\rho_{12}U_i^{k+N} + \delta(x)\delta_j^k = 0 \\ &QU^k_{j,ji} + \rho_{12}\omega^2 U^k_{i}{}_{,tt} + RU^{k+N}_{j,ji} + \rho_{22}\omega^2 U_i^{k+N} + \delta(x)\delta^k_{j+N} = 0 \\ &j = 1, \dots, 2N, \quad k = 1, \dots, 2N. \end{aligned} \tag{10}$$

Since fundamental solutions are not unique, we'll construct such, which tend to zero at infinity:

$$U_i^j(x,\omega) \to 0 \text{ at } \|x\| \to \infty \tag{11}$$

and satisfy the radiation condition of type of Sommerfeld radiation conditions [10]. Matrix of such fundamental equations is named *Green tensor* of Eq. (8).

5. Fourier transform of fundamental solutions

To construct $U_m^j(x, \omega)$, we use Fourier transformation, which for regular functions has the form:

$$F[\varphi(x)] = \overline{\varphi}(\xi) = \int_{R^N} \varphi(x) e^{i(\xi,x)} dx_1 \dots dx_N$$
$$F^{-1}[\overline{\varphi}(\xi)] = \varphi(x) = \frac{1}{(2\pi)^N} \int_{R^N} \overline{\varphi}(\xi) e^{-i(\xi,x)} d\xi_1 \dots d\xi_N$$

where $\xi = (\xi_1, \dots, \xi_N)$ are Fourier variables.

Let us apply Fourier transformation to Eq. (10) and use property of Fourier transform of derivatives [10]:

$$\frac{\partial}{\partial x_j} \leftrightarrow -i\xi_j \tag{12}$$

Then we get the system of 2 N linear algebraic equations for Fourier components of this tensor:

$$\begin{gathered} -(\lambda+\mu)\xi_j\xi_j\overline{U}_j^k - \mu\|\xi\|^2\overline{U}_j^k - Q\xi_j\xi_j\overline{U}^k_{j+N} + \rho_{11}\omega^2\overline{U}_j^k + \rho_{12}\omega^2\overline{U}^k_{j+N} + \delta_j^k = 0 \\ -Q\xi_j\xi_j\overline{U}_j^k - R\xi_j\xi_l\overline{U}^k_{j+N} + \rho_{12}\omega^2\overline{U}_j^k + \rho_{22}\omega^2\overline{U}^k_{j+N} + \delta^k_{j+N} = 0 \\ j = 1, \dots, N, \qquad k = N+1, \dots, 2N \end{gathered} \tag{13}$$

By using gradient divergence method, this system has been solved by us. For this the next basic function were introduced:

$$\begin{gathered} f_{0k}(\xi,\omega) = \frac{1}{c_k{}^2\|\xi\|^2 - \omega^2}, \\ f_{jk}(\xi,\omega) = \frac{f_{(\ j-1)k}(\xi,\omega)}{-i\omega}, \quad j = 1, 2; \end{gathered} \tag{14}$$

and the next theorem has been proved [11, 12].

Theorem 1. *Components of Fourier transform of fundamental solutions have the form:*

$$for\ j = \overline{1,N}, \qquad k = \overline{1,N},$$
$$\overline{U}_j^k = \left(-i\xi_j\right)(-i\xi_k)[\beta_1 f_{21} + \beta_2 f_{22} + \beta_3 f_{23}] +$$
$$+\frac{1}{\alpha_2}\left(\rho_{12}\delta^k_{j+N} - \rho_{22}\delta_j^k\right)f_{03},$$
$$\overline{U}^k_{j+N} = \left(-i\xi_j\right)(-i\xi_k)[\gamma_1 f_{21} + \gamma_2 f_{22} + \gamma_3 f_{23}] -$$
$$-\frac{\mu}{\alpha_2}\delta^k_{j+N}\|\xi\|^2 f_{23} - \frac{1}{\alpha_2}\left(\rho_{11}\delta^k_{j+N} + \rho_{12}\delta_j^k\right)f_{03};$$

$$for \quad j = 1, \dots, N \qquad k = N+1, \dots, 2N$$

$$\overline{U}_j^k = \left(-i\xi_j\right)(-i\xi_{k-N})\left[\eta_1 f_{21} + \eta_2 f_{22} + \eta_3 f_{23}\right] +$$

$$+\frac{1}{\alpha_2}\left(\rho_{12}\delta^k_{j+N} - \rho_{22}\delta^k_j\right) f_{03}$$

$$\overline{U}^k_{j+N} = \left(-i\xi_j\right)(-i\xi_{k-N})[\varsigma_1 f_{21} + \varsigma_2 f_{22} + \varsigma_3 f_{23}] -$$

$$-\frac{\mu}{\alpha_2}\delta^k_{j+N}\|\xi\|^2 f_{23} - \frac{1}{\alpha_2}\left(\rho_{11}\delta^k_{j+N} + \rho_{12}\delta^k_j\right) f_{03}$$

where the next constants have been introduced as:

$$D_1 = \frac{1}{\alpha_2 \upsilon_{12}}, \quad \upsilon_{lm} = c_l^2 - c_m^2, q_1 = Q\rho_{12} - (\lambda+\mu)\rho_{12}, \quad q_2 = \rho_{11}R - Q\rho_{12},$$

$$d_1 = (\lambda+\mu)\rho_{22} - Q\rho_{12}, \quad d_2 = Q\rho_{22} - R\rho_{12}, \quad d_{3j} = \rho_{12}c_j^2 - Q \quad (j = 1, 2)$$

$$\beta_j = (-1)^{(j+1)}\frac{D_1 c_j^2}{\alpha_2 \upsilon_{3j}}\left(d_1 b_{sj} + d_2 d_{3j}\right), \quad \beta_3 = -\frac{c_3^2}{\alpha_2 \upsilon_{31}\upsilon_{32}}(d_1 b_{s3} + d_2 d_{33});$$

$$\gamma_j = (-1)^{j+1}\frac{D_1 c_j^2}{\alpha_2 \upsilon_{3j}}\left(q_1 b_{fj} + q_2 d_{3j}\right), \quad \gamma_3 = -\frac{D_1 c_3^2 \upsilon_{12}}{\alpha_2 \upsilon_{31}\upsilon_{32}}\left(q_1 b_{f3} + q_2 d_{33}\right);$$

$$\eta_j = (-1)^{j+1}\frac{D_1 c_j^2}{\alpha_2 \upsilon_{3j}}\left(d_j d_{3j} + d_2 b_{js}\right), \quad \eta_3 = -\frac{c_3^2 \upsilon_{12}}{\alpha_2 \upsilon_{31}\upsilon_{32}}(d_1 d_{33} + d_2 b_{3s});$$

$$\varsigma_j = (-1)^{j+1}\frac{D_1 c_j^2}{\alpha_2 \upsilon_{3j}}\left(q_1 d_{3j} + q_2 b_{(4-j)s}\right), \quad \varsigma_3 = -\frac{c_3^2 \upsilon_{12}}{\alpha_2 \upsilon_{31}\upsilon_{32}}\left(q_1 d_{33} + q_2 b_{3s}\right)$$

$$b_{fj} = \rho_{22}\upsilon_{fj}, b_{sj} = \rho_{11}\upsilon_{js}.$$

This form is very convenient for constructing originals of Green tensor.

6. Stationary Green tensor construction: radiation conditions

In this case let us construct the originals of the basic function but only over ξ by constant frequency:

$$\Phi_{0m}(x,\omega) = F_\xi^{-1}\left[f_{0m}(\xi,\omega)\right]$$

which, in accordance with its definition (14), satisfies the equation:

$$\left(c_m^2\|\xi\|^2 - \omega^2\right) f_{0m} = 1 \tag{15}$$

Using property (12) for derivatives from here, we get Helmholtz equation for fundamental solution (accurate within a factor c_m^{-2}):

$$\left(\Delta + k_m^2\right)\Phi_{0m} + c_m^{-2}\delta(x) = 0, \quad k_m = \frac{\omega}{c_m} \tag{16}$$

Fundamental solutions of Helmholtz equation, which satisfy Sommerfeld conditions of radiation:

$$\text{at } r \to \infty$$

$$\Phi'_{0m}(r) - ik_m \Phi_{0m}(r) = O\left(r^{-1}\right), \qquad N = 3,$$

$$\Phi'_{0m}(r) - ik_m \Phi_{0m}(r) = O\left(r^{-1/2}\right), \quad N = 2.$$

are well known [10]. They are unique. Using them, we obtain:

for $N = 3$

$$\Phi_{0m} = \frac{1}{4\pi r c^2} e^{ik_m r}, \quad k_m = \frac{\omega}{c_m};$$

for $N = 2$

$$\Phi_{0m} = \frac{i}{4c^2} H_0^{(1)}(k_m r),$$

where $H_j^{(1)}(k_m r)$ is the cylindrical Hankel function of the first kind:

for $N = 1$

$$\Phi_{0m} = \frac{\sin k_m |x|}{2k_m c_m^2}.$$

These functions (subject to factor $e^{-i\omega t}$) describe harmonic waves which move from the point $x = 0$ to infinity and decay at infinity.

The last property is true only for $N = 2{,}3$. In the case $N = 1$, all fundamental solutions of Eq. (16):

$$\left(\frac{d^2}{dx^2} + k_m^2\right)\Phi_{0m} + c_m^{-2}\delta(x) = 0,$$

do not decay at infinity.

From Theorem 1, the next theorem follows.

Theorem 2. *The components of Green tensor of Biot's equations at stationary oscillations with frequency ω, which satisfy the radiation conditions, have the form:*

for $j = \overline{1,N}, \quad k = \overline{1,N},$

$$U_j^k(x,\omega) = -\omega^{-2} \sum_{m=1}^{3} \beta_m \frac{\partial^2 \Phi_{0m}}{\partial x_j \partial x_k} + \frac{1}{\alpha_2}\left(\rho_{12}\delta^k_{j+N} - \rho_{22}\delta_j^k\right)\Phi_{03},$$

$$U^k_{j+N}(x,\omega) = -\omega^{-2} \sum_{m=1}^{3} \gamma_m \frac{\partial^2 \Phi_{0m}}{\partial x_j \partial x_k} +$$

$$+ \frac{\mu \delta^k_{j+N}}{\alpha_2 \omega^2}\left(c_3^{-2}\delta(x) + k_3^2 \Phi_{0m}\right) - \frac{\rho_{11}\delta^k_{j+N} + \rho_{12}\delta_j^k}{\alpha_2}\Phi_{03};$$

for $j = 1, \dots, N \quad k = N+1, \dots, 2N$

$$U_j^k(x,\omega) = -\omega^{-2}\sum_{m=1}^{3}\eta_m \frac{\partial^2\Phi_{0m}}{\partial x_j \partial x_k} + \frac{1}{\alpha_2}\left(\rho_{12}\delta^k_{j+N} - \rho_{22}\delta^k_j\right)\Phi_{03}$$

$$U^k_{j+N}(x,\omega) = -\omega^{-2}\sum_{m=1}^{3}\varsigma_m \frac{\partial^2\Phi_{0m}}{\partial x_j \partial x_k} +$$

$$+\frac{\mu}{\alpha_2\omega^2}\left(c_3^{-2}\delta(x) + k_3^2\Phi_{0m}\right)\delta^k_{j+N} - \frac{1}{\alpha_2}\left(\rho_{11}\delta^k_{j+N} + \rho_{12}\delta^k_j\right)\Phi_{03}$$

where

for $N = 1$

$$\frac{d^2\Phi_{0m}}{dx^2} = \frac{1}{2c_m^2 k_m}\left({k_m}^2(\sin k_m|x|) - 2k_m\delta(x)\right);$$

for $N = 2$

$$\frac{\partial^2\Phi_{0m}}{\partial x_j \partial x_k} = -\frac{i}{4c_m^2}\left(0.5k_m^2(H_0(k_m r) - H_2(k_m r))r,_j r,_k + kH_1^1(k_m r)r,_{jk}\right);$$

for $N = 3$

$$\frac{\partial\Phi_{0m}}{\partial x_j \partial x_k} = \frac{1}{4\pi r c_m^2}e^{ikr}\left\{r,_j r,_k\left(\left(ik_m - \frac{1}{r}\right)^2 + \frac{1}{r^2}\right) + r,_{jk}\left(ik_m - \frac{1}{r}\right)\right\};$$

$$k_m = \frac{\omega}{c_m}, r = \|x\|, \quad r,_j = \frac{x_j}{r}, \quad r,_{ij} = \frac{1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)..$$

Proof. By using originals of basic functions, property (12) of derivatives, we can obtain from formulas for U_j^k in Theorem 1 the originals of all addends, besides that which contain factor $\|\xi\|^2$. But using (16) we have:

$$-\Delta\Phi = c_m^{-2}\delta(x) + k_m^2\Phi_{0m} \quad \leftrightarrow \quad \|\xi\|^2 f_{0m} = c_m^{-2} + k_m^2 f_{0m}$$

Then formulas of Theorem 2 follow from formulas of Theorem 1.

7. Generalized solutions by arbitrary periodic forces

Under the action of arbitrary mass forces with frequency ω in Biot's medium, the solution for complex amplitudes has the form of a tensor functional convolution:

$$u_j(x,t) = U_j^k(x,\omega) * F_k(x)e^{-i\omega t}, j,k = \overline{1,2N} \tag{17}$$

Note that mass forces may be different from the space of generalized vector function, singular and regular. Since Green tensor is singular and contains delta-functions, this convolution is calculated on the rule of convolution in generalized function space. If a support of acting forces are bounded (contained in a ball of finite radius), then all convolutions exist. If supports are not bounded, then the existence conditions of convolutions in formula (17) requires some limitations on behavior of forces at infinity which depends on a type of mass forces and space dimension.

The obtained solutions allow us to study the dynamics of porous water- and gas-saturated media at the action of periodic sources of disturbances of a sufficiently arbitrary form. In particular, they are applicable in the case of actions of certain forces on surfaces, for example, cracks, in porous media that can be simulated by simple and double layers on the crack surface.

There is another interesting feature of the Green tensor of the Biot's equations, which contains, as one of the terms, the delta-function that complicates the application of this tensor for solving boundary value problems based on analogues of Green formulas for elliptic systems of equations or the boundary element method. Here, when constructing the model, the viscosity of the liquid is not taken into account, which, apparently, leads to the presence of such terms, and it requires improvement of this model taking into account a viscosity.

8. Green tensor of Biot's equations by non-stationary motion

To construct the non-stationary Green tensor, at first we also construct the originals of the basic functions in an initial space-time:

$$\mathbf{\Phi}_{0m}(x,t) = F^{-1}\left[f_{0m}(\xi,\omega)\right] = F^{-1}\left[\left(c_m^2\|\xi\|^2 - \omega^2\right)^{-1}\right]$$

They are originals of the classic wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c_m^2\Delta\right)\mathbf{\Phi}_{0m} = \delta(t)\delta(x) \tag{18}$$

Depending on the dimension of a space, solutions of this wave equation that satisfy the radiation conditions have the following form [10]:

$$\mathbf{\Phi}_{0m}(x,t) = \frac{1}{4\pi c_m^2 r}\delta\left(t - \frac{r}{c_m}\right), \quad N = 3; \tag{19}$$

$$\mathbf{\Phi}_{0m}(x,t) = \frac{1}{2\pi c_m}\frac{H(ct-r)}{\sqrt{c_m^2t^2 - r^2}}, \quad N = 2; \tag{20}$$

$$\mathbf{\Phi}_{0m}(x,t) = \frac{1}{2}H(c_m t - |x|), \quad N = 1. \tag{21}$$

Here $H(t)$ is the Heaviside function, and singular function $\delta(t - r/c_m)$ is the *simple layer on the sound cone* $r = c_m t$, $r = \|x\|$.

Using regularization of the general function ω^{-1}in the space of distribution [10]:

$$H(t)\delta(x) \leftrightarrow \frac{1}{-i(\omega + i0)}$$

and the properties of Fourier transform of generalized functions convolution:

$$h = f * g \leftrightarrow \overline{h} = \overline{f} \times \overline{g}$$

It is easy to show that the next lemma is true.

Lemma. *The originals of the primitives of the basic functions satisfying the radiation conditions are representable in the following form:*

for N = 3

$$\begin{aligned}\mathbf{\Phi}_{1m}(x,t)&=\mathbf{\Phi}_{0m}(x,t)*H(t)\delta(x)=\frac{H(c_m t-r)}{4\pi c^2 r},\\ \mathbf{\Phi}_{2m}(x,t)&=\mathbf{\Phi}_{1m}(x,t)*H(t)\delta(x)=\frac{(c_m t-r)_+}{4\pi c^3 r};\end{aligned}\tag{22}$$

for N = 2

$$\begin{aligned}\mathbf{\Phi}_{1m}(x,t)&=\frac{1}{2\pi c^2}\ln\left(\frac{c_m t+\sqrt{c_m^2t^2-r^2}}{r}\right),\\ \mathbf{\Phi}_{2m}(x,t)&=\frac{1}{2\pi c^3}\left(c_m t\ln\left(\frac{c_m t+\sqrt{c_m^2t^2-r^2}}{r}\right)-\sqrt{c_m^2t^2-r^2}\right);\end{aligned}\tag{23}$$

for N = 1

$$\begin{aligned}\mathbf{\Phi}_{1m}(x,t)&=0,5(c_m t-r)H(c_m t-r)\triangleq 0.5\,(c_m t-r)_+,\\ \mathbf{\Phi}_{2m}(x,t)&=\frac{1}{2c^2}(c_m t-r)^2H(c_m t-r)\triangleq\frac{1}{2c^2}\,(c_m t-r)_+^2.\end{aligned}\tag{24}$$

Using these functions and the properties of the Fourier transform, we obtain the components of the Green tensor from the formulas of Theorem 1. We formulate the result in the next theorem.

Theorem 3. *The components of Green tensor of motion equations of two-component Biot's medium have the following forms:*

For $j=\overline{1,N},\quad k=\overline{1,N},$

$$U_j^k(x,t)=\sum_{m=1}^{3}\beta_m\frac{\partial^2\mathbf{\Phi}_{2m}}{\partial x_j\partial x_k}+\frac{1}{\alpha_2}\left(\rho_{12}\delta^k_{j+N}-\rho_{22}\delta_j^k\right)\mathbf{\Phi}_{03}(x),$$

$$U^k_{j+N}(x,t)=\sum_{m=1}^{3}\gamma_m\frac{\partial^2\mathbf{\Phi}_{2m}}{\partial x_j\partial x_k}+\frac{\mu}{\alpha_2c_m^2}\delta^k_{j+N}(\mathbf{\Phi}_{0m}-t_+\delta(x))-$$

$$-\frac{1}{\alpha_2}\left(\rho_{11}\delta^k_{j+N}+\rho_{12}\delta_j^k\right)\mathbf{\Phi}_{03}(x)$$

For $j=\overline{1,N},\quad k=\overline{N+1,2N},$

$$U_j^k(x,t)=\sum_{m=1}^{3}\eta_m\frac{\partial^2\mathbf{\Phi}_{2m}}{\partial x_j\partial x_{k-N}}+\frac{1}{\alpha_2}\left(\rho_{12}\delta^k_{j+N}-\rho_{22}\delta_j^k\right)\mathbf{\Phi}_{03}(x),$$

$$U^k_{j+N}(x,t)=\sum_{m=1}^{3}\varsigma_m\frac{\partial^2\mathbf{\Phi}_{2m}}{\partial x_j\partial x_{k-N}}+\frac{\mu}{\alpha_2c_m^2}\delta^k_{j+N}(\mathbf{\Phi}_{0m}-t_+\delta(x))-$$

$$-\frac{1}{\alpha_2}\left(\rho_{11}\delta^k_{j+N}+\rho_{12}\delta_j^k\right)\mathbf{\Phi}_{03}(x)$$

Here
for N = 1

$$\frac{d^2\mathbf{\Phi}_{2m}}{dx^2}=\frac{H(c_m t-|x|)}{c_m^2}-\frac{2}{c_m^2}(c_m t-|x|)_+\delta(x)\tag{25}$$

for $N = 2$

$$\frac{\partial^2 \mathbf{\Phi}_{2m}}{\partial x_j \partial x_k} = \frac{H(c_m t - r)}{2\pi c_m^3 r^3} \left(\frac{2c_m^2 t^2 - r^2}{\sqrt{c_m^2 t^2 - r^2}} r,_k r,_j - \delta_{jk} \sqrt{c_m^2 t^2 - r^2} \right) \tag{26}$$

for $N = 3$

$$\frac{\partial^2 \mathbf{\Phi}_{2m}}{\partial x_j \partial x_k} = \frac{t}{4\pi c_m^2 r^2} \left(\delta(c_m t - r) r,_k r,_j - \frac{tH(c_m t - r)}{r} \left(\delta_{jk} - 3r,_k r,_j \right) \right), \tag{27}$$

$$r,_j = x_j / r.$$

9. Generalized solutions of Biot's equations by non-stationary forces

Using the properties of Green tensor, we obtain generalized solutions of non-stationary Biot's equations under the action of arbitrary mass forces in the Biot's medium, which satisfy the radiation condition at infinity. They have the form of tensor functional convolution:

$$u_j(x,t) = U_j^k(x,t) * F_k(x,t), \quad j,k = 1, \dots, 2N \tag{28}$$

It's taken according to the rules of convolution of generalized functions depending on the type of mass forces [10].

In order to get the classic solution, we must present formulas (28) in regular integral forms. For this, let us present matrix of Green tensor as sum of regular functions and singular functions, which contain delta-function:

$$U(x,t) = U_{reg}(x,t) + U_{sing}(t)\delta(x).$$

Then also write:

$$u(x,t) = u1(x,t) + u2(x,t) \tag{29}$$

Here $u1(x,t)$ is representable by regular mass forces in the integral form:

$$u1(x,t) = H(t) \int_o^t d\tau \int_{R^N} U_{reg}(x-y,\tau) \times \begin{pmatrix} F_s(y,t-\tau) \\ F_f(y,t-\tau) \end{pmatrix} dy$$

The convolution with singular part is equal to:

$$u2(x,t) = H(t) \int_o^t U_{sing}(\tau) \times \begin{pmatrix} F_s(x,t-\tau) \\ F_f(x,t-\tau) \end{pmatrix} d\tau$$

In 3D space, there are convolutions with simple layers on sound cones (see (27)). To construct their integral presentation, use this rule:

$$\alpha(x,t)\delta(c_m t - r) * F(x,t) =$$

$$= H(t) \int_0^t d\tau \int_{\|y-x\| = c_m \tau} \alpha(x-y,\tau) F\left(y, t - \frac{\|y-x\|}{c_m}\right) dS(y)$$

Here the internal integral is taken over sphere with center in the point x, and its radius is equal to $c_m\tau$.

If components of acting forces $F(x,t)$ are double differentiable vector function, it is convenient to use the property of differentiation of convolution [10]:

$$\frac{\partial^2 \mathbf{\Phi}_{2m}}{\partial x_j \partial x_k} * F(x,t) = \frac{\partial^2}{\partial x_j \partial x_k} (\mathbf{\Phi}_{2m} * F(x,t)) = \mathbf{\Phi}_{2m}(x,t) * \frac{\partial^2 F}{\partial x_j \partial x_k}$$

Substituting the formulas of Theorem 4 into (29), we obtain displacements and stresses of skeleton and liquid in Biot's medium in spaces of dimension N = 1, 2, 3. Calculation of these convolutions by using these formulas essentially depends on the form of acting forces and gives possibility to construct regular presentation of generalized solution for wide class of acting forces, which are the classic solution of Biot's equation.

10. Calculation of the stress state of Biot's medium

Using Biot's law (2), we can define the generalized stresses in skeleton and a pressure in a liquid:

$$\begin{gathered} \sigma_{ij} = \left(\lambda \partial_l U_l^k * F_{ks} + Q \partial_l U_l^{k+N} * F_{kf}\right)\delta_{ij} + \\ + \mu\left(\partial_i U_j^k * F_{ks} + \partial_j U_i^{k+N} * F_{kf}\right) \\ \sigma = -mp = R \partial_l U_l^{k+N} * F_{kf} + Q \partial_l U_l^k * F_{ks} \end{gathered} \tag{30}$$

These formulas also can be written in integral form by using the same rules. But we can apply here the next lemma, which was proved in [11].

Lemma. *Fourier transformations of the divergences of Green tensor have the next form:*

by $k = 1, \dots, N$

$$\begin{aligned} &F\left[\partial_j U_j^k\right] = D_1 i\xi_k \left(b_{f1} f_{01}(\xi,\omega) - b_{f2} f_{02}(\xi,\omega)\right) \\ &F\left[\partial_j U^k_{j+N}\right] = D_1 i\xi_k (d_{31} f_{01}(\xi,\omega) - d_{32} f_{02}(\xi,\omega)) \\ &j = 1, \dots, N. \end{aligned}$$

by $k = N+1, \dots, 2N$

$$F\left[\partial_j U_j^k\right] = i\xi_{k-N} D_1 \left(d_{31} f_{01}(\xi,\omega) - d_{32} f_{02}(\xi,\omega)\right) F\left[\partial_j U^k_{j+N}\right] = i\xi_{k-N} D_1 (b_{s1} f_{01}(\xi,\omega) - b_{s2} f_{02}(\xi,\omega)) \qquad j =$$

From this lemma, we can prove easily the next theorem.

Theorem 4. *Divergences of elastic and liquid displacement of Green tensor have the next form:*

for $k = 1, \dots, N$

$$\begin{aligned} \partial_j U_j^k &= -D_1 \left(b_{f1} \partial_k \mathbf{\Phi}_{01} - b_{f2} \partial_k \mathbf{\Phi}_{02}\right) \\ \partial_j U^k_{j+N} &= -D_1 (d_{31} \partial_k \mathbf{\Phi}_{01} - d_{32} \partial_k \mathbf{\Phi}_{02}) \end{aligned}$$

$j = 1, \dots, N.$

for $k = N+1, \ldots, 2N$

$$\partial_j U_j^k = -D_1(d_{31}\partial_{k-N}\boldsymbol{\Phi}_{01} - d_{32}\partial_{k-N}\boldsymbol{\Phi}_{02})$$

$$\partial_j U^k_{j+N} = -D_1(b_{s1}\partial_{k-N}\boldsymbol{\Phi}_{01} - b_{s2}\partial_{k-N}\boldsymbol{\Phi}_{02})$$

$$j = 1, \ldots, N.$$

Substituting these formulas in (30), we define the stresses in the skeleton and the pressure in the liquid of Biot's medium.

If we paste $\Phi_{02}(x, \omega)$ instead of $\boldsymbol{\Phi}_{02}(x, t)$ in formulas of this theorem, then formula (30) expresses complex amplitudes of stress tensor and pressure by periodic oscillations. It is used to determine stresses and pressure by solving the periodic problems (4).

11. Conclusion

The obtained solutions give possibility to study the dynamics of porous water- and gas-saturated media and rods under actions of disturbance sources of different forms and can be used for solutions of boundary value problems in porous media by using boundary element method.

These solutions can be used for describing wave processes by explosions and earthquakes. In these cases mass forces are described by using singular generalized function, such as multipoles, simple and double layers, and others.

Acknowledgements

This work was financially supported by the Ministry of Education and Science of the Republic of Kazakhstan (Grant AP05132272).

Author details

Lyudmila Alexeyeva[1*] and Yergali Kurmanov[1,2]

1 Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

2 Al Farabi Kazakh National University, Almaty, Kazakhstan

*Address all correspondence to: alexeeva@math.kz

References

[1] Biot MA. Theory of propagation of elastic waves in a fluid-saturated porous solid. 1. Low frequency range. The Journal of the Acoustical Society of America. 1956;**28**(2):168-178

[2] Biot MA. Theory of propagation of elastic waves in a fluid-saturated porous solid. 2. Higher frequency range. The Journal of the Acoustical Society of America. 1956;**28**(2):178-191

[3] Biot MA. Mechanics of deformation and propagation of acoustic waves in a porous medium. Mechanics. 1963;**6**(28): 103-110

[4] Nikolayevsky VN. Mechanics of Porous and Fractured Media. Moscow: Nauka; 1984. p. 232

[5] Horoshun LP. To the theory of saturated porous media. International Journal of Applied Mechanics. 1976; **12**(12):36-48

[6] Rakhmatullin KA, Saatov YU, Filippov IG, Artykov TU. Waves in Two-Component Media. Tashkent: Nauka UzSSR; 1974. p. 266

[7] Saatov YU. Plane Problems of Mechanics of Elastic-Porous Media. Tashkent: Nauka UzSSR; 1972. p. 248

[8] Yerzhanov ZS, Aitaliev SM, Alexeyeva LA. Dynamics of Tunnels and Underground Pipelines. Alma-Ata: Nauka KazSSR; 1989. p. 240

[9] Alexeyeva LA, Shershnev VV. Fundamental solutions of the equations of motion equation of Biot are medium. In: Reports of National Academy of Sciences of Rep. Kazakhstan. Vol. 1. 1994. pp. 3-6

[10] Vladimirov VS. Generalized Functions in Mathematical Physics. Moscow: Mir; 1978. p. 282

[11] Alexeyeva LA, Kurmanov EB. Fundamental and generalized solutions of the two-component medium M. Bio 1. The Fourier transform of fundamental solutions and their regularization. Mathematical Journal. 2017;**17**(2):13-30

[12] Alexeyeva LA, Kurmanov EB. Fourier transform of fundamental solutions for the motion equations of two-component Biot's media. AIP Conference Proceedings. 2017;**1880**:1. DOI: 10.1063/1.5000675