Chapter

Fixed Point Theorems of a New Generalized Nonexpansive Mapping

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Abstract

This paper introduces a $T-(D_a)$ mapping that is weaker than the nonexpansive mapping. It introduces several iterations for the fixed point of the $T-(D_a)$ mapping. It gives fixed point theorems and convergence theorems for the $T-(D_a)$ mapping in Banach space, instead of uniformly convex Banach space. This paper gives some basic properties on the $T-(D_a)$ mapping and gives the example to show the existence of $T-(D_a)$ mapping. The results of this paper are obtained in general Banach spaces. It considers some sufficient conditions for convergence of fixed points of mappings in general Banach spaces under higher iterations.

Keywords: iteration, convergence theorems, nonexpansive mapping, fixed point

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1. Introduction

In this paper, E is a Banach space, C is a nonempty closed convex subset of E, and $Fix(T) = x \in C : Tx = x$.

Definition 1. T is contraction mapping if there is $r \in [0, 1)$

$$\|Tx - Ty\| \leq r\|x - y\| \text{ for all } x \ \ y \in C.$$

Definition 2. T is nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x \ \ y \in C.$$

Definition 3. T is quasinonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x \in C, y \in F(T).$$

Definition 4. $T : C \to C$ is a $T-(D_a)$ mapping on a subset C, if there is $a \in \left(\frac{1}{2}, 1\right)$, $\|Tx - Ty\| \leq \|x - y\|$ for all $\alpha \in [a, 1], x \in C, y \in C(T, x, \alpha)$, where $C(T, x, \alpha) = \{y \in C | y = (1-\alpha)p + \alpha Tp, p \in C, \|Tp - p\| \leq \|Tx - x\|\}$.

In 2008 Suzuki [1] defined a mapping T in Banach space: $\frac{1}{2}\|Tx - Ty\| \leq \|x - y\|$ implies $\|Tx - Ty\| \leq \|x - y\|$. And T is said to satisfy condition (C). Suzuki [1] showed that the mapping satisfying condition (C) is weaker than nonexpansive mapping and stronger than quasinonexpansive mapping.

Suzuki [1] proved the theorem T is a mapping in Banach space, T satisfies condition (C), and $\{x_n\}$ is the sequence defined by the iteration process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases} \tag{1}$$

then $\{x_n\}$ converges to a fixed point of T.

Suzuki [1] gave this convergence theorem in an ordinary Banach space, and the mapping satisfying condition (C) is weaker than nonexpansive mapping.

In 2016, Thakur [2] proved the theorem T is a mapping in uniformly convex Banach space, T satisfies condition (C), and $\{x_n\}$ is the sequence defined by iteration process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Ty_n, \\ y_n = Tz_n, \\ z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases} \tag{2}$$

then $\{x_n\}$ converges to a fixed point of T.

Thakur [2] claimed that the rate of iteration is fastest of known iterations. However, the disadvantage is that their results must be in uniformly convex Banach space, instead of the ordinary Banach space.

The aim of this article is there exists a generalized nonexpansive mapping, which makes the sequence generated by Thakur's iteration converge to a fixed point in a general Banach space.

The following propositions are obvious:

Proposition 1. If T is nonexpansive, then T satisfies condition (D_a).

Proposition 2. If T is $T-(D_a)$ mapping, then T is quasinonexpansive.

Proposition 3. Suppose $T: C \to C$ is a $T-(D_a)$ mapping. Then, for $x, y \in C$:

(1) $\|T^2x - Tx\| \le \|Tx - x\|$ for all $x \in C$.

(2) $\|T^2x - Ty\| \le \|Tx - y\|$ or $\|T^2y - Tx\| \le \|Ty - x\|$ for all $x, y \in C$.

Proof:

(1) Since $\|Tx - x\| \le \|Tx - x\|$, $Tx \in C(T, x, 1)$, we have $\|T^2x - Tx\| \le \|Tx - x\|$.

(2) For all $x, y \in C$, $\|Tx - x\| \le \|Ty - y\|$ or $\|Ty - y\| \le \|Tx - x\|$.

Then $Tx \in C(T, y, \alpha)$ or $Ty \in C(T, x, \alpha)$.

It follows that $\|T^2x - Ty\| \le \|Tx - y\|$ or $\|T^2y - Tx\| \le \|Ty - x\|$.

Example 1

$$Tx = \begin{cases} \begin{pmatrix} 1.1 & x_2 \\ x_3 & x_4 \end{pmatrix}, & x_1 = 3, \\ \begin{pmatrix} 0 & x_2 \\ x_3 & x_4 \end{pmatrix}, & x_1 \ne 3, \end{cases}$$

where

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, x_1 \in [0, 3], x_2 \in [0, 0.01], x_3 \in [0, 0.01], x_4 \in [0, 0.01].$$

$$\|x\|_1 = \max\{|x_1| + |x_3|, |x_2| + |x_4|\}$$

Set

$$x = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 2.5 & 0 \\ 0 & 0 \end{pmatrix}$$

We see that

$$\|Tx - Ty\|_1 = 1.1 > \|x - y\|_1.$$

Hence, T is not a nonexpansive mapping.
To verify that T is a $T-(D_a)$ mapping, consider the following cases:
Case 1:

$$\alpha \in \left[\frac{11}{19}, 1\right], x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, x_1 \neq 3.$$
$$y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in C(T, x, \alpha),$$

then $y_1 \neq 3$. We have

$$\|Ty - Tx\| = \left\| \begin{pmatrix} 0 & y_2 - x_2 \\ y_3 - x_3 & y_4 - x_4 \end{pmatrix} \right\| \le \left\| \begin{pmatrix} y_1 - x_1 & y_2 - x_2 \\ y_3 - x_3 & y_4 - x_4 \end{pmatrix} \right\| = \|y - x\|$$

Case 2:

$$\alpha \in \left[\frac{11}{19}, 1\right], x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, x_1 = 3.$$
$$y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in C(T, x, \alpha),$$

then $y_1 \in [0, 1.9]$. We have

$$\|Ty - Tx\| = \left\| \begin{pmatrix} 1.1 & y_2 - x_2 \\ y_3 - x_3 & y_4 - x_4 \end{pmatrix} \right\| \le 1.11 \le \|y - x\|$$

Hence, T is a $T-(D_a)$ mapping, and T is not nonexpansive.

2. Fixed point

In this section, we prove convergence theorems for fixed point of the $T-(D_a)$ mapping in Banach space.

Lemma 1. Let C be bounded convex subset of a Banach space B. Assume that $T : C \to C$ is $T - (D_a)$ mapping and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are sequences generated by iteration:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Ty_n, \\ y_n = Tz_n, \\ z_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \end{cases} \tag{3}$$

where $\frac{1}{2} < a \le \alpha_n \le b < 1$. Then

(1) $\|Tx_{n+1} - x_{n+1}\| \le \|Ty_n - y_n\| \le \|Tz_n - z_n\| \le \|Tx_n - x_n\|$.

(2) $\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ty_n - y_n\| = \lim_{n\to\infty} \|Tz_n - z_n\| = r \ge 0$.

Proof: (1) From Proposition 3 and $z_n = (1-\alpha_n)x_n + \alpha_n Tx_n$, we have

$$\begin{aligned} \|Tx_{n+1} - x_{n+1}\| &\le \|T^2 y_n - Ty_n\| \\ &\le \|Ty_n - y_n\| = \|T^2 z_n - Tz_n\| \\ &\le \|Tz_n - z_n\| \\ &= \|Tz_n - Tx_n + (1-\alpha_n)(Tx_n - x_n)\| \\ &\le \|z_n - x_n\| + (1-\alpha_n)\|Tx_n - x_n\| \\ &= \|Tx_n - x_n\|. \end{aligned}$$

(2) From (1), we have $0 \le \|Tx_{n+1} - x_{n+1}\| \le \|Tx_n - x_n\|$. So $\lim_{n\to\infty} \|Tx_n - x_n\| = r \ge 0$. Now, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ty_n - y_n\| = \lim_{n\to\infty} \|Tz_n - z_n\| = r \ge 0$.

Lemma 2. Assume that $T : C \to C$ is a $T - (D_a)$ mapping and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are sequences generated by iteration (3). $\frac{1}{2} < a \le \alpha_n \le b < 1$. Let $\{u_m\}$ satisfy $u_{3n-2} = x_n, u_{3n-1} = z_n, u_{3n} = y_n$. Then, for all $n \ge 1, p \ge 1$

$$\begin{aligned} &\left(1 + \sum_{k=3n-2}^{3n+p-3} \beta_k\right) \|Tu_{3n-2} - u_{3n-2}\| \le \|Tu_{3n-2+p} - u_{3n-2}\| \\ &+ \left(\prod_{k=n}^{n+p-1} \frac{2}{1-\alpha_k}\right) \left(\|Tu_{3n-2} - u_{3n-2}\| - \|Tu_{3n-2+3p} - u_{3n-2+3p}\|\right), \end{aligned} \tag{4}$$

where

$$\beta_k = \begin{pmatrix} \alpha_n & k = 3n-2 \\ 1 & k \ne 3n-2 \end{pmatrix}$$

Proof: From **Lemma 1**, we have

$$\begin{aligned} &\|Tx_{n+1} - x_{n+1}\| \\ &\le \|Tz_n - z_n\| \\ &= \|Tz_n - (1-\alpha_n)x_n - \alpha_n Tx_n\| \\ &\le (1-\alpha_n)\|Tz_n - x_n\| + \alpha_n \|Tz_n - Tx_n\| \\ &\le (1-\alpha_n)\|Tz_n - x_n\| + \alpha_n \|z_n - x_n\| \\ &= (1-\alpha_n)\|Tz_n - x_n\| + \alpha_n^2 \|Tx_n - x_n\|. \end{aligned}$$

So, for $p = 1$ and all $n \geq 1$

$$
\begin{aligned}
&(1+\beta_{3n-2})\|Tu_{3n-2} - u_{3n-2}\| \\
&= (1+\alpha_n)\|Tx_n - x_n\| \\
&\leq \|Tx_n - x_n\| + \left(\frac{1}{1-\alpha_n}\right)(\|Tx_n - x_n\| - \|Tx_{n+1} - x_{n+1}\|) \\
&= \|Tu_{3n-1} - u_{3n-2}\| + \left(\frac{1}{1-\alpha_n}\right)(\|Tu_{3n-2} - u_{3n-2}\| - \|Tu_{3n+1} - u_{3n+1}\|) \\
&\leq \|Tu_{3n-1} - u_{3n-2}\| + \left(\frac{2}{1-\alpha_n}\right)(\|Tu_{3n-2} - u_{3n-2}\| - \|Tu_{3n+1} - u_{3n+1}\|).
\end{aligned}
$$

(4) holds.

We make the inductive assumption that (4) holds for a given $p > 1$ and all $n > 0$ and obtain, upon replacing n with $n+1$

$$
\begin{aligned}
&\left(1 + \sum_{k=3n+1}^{3n+p} \beta_k\right)\|Tu_{3n+1} - u_{3n+1}\| \leq \|Tu_{3n+1+p} - u_{3n+1}\| \\
&+ \left(\prod_{k=n+1}^{n+p} \frac{2}{1-\alpha_k}\right)\big(\|Tu_{3n+1} - u_{3n+1}\| - \|Tu_{3n+1+3p} - u_{3n+1+3p}\|\big).
\end{aligned}
\tag{5}
$$

And obviously

$$k \geq 3n-2,\ \|Tu_{k+1} - Tu_k\| \leq \beta_k\|Tu_{3n-2} - u_{3n-2}\|, \tag{6}$$

$$k > t,\ \|Tu_k - Tu_t\| \leq \|u_k - u_t\|. \tag{7}$$

Case 1: $p = 3m$, $m \geq 1$. From (6) and (7)

$$
\begin{aligned}
&\|Tu_{3n+1+p} - u_{3n+1}\| \\
&= \|Tx_{n+m+1} - x_{n+1}\| \\
&= \|Tx_{n+m+1} - Ty_n\| \\
&\leq \|x_{n+m+1} - y_n\| \\
&= \|Ty_{n+m} - Tz_n\| \\
&\leq \|y_{n+m} - z_n\| \\
&= \|Tz_{n+m} - (1-\alpha_n)x_n - \alpha_n Tx_n\| \\
&\leq (1-\alpha_n)\|Tz_{n+m} - x_n\| + \alpha_n\|Tz_{n+m} - Tx_n\| \\
&\leq (1-\alpha_n)\|Tz_{n+m} - x_n\| \\
&+\alpha_n\big(\|Tz_{n+m} - Tx_{n+m}\| + \|Tx_{n+m} - Ty_{n+m-1}\| + \cdots + \|Ty_n - Tz_n\| + \|Tz_n - Tx_n\|\big) \\
&= (1-\alpha_n)\|Tu_{3n-1+p} - u_{3n-2}\| + \alpha_n \sum_{k=3n-2}^{3n-2+p} \|Tu_{k+1} - Tu_k\| \\
&\leq (1-\alpha_n)\|Tu_{3n-1+p} - u_{3n-2}\| + \alpha_n \sum_{k=3n-2}^{3n-2+p} \beta_k\|Tu_{3n-2} - u_{3n-2}\|.
\end{aligned}
$$

It follows that

$$\|Tu_{3n+1+p} - u_{3n+1}\| \leq (1-\alpha_n)\|Tu_{3n-1+p} - u_{3n-2}\| + \alpha_n \sum_{k=3n-2}^{3n-2+p} \beta_k\|Tu_{3n-2} - u_{3n-2}\| \tag{8}$$

Using (5) and (8), we have

$$\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\|Tu_{3n+1}-u_{3n+1}\|$$
$$\leq (1-\alpha_n)\|Tu_{3n-1+p}-u_{3n-2}\|+\alpha_n\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\|$$
$$+\left(\prod_{k=n+1}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n+1}-u_{3n+1}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

From $\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\leq\left(\prod_{k=n+1}^{n+p}\frac{1}{1-\alpha_k}\right)$ and $\|Tu_{3n+1}-u_{3n+1}\|\leq\|Tu_{3n-2}-u_{3n-2}\|$, we have

$$\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq (1-\alpha_n)\|Tu_{3n-1+p}-u_{3n-2}\|+\alpha_n\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\|$$
$$+\left(\prod_{k=n+1}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

Then

$$\frac{\left(1+\sum_{k=3n+1}^{3n+p}\beta_k-\alpha_n\sum_{k=3n-2}^{3n-2+p}\beta_k\right)}{1-\alpha_n}\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq\|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

It follows that

$$\left(1+\sum_{k=3n-2}^{3n-2+p}\beta_k\right)\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq\|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

Thus, for $n, p+1$, (4) holds.
Case 2: $p=3m+1, m\geq 0$. From (6) and (7), we have

$$\|Tu_{3n+1+p}-u_{3n+1}\|$$
$$=\|Tz_{n+m+1}-x_{n+1}\|$$
$$=\|Tz_{n+m+1}-Ty_n\|$$
$$\leq\|z_{n+m+1}-y_n\|$$
$$=\|(1-\alpha_{m+n+1})x_{m+n+1}+\alpha_{m+n+1}Tx_{m+n+1}-Tz_n\|$$

$$
\begin{aligned}
&\leq (1-\alpha_{m+n+1})\|x_{m+n+1}-Tz_n\| + \alpha_{m+n+1}\|Tx_{m+n+1}-Tz_n\| \\
&\leq (1-\alpha_{m+n+1})\|Ty_{m+n}-Tz_n\| + \alpha_{m+n+1}\|x_{m+n+1}-z_n\| \\
&\leq (1-\alpha_{m+n+1})(\|Ty_{m+n}-Tz_{m+n}\| + \|Tz_{m+n}-Tx_{n+m}\| + \cdots + \|Tx_{n+1}-Ty_n\| \\
&+\|Ty_n-Tz_n\|) + \alpha_{m+n+1}\|x_{m+n+1}-z_n\| \\
&= (1-\alpha_{m+n+1})\sum_{k=3n-1}^{3n-2+p}\|Tu_{k+1}-Tu_k\| + \alpha_{m+n+1}\|Ty_{m+n}-(1-\alpha_n)x_n-\alpha_n Tx_n\| \\
&\leq (1-\alpha_{m+n+1})\sum_{k=3n-1}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\| \\
&+\alpha_{m+n+1}\big((1-\alpha_n)\|Ty_{m+n}-x_n\| + \alpha_n\|Ty_{m+n}-Tx_n\|\big) \\
&\leq (1-\alpha_{m+n+1})\sum_{k=3n-1}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\| + \alpha_{m+n+1}(1-\alpha_n)\|Ty_{m+n}-x_n\| \\
&+\alpha_{m+n+1}\alpha_n\big(\|Ty_{m+n}-Tz_{m+n}\| + \|Tz_{m+n}-Tx_{m+n}\| + \cdots + \|Ty_n-Tz_n\| + \|Tz_n-Tx_n\|\big) \\
&= (1-\alpha_{m+n+1})\sum_{k=3n-1}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\| + \alpha_{m+n+1}(1-\alpha_n)\|Ty_{m+n}-x_n\| \\
&+\alpha_{m+n+1}\alpha_n\sum_{k=3n-2}^{3n-2+p}\|Tu_{k+1}-Tu_k\| \\
&\leq (1-\alpha_{m+n+1})\sum_{k=3n-1}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\| + \alpha_{m+n+1}(1-\alpha_n)\|Ty_{m+n}-x_n\| \\
&+\alpha_{m+n+1}\alpha_n\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_k-u_k\| \\
&= (1-\alpha_{m+n+1})\sum_{k=3n-1}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\| + \alpha_{m+n+1}(1-\alpha_n)\|Tu_{3n-1+p}-u_{3n-2}\| \\
&+\alpha_{m+n+1}\alpha_n\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_k-u_k\| \\
&= (1-\alpha_{m+n+1}+\alpha_{m+n+1}\alpha_n)\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\| \\
&-\alpha_n(1-\alpha_{m+n+1})\|Tu_{3n-2}-u_{3n-2}\| + \alpha_{m+n+1}(1-\alpha_n)\|Tu_{3n-1+p}-u_{3n-2}\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|Tu_{3n+1+p}-u_{3n+1}\| \\
&\leq (1-\alpha_{m+n+1}+\alpha_{m+n+1}\alpha_n)\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\| \\
&-\alpha_n(1-\alpha_{m+n+1})\|Tu_{3n-2}-u_{3n-2}\| + \alpha_{m+n+1}(1-\alpha_n)\|Tu_{3n-1+p}-u_{3n-2}\|
\end{aligned}
\tag{9}
$$

Using (5) and (9), we have

$$
\begin{aligned}
&\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\|Tu_{3n+1}-u_{3n+1}\| \\
&\leq (1-\alpha_{m+n+1}+\alpha_{m+n+1}\alpha_n)\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\|
\end{aligned}
$$

$$-\alpha_n(1-\alpha_{m+n+1})\|Tu_{3n-2}-u_{3n-2}\|+\alpha_{m+n+1}(1-\alpha_n)\|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n+1}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n+1}-u_{3n+1}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

From $\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\leq\left(\prod_{k=n+1}^{n+p}\frac{1}{1-\alpha_k}\right)$ and $\|Tu_{3n+1}-u_{3n+1}\|\leq\|Tu_{3n-2}-u_{3n-2}\|$, we have

$$\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq(1-\alpha_{m+n+1}+\alpha_{m+n+1}\alpha_n)\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tu_{3n-2}-u_{3n-2}\|$$
$$-\alpha_n(1-\alpha_{m+n+1})\|Tu_{3n-2}-u_{3n-2}\|+\alpha_{m+n+1}(1-\alpha_n)\|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n+1}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

Then

$$\frac{\left(1+\sum_{k=3n+1}^{3n+p}\beta_k+\alpha_n(1-\alpha_{m+n+1})-(1-\alpha_{m+n+1}+\alpha_{m+n+1}\alpha_n)\sum_{k=3n-2}^{3n-2+p}\beta_k\right)}{\alpha_{m+n+1}(1-\alpha_n)}$$
$$\|Tu_{3n-2}-u_{3n-2}\|\leq\|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

It follows that

$$\left(1+\sum_{k=3n-2}^{3n-2+p}\beta_k\right)\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq\|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n}^{n+p}\frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|\right).$$

Thus, for $n, p+1$, (4) holds.
Case 3: $p=3m+2, m\geq 0$. From (6) and (7), we have

$$\|Tu_{3n+1+p}-u_{3n+1}\|$$
$$=\|Ty_{n+m+1}-Ty_n\|$$
$$\leq\|y_{n+m+1}-y_n\|$$
$$=\|Tz_{n+m+1}-Tz_n\|$$
$$\leq\|z_{n+m+1}-z_n\|$$
$$\leq\|z_{n+m+1}-(1-\alpha_n)x_n-\alpha_nTx_n\|$$
$$\leq(1-\alpha_n)\|z_{n+m+1}-x_n\|+\alpha_n\|z_{n+m+1}-Tx_n\|$$
$$=(1-\alpha_n)\|(1-\alpha_{n+m+1})x_{n+m+1}+\alpha_{n+m+1}Tx_{n+m+1}-x_n\|$$
$$+\alpha_n\|(1-\alpha_{n+m+1})x_{n+m+1}+\alpha_{n+m+1}Tx_{n+m+1}-Tx_n\|$$

$$
\begin{aligned}
&\le (1-\alpha_n)((1-\alpha_{m+n+1})\|x_{n+m+1}-x_n\| + \alpha_{m+n+1}\|Tx_{n+m+1}-x_n\|)\\
&+\alpha_n((1-\alpha_{m+n+1})\|x_{n+m+1}-Tx_n\| + \alpha_{m+n+1}\|Tx_{n+m+1}-Tx_n\|)\\
&\le (1-\alpha_n)\alpha_{m+n+1}\|Tx_{n+m+1}-x_n\|\\
&+((1-\alpha_{m+n+1})(1-\alpha_n)+\alpha_{m+n+1}\alpha_n)\|x_{n+m+1}-x_n\|\\
&+\alpha_n(1-\alpha_{m+n+1})\|Ty_{n+m+1}-Tx_n\|\\
&\le (1-\alpha_n)\alpha_{m+n+1}\|Tx_{n+m+1}-x_n\|\\
&+((1-\alpha_{m+n+1})(1-\alpha_n)+\alpha_{m+n+1}\alpha_n)(\|Ty_{n+m}-Tz_{n+m}\|+\cdots+\|Tz_n-Tx_n\|+\|Tx_n-x_n\|)\\
&+\alpha_n(1-\alpha_{m+n+1})(\|Ty_{n+m+1}-Tz_{n+m+1}\|+\cdots+\|Ty_n-Tz_n\|+\|Tz_n-Tx_n\|)\\
&\le (1-\alpha_n)\alpha_{m+n+1}\|Tx_{n+m+1}-x_n\|\\
&+((1-\alpha_{m+n+1})(1-\alpha_n)+\alpha_{m+n+1}\alpha_n)\left(\sum_{k=3n-2}^{3n-3+p}\|Tu_{k+1}-Tu_k\|+\|Tu_{3n-2}-x_{3n-2}\|\right)\\
&+(1-\alpha_{m+n+1})\alpha_n\sum_{k=3n-2}^{3n-3+p}\|Tu_{k+1}-Tu_k\|\\
&\le (1-\alpha_n)\alpha_{m+n+1}\|Tx_{n+m+1}-x_n\|\\
&+((1-\alpha_{m+n+1})(1-\alpha_n)+\alpha_{m+n+1}\alpha_n)\sum_{k=3n-2}^{3n-2+p}\beta_k\|Tx_n-x_n\|\\
&+(1-\alpha_{m+n+1})\alpha_n\sum_{k=3n-2}^{3n-3+p}\beta_k\|Tx_n-x_n\|\\
&\le (1-\alpha_n)\alpha_{m+n+1}\|Tu_{3n-1+p}-u_{3n-2}\|\\
&+\left((1-\alpha_{m+n+1}+\alpha_n\alpha_{m+n+1})\sum_{k=3n-2}^{3n-2+p}\beta_k-\alpha_n(1-\alpha_{m+n+1})\right)\|Tu_{3n-2}-u_{3n-2}\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|Tu_{3n+1+p}-u_{3n+1}\|\\
&\le (1-\alpha_n)\alpha_{m+n+1}\|Tu_{3n-1+p}-u_{3n-2}\|\\
&+\left((1-\alpha_{m+n+1}+\alpha_n\alpha_{m+n+1})\sum_{k=3n-2}^{3n-2+p}\beta_k-\alpha_n(1-\alpha_{m+n+1})\right)\|Tu_{3n-2}-u_{3n-2}\|
\end{aligned}
\tag{10}
$$

Using (5) and (10), we have

$$
\begin{aligned}
&\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\|Tu_{3n+1}-u_{3n+1}\|\\
&\le (1-\alpha_n)\alpha_{m+n+1}\|Tu_{3n-1+p}-u_{3n-2}\|\\
&+\left((1-\alpha_{m+n+1}+\alpha_n\alpha_{m+n+1})\sum_{k=3n-2}^{3n-2+p}\beta_k-\alpha_n(1-\alpha_{m+n+1})\right)\|Tu_{3n-2}-u_{3n-2}\|\\
&+\left(\prod_{k=n+1}^{n+p}\frac{2}{1-\alpha_k}\right)(\|Tu_{3n+1}-u_{3n+1}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|).
\end{aligned}
$$

From $\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\le\left(\prod_{k=n+1}^{n+p}\frac{2}{1-\alpha_k}\right)$ and $\|Tu_{3n+1}-u_{3n+1}\|\le\|Tu_{3n-2}-u_{3n-2}\|$, we have

$$\left(1+\sum_{k=3n+1}^{3n+p}\beta_k\right)\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq (1-\alpha_n)\alpha_{m+n+1}\|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left((1-\alpha_{m+n+1}+\alpha_n\alpha_{m+n+1})\sum_{k=3n-2}^{3n-2+p}\beta_k-\alpha_n(1-\alpha_{m+n+1})\right)\|Tu_{3n-2}-u_{3n-2}\|$$
$$+\left(\prod_{k=n+1}^{n+p}\frac{2}{1-\alpha_k}\right)(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|).$$

Then

$$\frac{\left(1+\sum_{k=3n+1}^{3n+p}\beta_k-\left((1-\alpha_{m+n+1}+\alpha_n\alpha_{m+n+1})\sum_{k=3n-2}^{3n-2+p}\beta_k-\alpha_n(1-\alpha_{m+n+1})\right)\right.}{(1-\alpha_n)\alpha_{m+n+1}}\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq \|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n}^{n+p}\frac{2}{1-\alpha_k}\right)(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|).$$

It follows that

$$\left(1+\sum_{k=3n-2}^{3n-2+p}\beta_k\right)\|Tu_{3n-2}-u_{3n-2}\|$$
$$\leq \|Tu_{3n-1+p}-u_{3n-2}\|$$
$$+\left(\prod_{k=n}^{n+p}\frac{2}{1-\alpha_k}\right)(\|Tu_{3n-2}-u_{3n-2}\|-\|Tu_{3n+1+3p}-u_{3n+1+3p}\|).$$

Thus, for $n, p+1$, (4) holds. This completes the induction.

Lemma 3. $T: C \to C$ is a $T-(D_a)$ mapping, $\|Tx-x\| \leq \|Ty-y\|$. Then

$$\|x-Ty\| \leq 3\|Tx-x\|+\|x-y\|.$$

Proof: Since $\|Tx-x\| \leq \|Ty-y\|$, we have $Tx \in C(T, y, \alpha)$. Then

$$\|T^2x-Ty\| \leq \|Tx-y\|.$$

It follows that

$$\|x-Ty\| \leq \|x-Tx\|+\|T^2x-Tx\|+\|T^2x-Ty\|.$$

From **Proposition 3**, we have

$$\|x-Ty\| \leq 2\|Tx-x\|+\|Tx-y\| \leq 2\|Tx-x\|+\|Tx-x\|+\|x-y\| = 3\|Tx-x\|+\|x-y\|.$$

Theorem 1. Assume that $T: C \to C$ is a $T-(D_a)$ mapping and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are sequences generated by iteration (3), $\frac{1}{2} < a \leq \alpha_n \leq b < 1$. Then $\lim_{n\to\infty}\|Tx_n-x_n\| = 0$.

Proof: Since C is bounded, there must exists $d > 0$, for every $x \in C$, $\|x\| \leq d$. Let $\{u_m\}$ satisfy $u_{3n-2} = x_n$, $u_{3n-1} = z_n$, $u_{3n} = y_n$. From Lemma 1, $\lim_{k\to\infty}\|Tu_k-u_k\| = r \geq 0$. Assume $r > 0$. Let ε satisfy

$$e^{\frac{6}{1-b}\left(\frac{d}{r}+1\right)}\varepsilon < r$$

and choose n so that for every $p > 0$

$$\|Tu_{3n-2} - u_{3n-2}\| - \|Tu_{3n-2+3p} - u_{3n-2+3p}\| < \varepsilon.$$

Now choose p so that $r\left(\sum_{k=3n-2}^{3n+p-4} \beta_k\right) \le d \le r\left(\sum_{k=3n-2}^{3n+p-3} \beta_k\right)$.

Since $\frac{1}{2} < a \le \alpha_n \le b < 1$, for every k, t, we have $1 + \alpha_k < 3\alpha_k$, $\alpha_t < 2\alpha_k$. From **Lemma 2** and $r \le \|Tu_{3n-2} - u_{3n-2}\|$, we have

$$\begin{aligned}
d + r &\le r\left(1 + \sum_{k=3n-2}^{3n+p-3} \beta_k\right) \\
&\le \left(1 + \sum_{k=3n-2}^{3n+p-3} \beta_k\right)\|Tu_{3n-2} - u_{3n-2}\| \\
&\le \|Tu_{3n-2+p} - u_{3n-2}\| \\
&+ \left(\prod_{k=n}^{n+p-1} \frac{2}{1-\alpha_k}\right)\left(\|Tu_{3n-2} - u_{3n-2}\| - \|Tu_{3n-2+3p} - u_{3n-2+3p}\|\right) \\
&< d + \left(\prod_{k=n}^{n+p-1} \frac{2}{1-\alpha_k}\right)\varepsilon \\
&= d + e^{\sum_{k=n}^{n+p-1} \ln\left(1+\frac{1+\alpha_k}{1-\alpha_k}\right)}\varepsilon \\
&\le d + e^{\sum_{k=n}^{n+p-1} \frac{1+\alpha_k}{1-\alpha_k}}\varepsilon \\
&\le d + e^{\frac{3}{1-b}\sum_{k=n}^{n+p-1} \alpha_k}\varepsilon \\
&\le d + e^{\frac{6}{1-b}\sum_{k=3n-2}^{3n+p-3} \beta_k}\varepsilon \\
&\le d + e^{\frac{6}{1-b}\left(\sum_{k=3n-2}^{3n+p-4} \beta_k + 1\right)}\varepsilon \\
&< d + e^{\frac{6}{1-b}\left(\frac{d}{r}+1\right)}\varepsilon < d + r.
\end{aligned}$$

This is a contradiction. So $\lim_{k\to\infty} \|Tu_k - u_k\| = 0$. That is to say, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. This completes the proof.

Theorem 2. Assume that $T : C \to C$ is a $T - (D_a)$ mapping and $\{x_n\}$ is generated by iteration (3), $\frac{1}{2} < a \le \alpha_n \le b < 1$. Then the sequence $\{x_n\}$ converges to a fixed point of T.

Proof: Since C is compact, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ which converges to some $z \in C$. By **Lemma 3**, we have

$\|x_{n_k} - Tz\| \le 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|$. Since $\lim_{n_k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$ and $\lim_{n_k\to\infty} \|x_{n_k} - z\| = 0$, we have $\lim_{n_k\to\infty} \|x_{n_k} - Tz\| = 0$. This implies that $z = Tz$. On the other hand, from **Proposition 3**

$$\begin{aligned} \|x_{n+1} - z\| \le \|y_n - z\| \le \|z_n - z\| \\ \le \alpha_n \|Tx_n - z\| + (1-\alpha_n)\|x_n - z\| \\ \le \|x_n - z\|. \end{aligned}$$

So, $\lim_{n\to\infty} \|x_n - z\|$ exists. Therefore, $\lim_{n\to\infty} \|x_n - z\| = 0$. This completes the proof.

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