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Differential Electromagnetic Forms in Rotating Frames

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1. Introduction

Differential forms are completely antisymmetric homogeneous r-tensors on a differentiable n-manifold 0 ≤ r ≤ n belonging to the Grassmann algebra [1] and endowed by Cartan [2] with an exterior calculus. These differential forms found an immediate application in geometry and mechanics; introduced by Deschamps [3,4] in electromagnetism, they have known in parallel with the expansion of computers, an increasing interest [5-9] because Maxwell’s equations and the constitutive relations are put in a manifestly independent coordinate form.

In the Newton (3+1) space-time, with the euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$ the conventional Maxwell equations in which the $E$, $B$, $D$, $H$ fields are 3-vectors have, in absence of charge and current, the Gibbs representation

\[ \nabla \cdot B = 0, \quad \nabla \wedge E + \frac{1}{c^2} \partial_t B = 0 \] (1a)

\[ \nabla \cdot D = 0, \quad \nabla \wedge H - \frac{1}{c^2} \partial_t D = 0 \] (1b)

and, they also have the differential form representation ($\tilde{\alpha} = 1/c \alpha$) [5,7]

\[ d \wedge E + \tilde{\alpha} \cdot B = 0, \quad d \wedge B = 0 \] (2a)

\[ d \wedge H - \tilde{\alpha} \cdot D = 0, \quad d \wedge D = 0 \] (2b)

d = dx East + dy North + dz South is the exterior derivative, $E$, $H$ the differential 1-forms

\[ E = E_x \, dx + E_y \, dy + E_z \, dz, \quad H = H_x \, dx + H_y \, dy + H_z \, dz \] (3a)

and $B$, $D$ the differential 2-forms

\[ B = B_x(dy \wedge dz) + B_y(dz \wedge dx) + B_z(dx \wedge dy), \quad D = -[D_x(dy \wedge dz) + D_y(dz \wedge dx) + D_z(dx \wedge dy)] \] (3b)

We are interested here, for reasons to be discussed in Sec.(6) in a Frenet-Serret frame rotating around oz with a constant angular velocity requiring a relativistic processing. We shall prove that this situation leads to an Einstein space-time with a riemannian metric. As an introduction to this problem, we give a succinct presentation of differential electromagnetic forms in a Minkowski space-time with the metric $ds^2 = dx^2 + dy^2 + dz^2 - c^{-2} dt^2$. 


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2. Differential forms in Minkowski space-time [7]

In absence of charge and current, the Maxwell equations have the tensor representation [10, 11]

\[ \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \quad \text{a)} \]
\[ \partial_\nu F_{\mu\nu} = 0 \quad \text{b)} \]

(4)
The greek (resp. latin) indices take the values 1,2,3,4 (resp. 1,2,3) with \( x^1 = x, x^2 = y, x^3 = z, x^4 = ct, \partial_1 = \partial / \partial x_1, \partial_4 = 1/c \partial / \partial t \) and the summation convention is used. The components of the tensors \( F_{\mu\nu} \) and \( F^{\mu\nu} \) are with the 3D Levi-Civita tensor \( \varepsilon_{ijk} \)

\[ B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}, \quad E_i = -F_{i4}, \quad H_i = \frac{1}{2} \varepsilon_{ijk} F^{jk}, \quad D_i = -F^{i4} \]

(5)

and in vacuum

\[ D = \varepsilon_0 E, \quad H = \mu_0^{-1} B, \quad (\varepsilon_0 \mu_0)^{1/2} = 1/c \]

(5a)

Let \( d \) be the exterior derivative operator

\[ d = (\partial_x dx + \partial_y dy + \partial_z dz + \partial_t dt) \wedge \]

(6)

and \( F = E + B \) be the two-form in which:

\[ E = E_x (dx \wedge dt) + E_y (dy \wedge dt) + E_z (dz \wedge dt) \]
\[ B = B_x (dy \wedge dz) + B_y (dz \wedge dx) + B_z (dx \wedge dy) \]

(7)

Then the Maxwell equations (4a) have the differential 3-form representation \( d F = 0 \).

Similarly for \( G = D + H \) with:

\[ H = H_x (dx \wedge dt) + H_y (dy \wedge dt) + H_z (dz \wedge dt) \]
\[ D = -[D_x (dy \wedge dz) + D_y (dz \wedge dx) + D_z (dx \wedge dy)] \]

(8)

the differential 3-form representation of Maxwell’s equations (4b) is \( d G = 0 \).

To manage the constitutive relations (5a) the Hodge star operator [6,9] is introduced

\[ * (dx \wedge dt) = c^{-1} (dy \wedge dz), \quad * (dy \wedge dz) = c (dx \wedge dt) \]
\[ * (dy \wedge dt) = c^{-1} (dz \wedge dx), \quad * (dz \wedge dx) = c (dy \wedge dt) \]
\[ * (dz \wedge dt) = c^{-1} (dx \wedge dy), \quad * (dx \wedge dy) = c (dz \wedge dt) \]

(9)

Applying the Hodge star operator to \( F \) gives \( * F = * E + * B \) and one checks easily the relation \( G = \lambda_0 * F \) with \( \lambda_0 = (\varepsilon_0 / \mu_0)^{1/2} \) so that the Maxwell equations in the Minkowski vacuum, have the differential 3-form representation

\[ d F = 0, \quad d * F = 0 \]

(10)

3. Electromagnetism in a Frenet-Serret rotating frame

We consider a frame rotating with a constant angular velocity \( \Omega \) around oz. Then, using the Trocheris-Takeno relativistic description of rotation [12, 13], the relations between the
cylindrical coordinates \( R, \Phi, Z, T \) and \( r, \phi, z, t \) in the natural (fixed) and rotating frames are with \( \beta = \Omega R / c \)

\[
R = r, \quad \Phi = \phi \cosh \beta - ct / r \sinh \beta \\
Z = z, \quad cT = ct \cosh \beta - r \phi \sinh \beta
\] (11)

and a simple calculation gives the metric \( ds^2 \) in the rotating frame

\[
ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - (1 + B^2 - A^2) dr^2 - 2(A \sinh \beta + B \cosh \beta) c dt dr - \\
2(A \cosh \beta + B \sinh \beta) r dr d\phi
\] (12)

\[
A = \beta \sinh \beta \ c t / r + \beta \cosh \beta \ \phi + \sinh \beta \ \phi, \quad B = \beta \sinh \beta \ \phi + \beta \cosh \beta \ c t / r - \sinh \beta \ c t / r
\] (12a)

Using the notations \( x_4 = ct, \ x_3 = z, \ x_2 = \phi, \ x_1 = r \), we get from (12) \( ds^2 = g_{\mu \nu} dx^\mu dx^\nu \) with

\[
g_{44} = 1, \quad g_{33} = -1, \quad g_{22} = -r^2, \quad g_{11} = -(1 + B^2 - A^2)
\]

\[
g_{14} = g_{41} = 2(A \sinh \beta + B \cosh \beta), \quad g_{12} = g_{21} = 2(A \cosh \beta + B \sinh \beta)
\] (13)

The determinant \( g \) of \( g_{\mu \nu} \) is

\[
g = g_{33} [g_{11} g_{22} g_{44} - g_{11}^2 g_{44} - g_{44}^2 g_{22}] = R [g_{11} - g_{12} r^2 - g_{14}^2] \] (14)

but

\[
g_{12} r^2 + g_{14}^2 = 4(A^2 - B^2)
\] (14a)

and, taking into account the expression (13) of \( g_{11} \), we get finally

\[
g = r^2 [5(A^2 - B^2) - 1], \quad A^2 - B^2 = (\beta^2 - c^2 \ell^2 / r^2) (\beta^2 + \sinh^2 \beta + 2 \beta \sinh \beta \cosh \beta)
\] (15)

So, the rotating Frenet-Serret frame defines an Einstein space-time with the riemannian metric \( ds^2 = g_{\mu \nu} dx^\mu dx^\nu \), and in this Einstein space-time the Maxwell equations have the tensor representation[14, 15]

\[
\partial_\mu G_{\nu \rho} + \partial_\nu G_{\rho \mu} + \partial_\rho G_{\mu \nu} = 0 \quad (\text{a}) \quad \partial_\mu (|g|^{1/2} G_{\nu \rho}) = 0 \quad \text{(b)}
\] (16)

in which, using the cylindrical coordinates \( r, \phi, z, t \) with \( x_1 = r, \ x_2 = \phi, \ x_3 = z, \ x_4 = ct ; \ c_{\phi} \equiv \partial_\phi, \ c_\phi \equiv \partial_\phi, \ c_\phi \equiv \partial_\phi, \ c_\phi \equiv \partial_\phi \), the components of the electromagnetic tensors are

\[
G_{12} = FB_{\phi}, \quad G_{13} = -B_r, \quad G_{23} = r B_{\phi}, \quad G_{14} = -E_\phi, \quad G_{24} = -r E_\phi, \quad G_{34} = -E_z \\
G_{12} = H_z / r, \quad G_{13} = -H_\phi, \quad G_{23} = H_z / r, \quad G_{14} = D_\phi / r, \quad G_{24} = D_r, \quad G_{34} = D_z
\] (17)

To work with the differential forms, we introduce the exterior derivative

\[
d = (\partial_\rho dr + \partial_\phi d\phi + \partial_\phi dz + \partial_\phi dt) \wedge
\] (18)

(underlined expressions mean that they are defined with the cylindrical coordinates \( r, \phi, z, t \) and the two-forms \( F = \mathbb{E} + \mathbb{B} \) with

\[
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\[ E = E_r (dr \wedge cdt) + E_\phi (rd\phi \wedge cdt) + E_z (dz \wedge cdt) \]

\[ B = B_r (rd\phi \wedge dz) + B_\phi (dz \wedge dr) + B_z (dr \wedge rd\phi) \]

and writing \( g^{1/2} = rq \), \( q = [5(A^2-B^2)-1]^{1/2} \) the two-form \( \mathcal{G} = \mathcal{D} + \mathcal{H} \)

\[ \mathcal{D} = -q [D_r (rd\phi \wedge dz) + D_\phi (dz \wedge dr) + D_z (dr \wedge rd\phi)] \]

\[ \mathcal{H} = q[H_r (dr \wedge cdt) + H_\phi (rd\phi \wedge cdt) + H_z (dz \wedge cdt)] \]

Then, the Maxwell equations have the 3-form representation

\[ d\mathcal{F} = 0, \quad d\mathcal{G} = 0 \]

A simple calculation gives

\[ d\mathcal{F} = \left[ \partial_r (rB_r) + \partial_\phi (rB_\phi) + \partial_z (rB_z) \right] (dr \wedge d\phi \wedge dz) + \]

\[ \left[ \partial_\phi (rE_r - \partial_r (rE_\phi)) \right] (d\phi \wedge dz \wedge dt) + \]

\[ \left[ \partial_z (rE_\phi - \partial_\phi (rE_z)) \right] (dz \wedge dr \wedge dt) + \]

\[ \left[ \partial_t (rB_r) + c [\partial_t (rE_\phi - \partial_\phi (rE_z))] \right] (dr \wedge d\phi \wedge dt) \]

\[ * (dr \wedge cdt) = -qc^{-1} (rd\phi \wedge dz), \quad * (rd\phi \wedge dz) = q^{-1}c (dr \wedge cdt) \]

\[ * (dz \wedge dr) = q^{-1}c (rd\phi \wedge cdt), \quad * (dr \wedge rd\phi) = q^{-1}c (dz \wedge cdt) \]

The Hodge star operator needed to take into account the constitutive relations (5a) in vacuum is defined by the relation

\[ * (dr \wedge cdt) = -qc^{-1} (rd\phi \wedge dz), \quad * (rd\phi \wedge dz) = q^{-1}c (dr \wedge cdt) \]

Applying (22) to \( \mathcal{F} \) gives \( *\mathcal{F} = *\mathcal{E} + *\mathcal{B} \) and it is easily checked that \( \mathcal{G} = \lambda_0 *\mathcal{F} \) with \( \lambda_0 = (\epsilon_0/\mu_0)^{1/2} \) so that in vacuum \( d\mathcal{F} = 0, \quad d *\mathcal{F} = 0 \).

4. Wave equations in vacuum

4.1 Minkowski space-time

The wave equations satisfied by the electromagnetic field (in absence of charges and currents) are obtained from differential forms with the help of the Laplace-De Rham operator [6,8]

\[ L = (d^* d^* + *d^* d) \wedge \]

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requiring the Hodge star operators for the n-forms, \( n = 1, 2, 3 \). They are given in Appendix A where, using the exterior derivative (6), and assuming \( E_x = E_y = 0 \) so that the two-form (7) becomes \( E_z = E_z (dz \wedge cd t) \), we get

\[
L \ E_x = (\Lambda - c^2 \partial_4^2) \ E_x (dz \wedge cd t), \quad \Lambda = \partial_4 \partial_4 + \partial_4 \partial_4
\]

(24)

A similar relation exists for \( E_x, E_y \) and for the components of the \( B \)-field so that, we get finally for the 2-form \( F \):

\[
L \ F = (\Lambda - c^2 \partial_4^2) \ F_{\mu \nu} (dx^\mu \wedge dx^\nu)
\]

(25)

so that the wave equation has the 2-form representation \( L \ F = 0 \).

### 4.2 Einstein space-time

#### 4.2.1 Cartesian frame

In a riemannian cartesian frame, the components of the electromagnetic field tensor \( F_{\mu \nu} \) are solutions of the tensor wave equation [16]

\[
g^{\alpha \beta} \nabla_\alpha V_\beta F_{\mu \nu} - 2 R_{\mu \rho \nu \sigma} F^{\rho \sigma} + R_{\mu \nu} F_{\rho \sigma} - R_{\nu \sigma} F_{\rho \mu} = 0
\]

(26)

where \( V_\alpha \) is the covariant derivative, \( R_{\mu \rho \nu \sigma} \) and \( R_{\mu \nu} \) the Riemann curvature and Ricci tensors defined in terms of Christoffel symbols \( \Gamma_{\alpha \beta \rho} \), \( \Gamma^{\alpha \beta}_{\mu} \)

\[
\Gamma_{\beta \mu \nu} = \frac{1}{2} \left( \partial_\mu g_{\nu \alpha} + \partial_\nu g_{\alpha \mu} - \partial_\alpha g_{\mu \nu} \right), \quad \Gamma_{\alpha \beta \mu} = g^{\rho \sigma} \Gamma_{\alpha \beta \rho \mu}
\]

(27)

by the relations in which \( \partial_4 = \partial_x, \partial_2 = \partial_y, \partial_3 = \partial_z, \partial_4 = 1/c \partial_x \)

\[
R_{\alpha \beta \mu \nu} = \partial_\mu \Gamma_{\alpha \beta \nu} - \partial_\nu \Gamma_{\alpha \beta \mu} + \Gamma_{\alpha \gamma \mu} \Gamma_{\gamma \beta \nu} - \Gamma_{\alpha \gamma \nu} \Gamma_{\gamma \beta \mu}
\]

\[
R^{\alpha \beta}_{\mu \nu} = g^{\alpha \gamma} g^{\beta \rho} R_{\rho \gamma \mu \nu}, \quad R^\alpha_{\mu} = R^{\alpha \alpha}_{\mu \nu}
\]

(28)

Now, it is proved [6] that the Laplace-De Rham operator \( L = d^*d^* + d*d \) applied to the two form (7) written

\[
E = F_1 (dx^1 \wedge dx^2) + F_2 (dx^2 \wedge dx^3) + F_3 (dx^3 \wedge dx^4)
\]

(29)

in which \( dx^1 = dx, \ dx^2 = dy, \ dx^3 = dz, \ dx^4 = cd t \) gives the component \( F_4 \)

\[
L \ E = \frac{1}{2} (g^{\alpha \beta} \nabla_\alpha V_\beta F_{4 \mu} - 2 R_{\mu \rho \nu \sigma} F^{\rho \sigma} + R_{\mu \nu} F_{\rho \sigma} - R_{\nu \sigma} F_{\rho \mu}) (dx^\mu \wedge dx^\nu)
\]

(30)

\[
L \ E = 0 \) gives the 2-form representation of the wave equation in the Einstein space-time with cartesian coordinates

A similar result is obtained for \( B \) writing \(-1/2 F_{ij} \) (dx^i \wedge dx^j) the \( B \) magnetic two-form (7) so that

\[
L B = -1/4 (g^{\alpha \beta} \nabla_\alpha V_\beta F_{ij} - 2 R_{ij \rho \sigma} F^{\rho \sigma} + R^{ij}_{\rho \sigma} F_{\rho \sigma} - R^{ij}_{\rho \sigma} F_{\rho \sigma}) (dx^i \wedge dx^j)
\]

(30a)

Summing (30) and (30a) gives

\[
L F = \frac{1}{2} \left[ g^{\alpha \beta} \nabla_\alpha V_\beta F_{\mu \nu} - 2 R_{\mu \rho \nu \sigma} F^{\rho \sigma} + R_{\mu \nu} F_{\rho \sigma} - R_{\nu \sigma} F_{\rho \mu} \right] (dx^\mu \wedge dx^\nu)
\]

(31)

In the Minkowski cartesian frame where \( g_{44} = 0, \ g_{ij} = -1 \), Eq.(31) reduces to (25).
4.2.2 Frenet-Serret frame

In the Frenet-Serret frame, the Laplace-De Rham operator is defined with the exterior derivative operator (18) and to get a relation such as (30) on the components of the electric field requires some care. First with the greek indices associated to the polar coordinates as previously, one has first to get the Christoffel symbols needed to define the covariant derivative and according to (28), the Riemann curvature and Ricci tensors, a job performed in Appendix B, we are now in position to transpose (30) to a rotating cylindrical frame. To this end, the electric two-form (19a) with

\[
\text{d}x^1 \wedge \text{d}x^4 = \text{d}r \wedge \text{d}t, \quad \text{d}x^2 \wedge \text{d}x^4 = \text{d}\phi \wedge \text{d}t, \quad \text{d}x^3 \wedge \text{d}x^4 = \text{d}z \wedge \text{d}t
\]  

(32)

is written

\[
E = E_r (\text{d}x^1 \wedge \text{d}x^4) + rE_\phi (\text{d}x^2 \wedge \text{d}x^4) + E_z (\text{d}x^3 \wedge \text{d}x^4)
\]  

(33)

but \(E_r, rE_\phi, E_z\) are the \(G_{i4}\) components of the \(G_{\mu\nu}\) tensor (17) so that leaving aside a minus sign

\[
E = G_{14} (\text{d}x^1 \wedge \text{d}x^4) + G_{24} (\text{d}x^2 \wedge \text{d}x^4) + G_{34} (\text{d}x^3 \wedge \text{d}x^4)
\]  

(34)

and we get

\[
\mathcal{L} E = \frac{1}{2}( (\text{\text{g} \alpha\beta} \nabla_\alpha \nabla_\beta G_{i4} - 2R_{i4\alpha\rho} G^{\rho\alpha} + R^i_{\alpha\rho} G_{\rho4} - R^i_{\alpha\rho} G_{\rho4}) (\text{d}x^1 \wedge \text{d}x^4)
\]  

(35)

so that the components of the electric field are solutions of the two-form equation \(\mathcal{L} E = 0\) in the Frenet-Serret rotating frame. For the other components of the electromagnetic field, it comes

\[
\mathcal{L} F = \frac{1}{2}( (\text{\text{g} \mu\nu} \nabla_\mu \nabla_\nu G_{\mu\nu} - 2R_{\mu\nu\rho\sigma} G^{\rho\sigma} + R^\mu_{\rho\sigma} G_{\nu\rho} - R^\mu_{\rho\sigma} G_{\nu\rho}) (\text{d}x^\mu \wedge \text{d}x^\nu)
\]  

(36)

We have only considered the two-form \(F\) because in vacuum \(G = \lambda \delta^* F\).

5. To solve differential form equations

The local 2-form representation (2) of Maxwell’s equations follows, as a consequence of the Stokes’s theorem, from the Maxwell-Ampère and Maxwell-Faraday integral relations. Then, coming back to these theorems, to solve differential form equations is tantamount to perform the integrals

\[
I = \int_M \omega
\]  

(37)

in which \(\omega\) is a n-form, for instance \(F\) or \(G\), and \(M\) an oriented manifold with the same n-dimension as the degree of the \(\omega\) form [5].

In the 3D-space, the numerical evaluation of (37) is based on the finite element technique, largely used [17, 18] in the simulation of partial differential equations. The manifold \(M\) is described by a chain of simplexes made for instance of triangular surfaces, tetrahedral volumes… on which the Whitney forms [5,19,20] gives a manageable description of the n-form \(\omega\). A simple example may be found in [19] and a through discussion of the technique in [20]. These solutions may be called weak in opposition to the strong solutions of the Maxwell’s equations (1).
The numerical process just described is limited to the 3D-space but in the 4D space-time, in particular for the Frenet-Serret frame, $\omega$ depends on $dt$ so that $M$ has to be defined in terms of 2-cells, 3-cells, 4-cells of space-time [21] and the Whitney forms must be generalized accordingly. It does not seem that computational works have been made in this domain.

6. Discussion

Differential electromagnetic forms are usually managed in a Newton space and more rarely in a Minkowski space-time although, in this case, the comparison between tensors and differential forms is very enlightening [7]. This formalism is analyzed here in an Einstein space-time with a Riemann metric, particularly that of a Frenet-Serret frame. From a theoretical point of view, except for some more intricate relations due to Riemann, Ricci tensors and Christoffel symbols there is no difficulty to go from Newton to Einstein differential forms. The situation is different from a computational point of view, since as mentioned in Sec.5, an important work has still to be performed to get the solutions of the differential form equation in an Einstein space-time.

This work may be considered as a first step in a complete analysis of electromagnetic differential forms in an Einstein space-time. The subjects to be discussed go from the presence of charges and currents (left aside here) to boundary conditions with between the introduction of potentials, the energy conveyed by the electromagnetic field and so on. This extension could be performed in the style used in [7] to analyze the electromagnetic differential forms in a Minkowski space-time. In addition, it would make possible an interesting comparison (already sketched in Sec.4.1) with the electromagnetic tensor formalism of the General Relativity [15].

Now, why to take an interest in rotating frames? A first response could have been “Universal” assumed cylindrical. But, although Einstein and Romer (also Levi Civita) have obtained some exact cylindrical wave solutions of the general relativity equations [22], this cosmos has been superseded by a spherical world (nevertheless, because of its particular properties, some works are still devoted to the Levi Civita world [23]. A second response comes from the analysis of the Wilsons’ experiments in which was measured the electric potential between the inner and outer surfaces of a cylinder rotating in an external axially directed magnetic field: an analysis with many different approaches [6,23,24,25] (the Trocheris-Takeno description of rotations is used in [25]). Finally, a third response is provided by the increasing at-tention paid to paraxial optical beams with a helicoidal geometrical structure [26], [27] leading to a discussion of light propagation in rotating media: a problem object of some disputes [28-32].The relativistic theory of geometrical optics [15] is still a challenge to which it would be interesting to see what could be the differential form contribution.

Appendix A: Minkowski space-time in vacuum

The four dimensional Hodge operator for Minkowski space-time is defined as follows [8]:

zero-forms and four-forms

$$* (dx \wedge dy \wedge dz \wedge cdt) = -1,$$

$$*1 = (dx \wedge dy \wedge dz \wedge cdt) \quad (A.1)$$

one-forms and three-forms
\( * (dx \wedge dy \wedge dz) = -c dt \), \quad * c dt = -(dx \wedge dy \wedge dz) \)

\( *(dy \wedge dz \wedge c dt) = -dx \), \quad *dx = -(cdt \wedge dy \wedge dz) \)

\( *(dz \wedge dx \wedge c dt) = -dy \), \quad *dy = -(cdt \wedge dz \wedge dx) \)

\( *(dx \wedge dy \wedge c dt) = -dz \), \quad *dz = -(cdt \wedge dx \wedge dy) \) \hspace{1cm} (A.2)

Two forms

\( *(dy \wedge dz) = -(dx \wedge c dt) \), \quad *(dx \wedge dz) = (dy \wedge dz) \)

\( *(dz \wedge dx) = -(dy \wedge c dt) \), \quad *(dy \wedge dx) = (dz \wedge dx) \)

\( *(dx \wedge dy) = -(dz \wedge c dt) \), \quad *(dz \wedge dx) = (dx \wedge dy) \) \hspace{1cm} (A.3)

Let us assume \( E_x = E_y = 0 \), then the electric two-form (7) becomes

\[ E = E_z (dz \wedge c dt) \] \hspace{1cm} (A.4)

Applying the exterior derivative operator (6) to (B.4) and using (B.2) give

\[ *d E = \partial_3 E_z \wedge dy - \partial_1 E_y \wedge dx \] \hspace{1cm} (A.5)

and

\[ d^* E = \partial_3 E_z (dxy) + \partial_3 \partial_1 E_z (dxy) + \partial_1 \partial_3 E_z (dcy) + \]

\[ -[\partial_3^2 E_z (dxy) + \partial_3 \partial_1 E_z (dxy) + \partial_3 \partial_1 E_z (dcy)] \] \hspace{1cm} (A.6)

so that according to (A.3)

\[ *d^* E = - \partial_2 E_z (dyc) + \partial_1 \partial_3 E_z (dcy) - \frac{1}{c} \partial_3 \partial_1 E_z (dzc) + \]

\[ -[\partial_3^2 E_z (dyc) - \partial_3 \partial_1 E_z (dcy) - \frac{1}{c} \partial_3 \partial_1 E_z (dyc)] \] \hspace{1cm} (A.7)

1ow, using (A.3) and (A.2), we also have

\[ *d *E = - \partial_2 E_z \wedge cdt - \partial_1 E_y \wedge dz \] \hspace{1cm} (A.8)

and

\[ d^*d *E = - \partial_1 \partial_3 E_z (dcy) - \partial_1 \partial_3 E_z (dzy) - \partial_3 E_z (dzc) + \]

\[ - \frac{1}{c} [\partial_3^2 E_z (dcy) + \partial_3 \partial_1 E_z (dzy) + \partial_3 \partial_1 E_z (dzc)] \] \hspace{1cm} (A.9)

Summing (A.7) and (A.9) gives

\[ (*d*d + d^*d *)E = - (\partial_3^2 + \partial_1^2 + \partial_1^2 - c^2 \partial_3^2) E_z (dz \wedge cdt) \] \hspace{1cm} (A.10)

Appendix B: Christoffel symbols

The Christoffel symbols are defined in terms of the \( g_{\mu \nu} \)'s by the well known relations \[14-16\]

\[ \Gamma_{\beta \mu \nu} = \frac{1}{2} (\partial_\nu g_{\beta \mu} + \partial_\mu g_{\beta \nu} - \partial_\beta g_{\mu \nu}), \quad \Gamma^\alpha_{\mu \nu} = g^{\alpha \beta} \Gamma_{\beta \mu \nu} \] \hspace{1cm} (B.1)
In these expressions, the greek indices take the values 1,2,3,4 corresponding in a cylindrical frame to the coordinates \( x^1 = r, x^2 = \phi, x^3 = z, x^4 = ct \), while \( \partial_1 = \partial_r, \partial_2 = 1/r \partial_\phi, \partial_3 = \partial_z, \partial_4 = 1/c \partial_t \).

The relations (13) give the components \( g_{\mu \nu} \) of the metric tensor for a Frenet-Serret rotating frame and:

\[
g_{44} = 1, \quad g_{33} = -1, \quad g_{22} = -r^2, \quad g_{11} = u(r,\phi,t), \quad g_{12} = g_{21} = v(r,\phi,t), \quad g_{14} = g_{41} = w(r,\phi,t) \quad (B.2)
\]

the explicit expressions of the functions \( u,v,w \) are to be found in (13), no \( g_{\mu \nu} \) depends on \( z \).

Then, the non-null components of the Christoffel symbols are given for \( \mu \leq \nu \) (because of the \( \mu \nu \)-symmetry)

\[
\begin{align*}
\Gamma_{1,11} &= \frac{1}{2} \partial_r u, & \Gamma_{1,12} &= \frac{1}{2} \partial_\phi u, & \Gamma_{1,14} &= \frac{1}{2} \frac{1}{c} \partial_t u, \\
\Gamma_{1,24} &= \frac{1}{2} \partial_\phi v + \frac{1}{2r} \partial_r w, & \Gamma_{1,44} &= \frac{1}{c} \frac{1}{c} \partial_t w, & \Gamma_{2,11} &= \partial_r v - \frac{1}{2} \partial_\phi u, \\
\Gamma_{2,12} &= -\frac{1}{2r}, & \Gamma_{2,14} &= -\frac{1}{2r} \partial_r w - \frac{1}{2} \partial_\phi v, & \Gamma_{3,11} &= \partial_z w - \frac{1}{2} \partial_\phi u, \\
\Gamma_{3,12} &= \frac{1}{2} \partial_z w - \frac{1}{2} \partial_\phi v & (B.3)
\end{align*}
\]

The latin indices taking the values 1,2,3, the covariant derivatives of the components \( E_1 = E_r, E_2 = E_\phi, E_3 = E_z \) of the electric field are

\[
\begin{align*}
\nabla_1 E_1 &= \partial_r E_1 - \Gamma_{1,1} E_k \\
\nabla_2 E_1 &= \frac{1}{r} \partial_\phi E_1 - \Gamma_{1,2} E_k \\
\nabla_3 E_1 &= \partial_z E_1 - \Gamma_{1,3} E_k
\end{align*}
\]

Underlined expressions mean they are defined with the cylindrical coordinates \( r,\phi, z,t \).

Making in (B.2), \( u = v = w = 0 \), gives the metric of the Minkowski frame with polar coordinates and according to (B.3) the only nonnull Christoffel symbols are

\[
\Gamma_{1,22} = -r, \quad \Gamma_{2,12} = \Gamma_{2,21} = -r \quad (B.5)
\]

7. References

In the recent decades, there has been a growing interest in micro- and nanotechnology. The advances in nanotechnology give rise to new applications and new types of materials with unique electromagnetic and mechanical properties. This book is devoted to the modern methods in electrodynamics and acoustics, which have been developed to describe wave propagation in these modern materials and nanodevices. The book consists of original works of leading scientists in the field of wave propagation who produced new theoretical and experimental methods in the research field and obtained new and important results. The first part of the book consists of chapters with general mathematical methods and approaches to the problem of wave propagation. A special attention is attracted to the advanced numerical methods fruitfully applied in the field of wave propagation. The second part of the book is devoted to the problems of wave propagation in newly developed metamaterials, micro- and nanostructures and porous media. In this part the interested reader will find important and fundamental results on electromagnetic wave propagation in media with negative refraction index and electromagnetic imaging in devices based on the materials. The third part of the book is devoted to the problems of wave propagation in elastic and piezoelectric media. In the fourth part, the works on the problems of wave propagation in plasma are collected. The fifth, sixth and seventh parts are devoted to the problems of wave propagation in media with chemical reactions, in nonlinear and disperse media, respectively. And finally, in the eighth part of the book some experimental methods in wave propagations are considered. It is necessary to emphasize that this book is not a textbook. It is important that the results combined in it are taken “from the desks of researchers”. Therefore, I am sure that in this book the interested and actively working readers (scientists, engineers and students) will find many interesting results and new ideas.

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