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On Conformal Anti-Invariant Submersions Whose Total Manifolds Are Locally Product Riemannian

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Abstract

The aim of this chapter is to study conformal anti-invariant submersions from almost product Riemannian manifolds onto Riemannian manifolds as a generalization of anti-invariant Riemannian submersion which was introduced by B. Sahin. We investigate the integrability of the distributions which arise from the definition of the new submersions and the geometry of foliations. Moreover, we find necessary and sufficient conditions for this submersion to be totally geodesic and in order to guarantee the new submersion, we mention some examples of such submersions.

Keywords: conformal submersion, almost product Riemannian manifold, vertical distribution, conformal anti-invariant submersion

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1. Introduction

Immersions and submersions, which are special tools in differential geometry, also play a fundamental role in Riemannian geometry, especially when the involved manifolds carry an additional structure (such as contact, Hermitian and product structure). In particular, Riemannian submersions (which we always assume to have connected fibers) are fundamentally important in several areas of Riemannian geometry. For instance, it is a classical and important problem in Riemannian geometry to construct Riemannian manifolds with positive or non-negative sectional curvature. Riemannian submersions between Riemannian manifolds are important geometric structures. Riemannian submersions between Riemannian manifolds were studied by O'Neill [1] and Gray [2]. In [3], the Riemannian submersions were considered between almost Hermitian manifolds by Watson under the name of almost Hermitian submersions.
In this case, the Riemannian submersion is also an almost complex mapping and consequently the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total manifold of the submersion. The study of anti-invariant Riemannian submersions from almost Hermitian manifolds was initiated by Şahin [4]. In this case, the fibers are anti-invariant with respect to the almost complex structure of the total manifold. This notion extended to different total spaces see: [5–14].

On the other hand, as a generalization of Riemannian submersion, horizontally conformal submersions are defined as follows [15]: Suppose that \( (M, g_M) \) and \( (B, g_B) \) are Riemannian manifolds and \( \pi : M \to B \) is a smooth submersion, then \( \pi \) is called a horizontally conformal submersion, if there is a positive function \( \lambda \) such that

\[
\lambda^2 g_M(X, Y) = g_B(\pi_*X, \pi_*Y)
\]

for every \( X, Y \in \Gamma(\ker \pi^* \perp \ker \pi^*) \). It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with \( \lambda = 1 \). We note that horizontally conformal submersions are special horizontally conformal maps which were introduced independently by Fuglede [16] and Ishihara [17]. We also note that a horizontally conformal submersion \( \pi : M \to B \) is said to be horizontally homothetic if the gradient of its dilation \( \lambda \) is vertical, i.e.,

\[
\mathcal{H}(\text{grad} \lambda) = 0
\]

at \( p \in M \), where \( \mathcal{H} \) is the projection on the horizontal space \((\ker \pi^*)^\perp\). For conformal submersion, see: [15, 18, 19].

One can see that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps do not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties.

Recently, we introduced conformal anti-invariant submersions [20] and conformal semi-invariant submersions [21] from almost Hermitian manifolds, and gave examples and investigated the geometry of such submersions (see also [22, 23]). We showed that the geometry of such submersions is different from their counterpart anti-invariant Riemannian submersions and semi-invariant Riemannian submersions. In the present paper, we define and study conformal anti-invariant submersions from almost product Riemannian manifolds, give examples and investigate the geometry of the total space and the base space for the existence of such submersions.

Our work is structured as follows: Section 2 is focused on basic facts for conformal submersions and almost product Riemannian manifolds. The third section is concerned with definition of conformal anti-invariant submersions, investigating the integrability conditions of the horizontal distribution and the vertical distribution. In Section 4, we study the geometry of leaves of the horizontal distribution and the vertical distribution. In Section 5, we find necessary and
sufficient conditions for a conformal anti-invariant submersion to be totally geodesicness. The last section, we give some examples of such submersions.

2. Preliminaries

In this section we recall several notions and results which will be needed throughout the chapter.

Let $M$ be a $m$-dimensional manifold with a tensor $F$ of a type $(1,1)$ such that

$$F^2 = I, \ (F \neq I).$$

Then, we say that $M$ is an almost product manifold with almost product structure $F$. We put

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F).$$

Then we get

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad F = P - Q.$$ 

Thus $P$ and $Q$ define two complementary distributions $P$ and $Q$. We easily see that the eigenvalues of $F$ are $+1$ or $-1$. If an almost product manifold $M$ admits a Riemannian metric $g$ such that

$$g(FX, FY) = g(X, Y)$$

for any vector fields $X$ and $Y$ on $M$, then $M$ is called an almost product Riemannian manifold, denoted by $(M, g, F)$. Denote the Levi-Civita connection on $M$ with respect to $g$ by $\nabla$. Then, $M$ is called a locally product Riemannian manifold [24] if $F$ is parallel with respect to $\nabla$, i.e.,

$$\nabla_X F = 0, \quad X \in \Gamma(TM).$$

Conformal submersions belong to a wide class of conformal maps that we are going to recall their definition, but we will not study such maps in this paper.

**Definition 2.1 ([15])** Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then $\varphi$ is called horizontally weakly conformal or semi conformal at $x$ if either

(i) $d\varphi_x = 0$, or

(ii) $d\varphi_x$ maps horizontal space $\mathcal{H}_x = (\ker(d\varphi_x))^\perp$ conformally onto $T_{\varphi(x)}N$, i.e., $d\varphi_x$ is surjective and there exists a number $\Lambda(x) \neq 0$ such that
\[ h(\partial \phi, X, \partial \phi, Y) = \Lambda(x)g(X, Y) \quad (X, Y \in \mathcal{H}_x). \]  

(4)

Note that we can write the last equation more sufficiently as

\[ (\phi^*h)_x|_{\mathcal{H}_x} = \Lambda(x)g_x|_{\mathcal{H}_x \times \mathcal{H}_x}. \]

A point \( x \) is of type (i) in Definition if and only if it is a critical point of \( \phi \); we shall call a point of type (ii) a regular point. At a critical point, \( d\phi \) has rank 0; at a regular point, \( d\phi \) has rank \( n \) and \( \phi \) is submersion. The number \( \Lambda(x) \) is called the square dilation (of \( \phi \) at \( x \)); it is necessarily non-negative; its square root \( \lambda(x) = \sqrt{\Lambda(x)} \) is called the dilation (of \( \phi \) at \( x \)). The map \( \phi \) is called horizontally weakly conformal or semi conformal (on \( M \)) if it is horizontally weakly conformal at every point of \( M \). It is clear that if \( \phi \) has no critical points, then we call it a (horizontally) conformal submersion.

Next, we recall the following definition from [18]. Let \( \pi : M \to N \) be a submersion. A vector field \( E \) on \( M \) is said to be projectable if there exists a vector field \( \tilde{E} \) on \( N \), such that \( d\pi E_x(x) = \tilde{E}_{\pi(x)} \) for all \( x \in M \). In this case \( E \) and \( \tilde{E} \) are called \( \pi \)-related. A horizontal vector field \( Y \) on \( M \) is called basic, if it is projectable. It is well known fact, that is, \( Z \) is a vector field on \( N \), then there exists a unique basic vector field \( Z \) on \( M \), such that \( Z \) and \( \tilde{Z} \) are \( \pi \)-related. The vector field \( Z \) is called the horizontal lift of \( \tilde{Z} \).

The fundamental tensors of a submersion were introduced in [1]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O’Neill’s tensors \( T \) and \( A \) defined for vector fields \( E, G \) on \( M \) by

\[ A_E G = V^M_M E H G + H^M_M V G \]  

(5)

\[ T_E G = H^M_N V E + V^M_N H G \]  

(6)

where \( V \) and \( H \) are the vertical and horizontal projections (see [25]). On the other hand, from (5) and (6), we have

\[ V^M_N W = TV_W + \tilde{V} W \]  

(7)

\[ V^M_N X = H^M_N V M X + T V X \]  

(8)

\[ V^M_N V = AX V + V^M_N V \]  

(9)

\[ V^M_N Y = H^M_N Y + AX Y \]  

(10)

for \( X, Y \in \Gamma((\ker \pi)^\perp) \) and \( V, W \in \Gamma(\ker \pi) \), where \( \tilde{V} W = V^M_N W \). If \( X \) is basic, then \( H^M_N V M X = AX V \). It is easily seen that for \( x \in M, X \in \mathcal{H}_x \) and \( V_x \) the linear operators \( T_V, A_X : T_X M \to T_X M \) are skew-symmetric, that is
\[ g(T_v E, G) = -g(E, T_v G) \] and \[ g(A_x E, G) = -g(E, A_x G) \]

for all \( E, G \in T_x M \). We also see that the restriction of \( T \) to the vertical distribution \( T_{|V} \) is exactly the second fundamental form of the fibers of \( \pi \). Since \( T_{|V} \) is skew symmetric, we get \( \pi \) which has totally geodesic fibers if and only if \( T \equiv 0 \). For the special case when \( \pi \) is horizontally conformal we have the following:

**Proposition 2.1** ([18]) Let \( \pi : (M^n, g) \to (N^n, h) \) be a horizontally conformal submersion with dilation \( \nabla \) and \( X, Y \) be horizontal vectors, then

\[ A_X Y = \frac{1}{2} \left\{ V[X, Y] - \lambda^2 g(X, Y) \text{grad} \left( \frac{1}{\lambda^2} \right) \right\}. \quad (11) \]

We see that the skew-symmetric part of \( A_{|\ker \pi \ast} \) measures the obstruction integrability of the horizontal distribution \( \ker \pi \ast \).

Let \( (M, g_M) \) and \( (N, g_N) \) be Riemannian manifolds and suppose that \( \pi : M \to N \) is a smooth map between them. The differential of \( \pi \) can be viewed a section of the bundle \( \text{Hom}(T_M, \pi^{-1}T_N) \to M \), where \( \pi^{-1}T_N \) is the pullback bundle which has fibers \( \pi^{-1}(p) = T_{\pi(p)}N, p \in M \). \( \text{Hom}(T_M, \pi^{-1}T_N) \) has a connection \( \nabla \) induced from the Levi-Civita connection \( \nabla^M \) and the pullback connection. Then the second fundamental form of \( \pi \) is given by

\[ \nabla \pi \ast : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TN) \]

defined by

\[ (\nabla \pi \ast)(X, Y) = \nabla^M_X \pi \ast(Y) - \pi \ast(\nabla^M_X Y) \quad (12) \]

for \( X, Y \in \Gamma(TM) \), where \( \nabla^M \) is the pullback connection. It is known that the second fundamental form is symmetric.

**Lemma 2.1.** [26] Let \( (M, g_M) \) and \( (N, g_N) \) be Riemannian manifolds and suppose that \( \varphi : M \to N \) is a smooth map between them. Then we have

\[ \nabla^M_X \varphi \ast(Y) - \nabla^M_Y \varphi \ast(X) - \varphi \ast([X, Y]) = 0 \quad (13) \]

for \( X, Y \in \Gamma(TM) \).

Finally, we recall the following lemma from [15].

**Lemma 2.2.** Suppose that \( \pi : M \to N \) is a horizontally conformally submersion. Then, for any horizontal vector fields \( X, Y \) and vertical fields \( V, W \) we have.
\[(\nabla \pi_*)(X, Y) = X(\ln \lambda)_\pi_+ Y + Y(\ln \lambda)_\pi_+ X - g_M(X, Y)\pi_+ (\text{grad} \ln \lambda);
\]

\[(\nabla \pi_*)(V, W) = -\pi_+(T_V W);
\]

\[(\nabla \pi_*)(X, V) = -\pi_+(\nabla^M_X V) = -\pi_+(A_X V).
\]

3. Conformal anti-invariant submersions from almost product Riemannian manifolds

In this section, we define conformal anti-invariant submersions from an almost product Riemannian manifold onto a Riemannian manifold, investigating the geometry of distributions \(\ker \pi_*\) and \(\ker \pi_*^\perp\) and obtain the integrability conditions for the distribution \(\ker \pi_*^\perp\) for such submersions.

**Definition 3.1.** Let \((M_1, g_1, F)\) be an almost product Riemannian manifold and \((M_2, g_2)\) be a Riemannian manifold. A horizontally conformal submersion \(\pi : M_1 \to M_2\) with dilation \(\lambda\) is called conformal anti-invariant submersion if the distribution \(\ker \pi_*\) is anti-invariant with respect to \(F\), i.e., \(F \ker \pi_* \subseteq \ker \pi_*^\perp\).

Let \(\pi : (M_1, g_1, F) \to (M_2, g_2)\) is a conformal anti-invariant submersion from an almost product Riemannian manifold \((M_1, g_1, F)\) to a Riemannian manifold \((M_2, g_2)\). First of all, from Definition 3.1, we have \(F \ker \pi_*^\perp \cap \ker \pi_* \neq 0\). We denote the complementary orthogonal distribution to \(F \ker \pi_*^\perp \cap \ker \pi_*\) by \(\mu\). Then we have

\[(\ker \pi_*)^\perp = F(\ker \pi_*) \oplus \mu.\quad (14)
\]

**Proposition 3.1.** Let \((M_1, g_1, F)\) be an almost product Riemannian manifold and \((M_2, g_2)\) be a Riemannian manifold. Then \(\mu\) is invariant with respect to \(F\).

**Proof.** For \(Z \in \Gamma(\mu)\) and \(V \in \Gamma(\ker \pi_*)\), by using (2), we have \(g_1(FZ, FV) = 0\), which show that \(FZ\) is orthogonal to \(F \ker \pi_*\). On the other hand, since \(FV\) and \(Z\) are orthogonal we get \(g_1(FV, Z) = g_1(V, FZ) = 0\) which shows that \(FZ\) is orthogonal to \(\ker \pi_*\). This completes proof. \(\Box\)

For \(Z \in \Gamma((\ker \pi_*)^\perp)\), we have

\[FZ = BZ + CZ,\quad (15)\]

where \(BZ \in \Gamma(\ker \pi_*)\) and \(CZ \in \Gamma(\mu)\). On the other hand, since \(\pi_+((\ker \pi_*)^\perp) = TM_2\) and \(\pi_+\) is a conformal submersion, using (15) we derive \(\frac{1}{\pi}g_2(\pi_+ FV, \pi_+ CZ) = 0\) for any \(Z \in \Gamma((\ker \pi_*)^\perp)\) and \(V \in \Gamma(\ker \pi_*)\), which implies that
\[ TM_2 = \pi_*(F_{\ker \pi}) \oplus \pi_*(\mu). \] (16)

**Lemma 3.1.** Let \( \pi \) be a conformal anti-invariant submersion from a locally product Riemannian manifold \((M_1, g_1, F/C_0/C_1)\) onto a Riemannian manifold \((M_2, g_2/C_0/C_1)\). Then we have

\[ g_1(CW, FV) = 0 \] (17)

and

\[ g_1(V^M_Z CW, FV) = -g_1(CW, FA_2 V) \] (18)

for \( Z, W \in \Gamma((\ker \pi)^\perp) \) and \( V \in \Gamma(\ker \pi) \).

**Proof.** For \( W \in \Gamma((\ker \pi)^\perp) \) and \( V \in \Gamma(\ker \pi) \), using (2) we have

\[ g_1(CW, FV) = g_1(FW - BW, FV) = g_1(FW, FV) \]

due to \( BW \in \Gamma(\ker \pi) \) and \( FV \in \Gamma((\ker \pi)^\perp) \). Hence \( g_1(FW, FV) = g_1(W, V) = 0 \) which is (17). Since \( M_1 \) is a locally product Riemannian manifold, differentiating (3.4) with respect to \( Z \), we get

\[ g_1(V^M_Z CW, FV) = g_1(CW, FA_2 V) \]

for \( Z, W \in \Gamma((\ker \pi)^\perp) \) and \( V \in \Gamma(\ker \pi) \). Then using (9) we have

\[ g_1(V^M_Z CW, FV) = -g_1(CW, FA_2 V) - g_1(CW, FV V^M_Z V). \]

Since \( FV V^M_Z V \in \Gamma(F_{\ker \pi}) \), we obtain (18). \( \square \)

We now study the integrability of the distribution \((\ker \pi)^\perp\) and then we investigate the geometry of the leaves of \( \ker \pi \) and \((\ker \pi)^\perp\). We note that it is known that the distribution \( \ker \pi \) is integrable.

**Theorem 3.1.** Let \( \pi : (M_1, g_1, F) \rightarrow (M_2, g_2) \) is a conformal anti-invariant submersion from an almost product Riemannian manifold \((M_1, g_1, F)\) to a Riemannian manifold \((M_2, g_2)\). Then the following assertions are equivalent to each other;

(a) \((\ker \pi)^\perp\) is integrable,

(b) \[ \frac{1}{\lambda^2} g_2(V^\pi_W CZ - V^\pi_Z CW, \pi_*(\mu) FV) = g_1(A_2 BW - A_2 BZ - CW(\ln \lambda) Z + CZ(\ln \lambda) W, FV) \] (19)
for any $Z, W \in \Gamma\left((\ker \pi)^\perp\right)$ and $V \in \Gamma(\ker \pi)$.

\textbf{Proof.} For $W \in \Gamma\left((\ker \pi)^\perp\right)$ and $V \in \Gamma(\ker \pi)$, we see from Definition 3.1, $FW \in \Gamma\left((\ker \pi)^\perp\right)$ and $FW \in \Gamma(\ker \pi \oplus \mu)$. Thus using (2) and (3), for $Z \in \Gamma\left((\ker \pi)^\perp\right)$ we obtain

$$g_1([Z, W], V) = g_1(FW, FW) - g_1(FW, Z).$$

Further, from (15) we get

$$g_1([Z, W], V) = g_1(V^M_{Z}FW, FW) - g_1(V^M_WZ, FW).$$

Using (9), (10) and if we take into account $\pi$ is a conformal submersion, we arrive at

$$g_1([Z, W], V) = g_1(A_ZBW - A_WBZ, FW) + \frac{1}{\lambda^2}g_2(\pi_*(V^M_ZCW), \pi_*FW) - \frac{1}{\lambda^2}g_2(\pi_*(V^M_WCW), \pi_*FW).$$

Thus, from (12) and Lemma 2.2 we derive

$$g_1([Z, W], V) = g_1(A_ZBW - A_WBZ, FW) - \frac{1}{\lambda^2}g_2(\pi_*(V^M_ZCW), \pi_*FW).$$

Moreover, using (17), we obtain

$$g_1([Z, W], V) = g_1(A_ZBW - A_WBZ - \ln(\lambda)Z + C(\ln(\lambda)W, FW)
- \frac{1}{\lambda^2}g_2(V^\pi_W\pi_*CW - V^\pi_Z\pi_*CW, \pi_*FW)$$

which proves (a) $\Rightarrow$ (b).

From Theorem 3.1, we deduce the following characterization.

\textbf{Theorem 3.2.} Let $\pi$ be a conformal anti-invariant submersion from a locally product Riemannian manifold $(M_1, \tilde{g}_1, F)$ onto a Riemannian manifold $(M_2, g_2)$. Then any two conditions below imply the three;

\begin{enumerate}
  \item $(\ker \pi)_{\perp}$ is integrable.
  \item $\lambda$ is a constant on $\Gamma(\mu)$.
  \item $g_2(V^\pi_W\pi_*CW - V^\pi_Z\pi_*CW, \pi_*FW) = \lambda^2g_1(A_ZBW - A_WBZ, FW)$
\end{enumerate}
for $Z, W \in \Gamma(\langle \ker \pi \rangle^\perp)$ and $V \in \Gamma(\ker \pi)$. 

Proof. From Theorem 3.1, we have

$$g_1([Z, W], V) = g_1(A_2 BW - A_W BZ - CW(\ln \lambda)Z + CZ(\ln \lambda)W, FV)$$

$$- \frac{1}{\lambda^2} g_2(V^n_W \pi_* CZ - V^n_Z \pi_* CW, \pi_* FV).$$

for $Z, W \in \Gamma(\langle \ker \pi \rangle^\perp)$ and $V \in \Gamma(\ker \pi)$. Now, if we have (i) and (iii), then we arrive at

$$-g_1(\nabla \pi^\ast \nabla \lambda, CW) g_1(Z, FV) + g_1(\nabla \pi^\ast \nabla \lambda, CZ) g_1(W, FV) = 0. \quad (20)$$

Now, taking $W = FV$ in (20) for $V \in \Gamma(\langle \ker \pi \rangle^\perp)$, using (17), we get

$$-g_1(\nabla \pi^\ast \nabla \lambda, C(FV)) g_1(Z, FV) + g_1(\nabla \pi^\ast \nabla \lambda, CZ) g_1(FV, FV) = 0.$$

Hence $\lambda$ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

We say that a conformal anti-invariant submersion is a conformal Lagrangian submersion if $F(\ker \pi) = \ker \pi^\ast$. From Theorem 3.1, we have the following result.

Corollary 3.1. Let $\pi$ be a conformal Lagrangian submersion from a locally product Riemannian manifold $(M_1, g_1, F)$ onto a Riemannian manifold $(M_2, g_2)$. Then the following assertions are equivalent to each other:

i. $\langle \ker \pi \rangle^\perp$ is integrable
ii. $A_2 FW = A_W FZ$
iii. $(\nabla \pi_*) (Z, FW) = (\nabla \pi_*) (W, FZ)$

for $Z, W \in \Gamma(\langle \ker \pi \rangle^\perp)$.

Proof. For $Z, W \in \Gamma(\langle \ker \pi \rangle^\perp)$ and $V \in \Gamma(\ker \pi)$, we see from Definition 3.1, $FV \in \Gamma(\langle \ker \pi \rangle^\perp)$ and $FW \in \Gamma(\ker \pi)$. From Theorem 3.1 we have

$$g_1([Z, W], V) = g_1(A_2 BW - A_W BZ - CW(\ln \lambda)Z + CZ(\ln \lambda)W, FV)$$

$$- \frac{1}{\lambda^2} g_2(V^n_W \pi_* CZ - V^n_Z \pi_* CW, \pi_* FV).$$

Since $\pi$ is a conformal Lagrangian submersion, we derive

$$g_1([Z, W], V) = g_1(A_2 BW - A_W BZ, FV)$$

which shows (i) $\Leftrightarrow$ (ii). On the other hand, using Definition 3.1 and (9) we arrive at
Now, using (12) we obtain
\[
\frac{1}{\lambda^2} \nabla M_1 Z \pi \ast FV = \frac{1}{\lambda^2} \nabla M_1 Z \pi \ast FV (W, BZ) + \nabla \pi \ast FV (W, BZ) + \nabla \pi \ast FV (Z, BW),
\]
for \( Z, W \in \Gamma (\ker \pi \ast), \) and \( V \in \Gamma (\ker \pi \ast). \)

Proof. For \( Z, W \in \Gamma (\ker \pi \ast) \) and \( V \in \Gamma (\ker \pi \ast), \) by using (3), (9), (10), (14) and (15) we have
\[
g_1 (A_2 BW; FV) = g_1 (A_2 BW; FV) + g_1 (A_2 BW; FV) + g_1 (A_2 BW; FV).
\]

Since \( BZ, BW \in \Gamma (\ker \pi \ast), \) we derive
\[
g_1 (A_2 BW; FV) = g_1 (A_2 BW; FV) + g_1 (A_2 BW; FV) + g_1 (A_2 BW; FV) + g_1 (A_2 BW; FV)
\]
which tells that \((\text{ii}) \iff (\text{iii}).\) \(\square\)

4. Totally geodesic foliations

In this section, we shall investigate the geometry of leaves of \((\ker \pi \ast)\) and \((\ker \pi \ast)^\perp.\) For the geometry of leaves of the horizontal distribution \((\ker \pi \ast)^\perp,\) we have the following theorem.

Theorem 4.1. Let \(\pi : (M_1, g_1, F) \to (M_2, g_2)\) be a conformal anti-invariant submersion from an almost product Riemannian manifold \((M_1, g_1, F)\) to a Riemannian manifold \((M_2, g_2)\). Then the following assertions are equivalent to each other;

i. \((\ker \pi \ast)^\perp\) defines a totally geodesic foliation on \(M_1\).

ii. \[-\frac{1}{\lambda^2} g_2 (\nabla_2 \pi \ast CW; \pi \ast FV) = g_1 (A_2 BW - CW (\ln \lambda) Z + g_1 (Z, CW) \ln \lambda; FV)\]

for \(Z, W \in \Gamma ((\ker \pi \ast)^\perp)\) and \(V \in \Gamma (\ker \pi \ast).\)

Proof. For \(Z, W \in \Gamma ((\ker \pi \ast)^\perp)\) and \(V \in \Gamma (\ker \pi \ast),\) by using (3), (9), (10), (14) and (15) we have
\[
g_1 (\nabla_2^M Z; V) = g_1 (A_2 BW, FV) + g_1 (A_2 BW, FV).
\]

Since \(\pi\) is a conformal submersion, using (12) and Lemma 2.2 we arrive at
Moreover, using Definition 3.1 and (17) we obtain
\[ g_1(M_1 Z W, V) = g_1(A Z BW) - \frac{1}{\lambda^2} g_1(\text{Hgrad} \ln \lambda, Z)g_2(\pi_* CW, \pi_* FV) - \frac{1}{\lambda^2} g_1(\text{Hgrad} \ln \lambda, CW)g_2(\pi_*, \pi_* FV) + \frac{1}{\lambda^2} g_1(Z, CW)g_2(\pi_*(\text{Hgrad} \ln \lambda), \pi_* FV) + \frac{1}{\lambda^2} g_2(V_2 \pi_* CW, \pi_* FV). \]

which proves \((i) \iff (ii)\). \(\square\)

From Theorem 4.1, we also deduce the following characterization.

**Theorem 4.2.** Let \( \pi \) be a conformal anti-invariant submersion from a locally product Riemannian manifold \((M_1, g_1, F)/(C_0/C_1)\) onto a Riemannian manifold \((M_2, g_2)/(C_0/C_1)\). Then any two conditions below imply the three;

i. \((\ker \pi)^\perp\) defines a totally geodesic foliation on \(M_1\).

ii. \(\pi\) is horizontally homothetic submersion.

iii. \(g_2(V_2 \pi_* CW, \pi_* FV) = \lambda^2 g_1(A Z FW, BW)\)

for \(Z, W \in \Gamma((\ker \pi)^\perp)\) and \(V \in \Gamma(\ker \pi)\).

**Proof.** For \(Z, W \in \Gamma((\ker \pi)^\perp)\) and \(V \in \Gamma(\ker \pi)\), from Theorem 4.1, we have
\[ g_1(M_1 Z W, V) = g_1(A Z BW) - \frac{1}{\lambda^2} g_1(\text{Hgrad} \ln \lambda, Z)g_2(\pi_* CW, \pi_* FV). \]

Now, if we have (i) and (iii), then we obtain
\[ -g_1(\text{Hgrad} \ln \lambda, CW)g_1(Z, FW) + g_1(\text{Hgrad} \ln \lambda, FW)g_1(Z, CM) = 0. \]

(21)

Now, taking \(Z = CW\) in (4.1) and using (17), we get
\[ g_1(\text{Hgrad} \ln \lambda, FW)g_1(CM, CW) = 0. \]

Thus, \(\lambda\) is a constant on \(\Gamma(F \ker \pi)\). On the other hand, taking \(Z = FW\) in (25) and using (17) we derive
\[ g_1(\nabla \ln \lambda, C_W)g_1(F, F) = 0. \]

From above equation, \( \lambda \) is a constant on \( \Gamma(\mu) \). Similarly, one can obtain the other assertions. \( \square \)

For conformal Lagrangian submersion, we have the following result.

**Corollary 4.1.** Let \( \pi \) be a conformal Lagrangian submersion from a locally product Riemannian manifold \( (M_1, g_1, F) \) onto a Riemannian manifold \( (M_2, g_2) \). Then the following assertions are equivalent to each other:

1. \( (\ker \pi)\perp \) defines a totally geodesic foliation on \( M_1 \).
2. \( A_ZB_W = 0 \)
3. \( (\nabla \pi_\ast)(Z, F) = 0 \)

for \( Z, W \in \Gamma\left(\left(\ker \pi_\ast\right)^\perp\right) \) and \( V \in \Gamma(\ker \pi_\ast) \).

**Proof.** For \( Z, W \in \Gamma\left(\left(\ker \pi_\ast\right)^\perp\right) \) and \( V \in \Gamma(\ker \pi_\ast) \), from Theorem 4.1, we have

\[ g_1(A_Z B_W, V) = \frac{1}{\lambda^2} g_2(\pi_\ast(A_Z B_W), \pi_\ast F) = \frac{1}{\lambda^2} g_2(\pi_\ast(\nabla_M^Z W), \pi_\ast F). \]

Since \( \pi \) is a conformal Lagrangian submersion, we derive

\[ g_1(\nabla_M^Z W, V) = g_1(A_Z B_W, F) \]

which shows \( (i) \Leftrightarrow (ii) \). On the other hand, using Definition 3.1 and (9) we arrive at

\[ g_1(A_Z B_W, F) = \frac{1}{\lambda^2} g_2(\pi_\ast(A_Z B_W), \pi_\ast F) = \frac{1}{\lambda^2} g_2(\pi_\ast(\nabla_M^Z W), \pi_\ast F). \]

Now, using (12) we obtain

\[ \frac{1}{\lambda^2} g_2(\pi_\ast(\nabla_M^Z W), \pi_\ast F) = \frac{1}{\lambda^2} g_2(- (\nabla \pi_\ast)(Z, B_W) + \nabla_M^Z \pi_\ast B_W, \pi_\ast F) = - \frac{1}{\lambda^2} g_2((\nabla \pi_\ast)(Z, B_W), \pi_\ast F) \]

which tells that \( (ii) \Leftrightarrow (iii) \). \( \square \)

For the totally geodesicness of the foliations of the distribution \( \ker \pi_\ast \).

**Theorem 4.3.** Let \( \pi : (M_1, g_1, F) \rightarrow (M_2, g_2) \) is a conformal anti-invariant submersion from an almost product Riemannian manifold \( (M_1, g_1, F) \) to a Riemannian manifold \( (M_2, g_2) \). Then the following assertions are equivalent to each other:
i. $\ker \pi$ defines a totally geodesic foliation on $M_1$.

ii. \[ -\frac{1}{\lambda^2} g_2(\nabla_{FU}^\pi FV, \pi_* FCZ) = g_1(T_v FU, BZ) + g_1(U, V) g_1(\nabla_1 \ln \lambda, FCZ) \]

for $V, U \in \Gamma(\ker \pi)$ and $Z \in \Gamma((\ker \pi)^\perp)$.

**Proof.** For $Z \in \Gamma((\ker \pi)^\perp)$ and $V, U \in \Gamma(\ker \pi)$, by using (2), (3), (8) and (15) we get

\[ g_1(\nabla^M_1 V U, Z) = g_1(T_v FU, BZ) + g_1(\nabla^M_1 FU, CZ). \]

Since $\nabla^M_1$ is torsion free and $[V, FU] \in \Gamma(\ker \pi)$, we obtain

\[ g_1(\nabla^M_1 V U, Z) = g_1(T_v FU, BZ) + g_1(\nabla^M_1 FU, FCZ). \]

Using (3) and (10) we have

\[ g_1(\nabla^M_1 V U, Z) = g_1(T_v FU, BZ) + g_1(\nabla^M_1 FU, FCZ) \]

here we have used that $\mu$ is invariant. Since $\pi$ is a conformal submersion, using (12) and Lemma 2.2 we obtain

\[ g_1(\nabla^M_1 V U, Z) = g_1(T_v FU, BZ) + g_1(\nabla^M_1 FU, FCZ) \]

Moreover, using Definition 3.1 and (17), we obtain

\[ g_1(\nabla^M_1 V U, Z) = g_1(T_v FU, BZ) + g_1(U, V) g_1(\nabla_1 \ln \lambda, FCZ) + \frac{1}{\lambda^2} g_2(\nabla_{FU}^\pi FV, \pi_* FCZ). \]

which proves (i) $\iff$ (ii). $\square$

From Theorem 4.3, we deduce the following result.

**Theorem 4.4.** Let $\pi$ be a conformal anti-invariant submersion from a locally product Riemannian manifold $(M_1, g_1, F)$ onto a Riemannian manifold $(M_2, g_2)$. Then any two conditions below imply the three;

i. $\ker \pi$ defines a totally geodesic foliation on $M_1$

ii. $\lambda$ is a constant on $\Gamma(\mu)$

iii. \[ -\frac{1}{\lambda^2} g_2(\nabla_{FU}^\pi FV, \pi_* FCZ) = g_1(T_v FU, BZ) \]
for $V, U \in \Gamma(ker\pi)$ and $Z \in \Gamma(\left((ker\pi)_s\right)^\perp)$.

**Proof.** For $V, U \in \Gamma(ker\pi)$ and $Z \in \Gamma(\left((ker\pi)_s\right)^\perp)$, from Theorem 4.3 we have

$$g_1(V^\Gamma U, Z) = g_1(T_VFU, BZ) + g_1(U, V)g_1(H\text{grad ln} \lambda, FCZ) + \frac{1}{\lambda^2} g_2(V\pi^*\pi_*, FV, \pi_* FCZ).$$

Now, if we have (i) and (iii), then we obtain

$$g_1(U, V)g_1(H\text{grad ln} \lambda, FCZ) = 0.$$  

From above equation, $\lambda$ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

If $\pi$ is a conformal Lagrangian submersion, then (16) implies that $TM_2 = \pi_* (ker\pi)$. Hence we have the following corollary:

**Corollary 4.2.** Let $\pi$ be a conformal Lagrangian submersion from a locally product Riemannian manifold $(M_1, g_1; F)$ onto a Riemannian manifold $(M_2, g_2)$. Then the following assertions are equivalent to each other:

i. $ker\pi$ defines a totally geodesic foliation on $M_1$.

ii. $T_VFU = 0$

for $V, U \in \Gamma(ker\pi)$ and $Z \in \Gamma(\left((ker\pi)_s\right)^\perp)$.

**Proof.** From Theorem 4.3 we have

$$g_1(V^\Gamma U, Z) = g_1(T_VFU, BZ) + g_1(U, V)g_1(H\text{grad ln} \lambda, FCZ) + \frac{1}{\lambda^2} g_2(V\pi^*\pi_*, FV, \pi_* FCZ).$$

for $V, U \in \Gamma(ker\pi)$ and $Z \in \Gamma(\left((ker\pi)_s\right)^\perp)$. Since $\pi$ is a conformal Lagrangian submersion, we get

$$g_1(V^\Gamma U, Z) = g_1(T_VFU, BZ)$$

which shows $(i) \iff (ii)$. □

5. **Totally geodesicness of the conformal anti-invariant submersion**

In this section, we shall examine the totally geodesicness of a conformal anti-invariant submersion. We give a necessary and sufficient condition for a conformal anti-invariant submersion to be totally geodesic map. Recall that a smooth map $\pi$ between two Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$ [15].
Theorem 5.1. Let \( \pi : (M_1, g_1, F) \to (M_2, g_2) \) is a conformal anti-invariant submersion from an almost product Riemannian manifold \( (M_1, g_1, F) \) to a Riemannian manifold \( (M_2, g_2) \). \( \pi \) is totally geodesic map if and only if,

(a) \( \pi \) is a horizontally homothetic map,

(b) \( T_U F V = 0 \) and \( H V_{M_1}^M F V \in \Gamma(\text{Ker} \pi^*) \),

(c) \( A_Z F V = 0 \) and \( H \nabla_{M_1}^Z F V \in \Gamma(\text{Ker} \pi^*) \)

for \( Z, W, Z \in \Gamma(\text{Ker} \pi^*) \) and \( U, V \in \Gamma(\text{Ker} \pi^*) \).

Proof. (a) For any \( Z, W \in \Gamma(\mu) \), from Lemma 2.2 we derive

\[
(\nabla \pi^*)(Z, W) = Z(\ln \lambda)\pi^* W + W(\ln \lambda)\pi^* Z - g_1(Z, W)\pi^*(\text{grad} \ln \lambda).
\]

It is obvious that if \( \pi \) is a horizontally homothetic map, it follows that \( (\nabla \pi^*)(Z, W) = 0 \). Conversely, if \( (\nabla \pi^*)(Z, W) = 0 \), taking \( W = F Z \) in above equation, we get

\[
Z(\ln \lambda)\pi^* F Z + F Z(\ln \lambda)\pi^* Z - g_1(Z, F Z)\pi^*(\text{grad} \ln \lambda) = 0. \tag{22}
\]

Taking inner product in (31) with \( \pi^* F Z \), we obtain

\[
g_1(\text{grad} \ln \lambda, Z)\lambda^2 g_1(F Z, F Z) + g_1(\text{grad} \ln \lambda, F Z)\lambda^2 g_1(Z, F Z) - g_1(Z, F Z)\lambda^2 g_1(\text{grad} \ln \lambda, F Z) = 0. \tag{23}
\]

From (32), \( \lambda \) is a constant on \( \Gamma(\mu) \). On the other hand, for \( U, V \in \Gamma(\text{Ker} \pi^*) \), from Lemma 2.2 we have

\[
(\nabla \pi^*)(F U, F V) = F U(\ln \lambda)\pi^* F V + F V(\ln \lambda)\pi^* F U - g_1(F U, F V)\pi^*(\text{grad} \ln \lambda).
\]

Again if \( \pi \) is a horizontally homothetic map, then \( (\nabla \pi^*)(F U, F V) = 0 \). Conversely, if \( (\nabla \pi^*)(F U, F V) = 0 \), putting \( U \) instead of \( V \) in above equation, we derive

\[
2 F U(\ln \lambda)\pi^* F U - g_1(F U, F U)\pi^*(\text{grad} \ln \lambda) = 0. \tag{24}
\]

Taking inner product in (33) with \( \pi^* F U \) and since \( \pi \) is a conformal submersion, we have

\[
g_1(F U, F U)\lambda^2 g_1(\text{grad} \ln \lambda, F U) = 0.
\]

From above equation, \( \lambda \) is a constant on \( \Gamma(\text{Ker} \pi^*) \). Thus \( \lambda \) is a constant on \( \Gamma((\text{Ker} \pi^*)^\perp) \).

(b) For any \( U, V \in \Gamma(\text{Ker} \pi^*) \), using (3) and (12) we have
\[(\nabla \pi_*)(U, V) = \nabla^*_U \pi_* V - \pi_* (\nabla^*_U V) = -\pi_* (F \nabla^*_U F V).\]

Then from (7) and (8) we arrive at

\[(\nabla \pi_*)(U, V) = -\pi_* (F T_U F V + C H \nabla^*_U F V).\]

From above equation, \((\nabla \pi_*)(U, V) = 0\) if and only if
\[
\pi_* (F T_U F V + C H \nabla^*_U F V) = 0 \tag{25}\]

Since \(\pi\) is non-singular, this implies \(T_U F V = 0\) and \(H \nabla^*_U F V \in \Gamma(\ker \pi_*).\)

(c) For \(Z \in \Gamma(\mu)\) and \(V \in \Gamma(\ker \pi_*),\) from (3) and (12) we get

\[(\nabla \pi_*)(Z, V) = \nabla^*_Z \pi_* V - \pi_* (\nabla^*_Z V) = -\pi_* (F V^*_Z F V).\]

Using (9) and (10) we have

\[(\nabla \pi_*)(Z, V) = \pi_* (F A Z F V + C H V^*_Z F V).\]

Thus \((\nabla \pi_*)(Z, V) = 0\) if and only if
\[
\pi_* (F A Z F V + C H V^*_Z F V) = 0.
\]

Then, since \(\pi\) is a linear isomorphism between \((\ker \pi_*)\) and \(T M_2, (\nabla \pi_*)(Z, V) = 0\) if and only if \(A Z F V = 0\) and \(H V^*_Z F V \in \Gamma(\ker \pi_*).\) Thus proof is complete. \(\square\)

Here we present another result on conformal anti-invariant submersion to be totally geodesic.

**Theorem 5.2** Let \(\pi\) be a conformal anti-invariant submersion from a locally product Riemannian manifold \((M_1, g_1, F)\) onto a Riemannian manifold \((M_2, g_2).\) If \(\pi\) is a totally geodesic map then

\[\nabla^*_Z \pi_* W_2 = \pi_* (F (A Z F W_1 + W^*_Z B W_2 + A Z C W_2) + C (H V^*_Z F W_1 + A Z B W_2 + H V^*_Z C W_2))\]

for any \(Z \in \Gamma((\ker \pi_*)^+)\) and \(W = W_1 + W_2 \in \Gamma(T M),\) where \(W_1 \in \Gamma(\ker \pi_*)\) and \(W_2 \in \Gamma((\ker \pi_*)^+).\)
Proof. Using (3) and (12) we have

\[(\nabla \pi_\ast)(Z, W) = \nabla^\pi_\ast W - \pi_\ast(FV_M^2FW)\]

for any \(Z \in \Gamma((\ker \pi_\ast)^\perp)\) and \(W \in \Gamma(TM_1)\). Then from (9), (10) and (15) we get

\[(\nabla \pi_\ast)(Z, W) = \nabla^\pi_\ast W - \pi_\ast(F(A_2FW_1 + BFW_2 + CFW_1 + DFW_2) + EFW_1 + FFW_2 + GFW_1 + HFW_2)\]

for any \(W = W_1 + W_2 \in \Gamma(TM_1)\), where \(W_1 \in \Gamma((\ker \pi_\ast)\perp)\) and \(W_2 \in \Gamma((\ker \pi_\ast)^\perp)\). Thus taking into account the vertical parts, we find

\[(\nabla \pi_\ast)(Z, W) = \nabla^\pi_\ast W - \pi_\ast(F(A_2FW_1 + BFW_2 + CFW_1 + DFW_2) + EFW_1 + FFW_2 + GFW_1 + HFW_2)\]

which gives our assertion.

□

6. Examples

In this section, we now give some examples for conformal anti-invariant submersions from almost product Riemannian manifolds.

**Example 6.1.** Every anti-invariant Riemannian submersion is a conformal anti-invariant submersion with \(\lambda = I\), where \(I\) is the identity function [7].

We say that a conformal anti-invariant submersion is proper if \(\lambda \neq I\). We now present an example of a proper conformal anti-invariant submersion. Note that given an Euclidean space \(\mathbb{R}^4 \times \mathbb{R}^4\) with coordinates \((x_1, \ldots, x_4)\), we can canonically choose an almost product structure \(F\) on \(\mathbb{R}^4\) as follows:

\[F\left(\frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + a_2 \frac{\partial}{\partial x_3} + a_3 \frac{\partial}{\partial x_4}\right) = \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + a_2 \frac{\partial}{\partial x_3} + a_3 \frac{\partial}{\partial x_4},\]

\[a_1, \ldots, a_3 \in \mathbb{R}.\]

(26)

**Example 6.2.** Let \(\pi\) be a submersion defined by

\[\pi : \mathbb{R}^4 \to \mathbb{R}^2 \times \mathbb{R}^2 \quad (x_1, x_2, x_3, x_4) \to (\cos x_1 \sinh x_2, \sin x_1 \cosh x_2).\]
Then it follows that
\[ \ker \pi^* = \text{span} \{ V_1 = \partial x_3, \; V_2 = \partial x_4 \} \]

and
\[ (\ker \pi^*)^\perp = \text{span} \{ X_1 = \partial x_1, \; X_2 = \partial x_2 \}. \]

Hence, we have \( FV_1 = X_1 \) and \( FV_2 = X_2 \) imply that \( F(\ker \pi^*) = (\ker \pi^*)^\perp \). Also by direct computations, we get
\[
\pi_* X_1 = -\sin x_1 \sinh x_2 \partial y_1 + \cos x_1 \cosh x_2 \partial y_2, \\
\pi_* X_2 = \cos x_1 \cosh x_2 \partial y_1 + \sin x_1 \sinh x_2 \partial y_2.
\]

Hence, we have
\[
g_2(\pi_* X_1, \pi_* X_1) = (\sin^2 x_1 \sinh^2 x_2 + \cos^2 x_1 \cosh^2 x_2) g_1(X_1, X_1), \]
\[
g_2(\pi_* X_2, \pi_* X_2) = (\sin^2 x_1 \sinh^2 x_2 + \cos^2 x_1 \cosh^2 x_2) g_1(X_2, X_2),
\]

where \( g_1 \) and \( g_2 \) denote the standard metrics (inner products) of \( \mathbb{R}^4 \) and \( \mathbb{R}^2 \). Thus \( \pi \) is a conformal anti-invariant submersion with \( \lambda^2 = (\sin^2 x_1 \sinh^2 x_2 + \cos^2 x_1 \cosh^2 x_2) \).

**Example 6.3.** Let \( \pi \) be a submersion defined by
\[
\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^2, \;
( x_1, x_2, x_3, x_4 ) \rightarrow ( \frac{e^{x_3} \sin x_4}{\sqrt{2}} \partial y_1 + \frac{e^{x_3} \cos x_4}{\sqrt{2}} \partial y_2, \frac{e^{x_3} \cos x_4}{\sqrt{2}} \partial y_1 - \frac{e^{x_3} \sin x_4}{\sqrt{2}} \partial y_2).
\]

Then it follows that
\[ \ker \pi^* = \text{span} \{ V_1 = \partial x_3, \; V_2 = \partial x_4 \} \]

and
\[ (\ker \pi^*)^\perp = \text{span} \{ W_1 = \partial x_3, \; W_2 = \partial x_4 \}. \]

Hence we have \( FV_1 = W_1 \) and \( FV_2 = W_2 \) imply that \( F(\ker \pi^*) = (\ker \pi^*)^\perp \). Also by direct computations, we get
\[
\pi_* W_1 = \frac{e^{x_3} \sin x_4}{\sqrt{2}} \partial y_1 + \frac{e^{x_3} \cos x_4}{\sqrt{2}} \partial y_2, \\
\pi_* W_2 = \frac{e^{x_3} \cos x_4}{\sqrt{2}} \partial y_1 - \frac{e^{x_3} \sin x_4}{\sqrt{2}} \partial y_2.
\]
Hence, we have
\[g_2(\pi_1 W_1, \pi_1 W_1) = \left(\frac{e^x}{\sqrt{2}}\right)^2 g_1(W_1, W_1),\]
\[g_2(\pi_2 W_2, \pi_2 W_2) = \left(\frac{e^x}{\sqrt{2}}\right)^2 g_1(W_2, W_2),\]
where \(g_1\) and \(g_2\) denote the standard metrics (inner products) of \(\mathbb{R}^4\) and \(\mathbb{R}^2\). Thus \(\pi\) is a conformal anti-invariant submersion with \(\lambda = \left(\frac{e^x}{\sqrt{2}}\right)\).

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