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# Discrete-Time Sliding Mode Control with Outputs of Relative Degree More than One

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Additional information is available at the end of the chapter

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## Abstract

This work deals with sliding mode control of discrete-time systems where the outputs are defined or chosen to be of relative degrees more than one. The analysis brings forward important advancements in the direction of discrete-time sliding mode control, such as improved robustness and performance of the system. It is proved that the ultimate band about the sliding surface could be greatly reduced by the choice of higher relative degree outputs, thus increasing the robustness of the system. Moreover, finite-time stability in absence of uncertainties is proved for such a choice of higher relative degree output. In presence of uncertainties, the system states become finite time ultimately bounded in nature. The work presents in some detail the case with relative degree two outputs, deducing switching and non-switching reaching laws for the same, while for arbitrary relative degree outputs, it shows a general formalisation of a control structure specific for a certain type of linear systems.

**Keywords:** discrete time, sliding mode control, finite-time stability, robust control, ultimate band

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## 1. Introduction

Sliding mode control is a robust control technique, which is able to make the system insensitive towards a particular class of uncertainties in finite time. Such uncertainties, known as matched uncertainties, are those that appear along the input channel of the system and can be nullified by a simple switching control structure when the disturbance is bounded in nature. The switch happens about a surface in the space of the state variables and is called a sliding or a switching surface. The sliding variable  $s = s(x)$  denotes how far the system states are from the sliding surface  $S = \{x : s(x) = 0\}$ . The control brings the system monotonically towards the sliding

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surface, thus  $|s(t)|$  reducing until it becomes zero at a finite time. This is called the reaching phase. Once the system hits the surface, it stays there for all future times, thus making the system dynamics independent of the matched uncertainties and dependent only on the sliding surface parameters. Chosen appropriately, one can ensure that the system states become at least asymptotically stable during this phase called sliding motion of the system [15].

However, in practice, this beautiful property of sliding mode control could not be realized because of physical limitations of an actuator. Theoretically, the control needs to switch about the sliding surface with infinite frequency in order to be insensitive towards bounded matched uncertainties, but no real actuators can offer switching with infinite frequency. This causes chattering, which are high frequency actuator action giving rise to unmodelled dynamics excitation in the system as well as rapid degradation of the physical system. Moreover, measurements by sensors and control computation in a digital computer take place in finite-time intervals in modern times, thus ripping off the properties of continuous sliding mode control which made it theoretically so appealing.

To remove this gap between theory and practice, researchers developed the theory of discrete-time sliding mode control (DSMC) in [1–3, 16, 17, 19, 20, 22, 23]. Moreover, there are many inherently discrete-time systems that appear in nature as well as in engineering. For such discrete representation of a system, it was shown that the states of these systems can no longer hit the sliding surface and stay there in presence of disturbances. The best that can be achieved is ultimate boundedness of the system about the sliding surface in finite time. Hence, robustness of the system gets defined by the width of this ultimate band for discrete-time systems. It then becomes imperative that research takes place in the direction to reduce the width of the ultimate band, ensuring better robustness of the system. The work in this chapter is motivated by this objective and in the sequel it is shown how the choice of the relative degree of the output (or the sliding variable) to be greater than one, positively influences the robustness as well as the performance of the system as defined above. From this point and further in the chapter, the terms ‘output’ and ‘sliding variable’ will be used interchangeably, as sliding variable can be viewed as a constructed output of the system.

Traditionally, DSMC has been developed by taking outputs of relative degree one, i.e. there is only unit delay between the output and the input of the system. This has given rise to proposals of various reaching laws of the form  $s(k+1) = f(s(k))$ , where  $s(k)$  is the sliding variable at the  $k$ th time step. These reaching laws make  $|s(k)|$  approach an ultimate band about the sliding surface in finite time. One can readily calculate the control that does so from the reaching law, since  $s(k+1)$  contains the control  $u(k)$ , when calculated from the system model. The most well-known reaching laws are laid down in Refs. [2, 3, 17]. Of the above, the first two papers deal with non-switching reaching laws, whereas the third one had proposed a switching reaching law. Even to this day, reaching law propositions form an important area of work in discrete-time sliding mode control, with different reaching laws favouring the design of control for a particular type of system. Some of these reaching laws are found in Refs. [5–11, 21, 24, 25].

The unity relative degree assumed in all the above works is also their major limitation. While it is the normal case to consider, there is no real restriction on the choice of this relative degree. In some system structures, the output can be naturally of relative degree more than one. In

others, one can easily construct an output with higher relative degree and consider it to the sliding variable to go about the analysis. In the recent studies [13, 14] which constitute the content in this chapter, it is shown that when this apparent limitation is lifted, we get reduced width of ultimate band, thus increasing robustness, as well as finite-time stability during sliding in absence of uncertainties. The latter is an important achievement, as previously finite-time stability during sliding for discrete-time systems had not been achieved. Only in Ref. [18], such finite-time stability of states had been achieved during sliding, but with specific design of surface parameters. With relative degree more than one, this finite-time stability of the system states during sliding is always guaranteed for a wide range of choices of the surface parameters.

The chapter is written as follows: in Section 2, an idea on the relative degree of outputs for discrete-time systems is given, which is used in the theoretical developments in the remainder of the chapter. In Section 3, a detailed work with reaching law propositions is done for relative degree two outputs for general linear time-invariant (LTI) systems of order  $n$ . For arbitrary relative degree outputs, a generalized control structure is proposed for a specific form of LTI systems in Section 4, in which the relative degree  $r$  is equal to the order  $n$  of the system. Improved robustness and finite-time stability are proved for all cases in both the sections. Simulation examples are also shown in each section, which corroborate the theoretical developments. The chapter ends with discussing the main results and implications thereof.

## 2. Relative degree for discrete-time systems

The concept of relative degree is well understood for continuous-time systems. The definition can be written as follows:

**Definition 1:** For a continuous-time system

$$\dot{x} = f_c(t, x, u) \tag{1}$$

the output  $y(t)$  is said to be of relative degree  $r$  if  $y^r = g_r(t, x, u)$  and  $y^i = g_i(t, x) \forall 0 \leq i < r$ , where  $u(t)$  is the control input and  $y^p$  denotes the  $p^{\text{th}}$  time derivative of  $y$ .

The above definition means that the control first appears physically in the  $r$ th derivative of the output  $y(t)$  and not before that.

The concept of relative degree for discrete-time systems can be easily understood by making a parallel of the above definition in the discrete-time domain. The derivative operator in continuous time becomes the difference operator in discrete time. Each difference introduces a delay between the output and the input of the system. With this in mind, one can propose the definition of relative degree for discrete-time systems as follows:

**Definition 2:** For a discrete-time system

$$x(k+1) = f_d(k, x(k), u(k)) \tag{2}$$

the output  $y(k)$  is said to be of relative degree  $r$  if  $y(k+r) = h_r(k, x(k), u(k))$  and  $y(k+i) = h_i(k, x(k)) \forall 0 \leq i < r$ , where  $u(k)$  is the control input and  $y(k+p)$  denotes the  $p$  unit delays of  $y$ .

Physically, the above definition means that the control first appears in the  $r$ th delay of the output  $y(k)$  and not before that. For a simple LTI system  $(A, B, C)$ , this will mean that  $CA^{i-1}B = 0 \forall i = 1$  to  $(r-1)$  and  $CA^rB \neq 0$ .

### 3. Systems with relative degree two output

Let us consider a discrete-time LTI system in the regular form as

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) \\ x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k) + B_2f(k) \end{aligned} \quad (3)$$

where  $x_1(k) \in \mathbb{R}^{n-m}$  and  $x_2(k) \in \mathbb{R}^m$  are the  $n$  states and  $u(k) \in \mathbb{R}^m$  is the control input. The disturbance  $f(k) \in \mathbb{R}^m$  is assumed to be bounded as  $\|f(k)\| \leq f_m$ .

Obviously  $A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$ ,  $A_{12} \in \mathbb{R}^{(n-m) \times m}$ ,  $A_{21} \in \mathbb{R}^{m \times (n-m)}$ ,  $A_{22} \in \mathbb{R}^{m \times m}$  and  $B_2 \in \mathbb{R}^{m \times m}$ . Let us assume  $\det(B_2) \neq 0$ . Written in the standard form  $x(k+1) = Ax(k) + B(u(k) + f(k))$  for LTI systems, we shall have  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$  and  $B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$ .

#### 3.1. Asymptotic stability with relative degree one output

A relative degree one output for the discrete-time system as in Eq. (3) can be proposed as

$$s_1(k) = C_1^T x(k) = Cx_1(k) + I_m x_2(k) \quad (4)$$

where  $C \in \mathbb{R}^{m \times (n-m)}$  and the suffix 1 denotes relative degree one. Then

$$C_1^T B = [C \quad I_m] \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = B_2 \quad (5)$$

and we can calculate the control  $u(k)$  from

$$s_1(k+1) = C_1^T Ax(k) + C_1^T Bu_1(k) + C_1^T Bf(k) \quad (6)$$

using some relative degree one reaching law for  $s(k)$ , since  $B_2$  is non-singular.

Design of  $C$  is done considering closed-loop performance during sliding motion of the nominal system, i.e. system with  $f(k) = 0$ . When the system is sliding, output  $s_1(k)$  is zero, which makes  $x_2(k) = -Cx_1(k)$ . Hence, the closed loop during sliding becomes

$$x_1(k+1) = (A_{11} - A_{12}C)x_1(k) \quad (7)$$

which is traditionally made asymptotically stable by choosing  $\lambda_{\max}(A_{11} - A_{12}C) < 1$ . Since  $x_2(k)$  is algebraically related to  $x_1(k)$ , it also settles down to zero asymptotically.

### 3.2. Finite-time stability with relative degree two output

For the system in Eq. (3), a relative degree two output can be

$$s_2(k) = C_2^T x(k) = Cx_1(k) \quad (8)$$

where  $C \in \mathbb{R}^{m \times (n-m)}$  can be chosen same as in Eq. (4) or different, but satisfying the conditions in Theorem 1 below. The suffix 2 is used to denote relative degree two.

Now  $C_2^T B = [C \ 0] \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = 0$  clearly shows that

$$s_2(k+1) = C_2^T Ax(k) \quad (9)$$

as calculated from the system dynamics in Eq. (3) does not contain the control input  $u(k)$ . Then we need to further assume  $C_2^T B = [C \ 0] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = CA_{12}B_2$  to be non-singular so that the output  $s_2(k)$  is of relative degree two. Then we obtain

$$s_2(k+2) = C_2^T A^2 x(k) + C_2^T AB(u_2(k) + f(k)) \quad (10)$$

by adding one more delay to Eq. (9). The control input  $u(k)$  can now be obtained using Eq. (10).

**Theorem 1.** *If  $\text{Ker}(C) = 0$  and  $\det(CA_{12}) \neq 0$ , then the output  $s_2(k)$  with relative degree two as designed in Eq. (8) ensures finite-time stability of the states of the system in Eq. (3) during sliding, in absence of the disturbance  $f(k)$ .*

*Proof.* During sliding,  $s_2(k) = Cx_1(k) = 0$ . If  $\text{Ker}(C) = 0$ , it follows that  $x_1(k) = 0$  during sliding. Also, we have  $s_2(k+1) = CA_{11}x_1(k) + CA_{12}x_2(k) = 0$  during sliding which implies  $x_2(k) = -(CA_{12})^{-1}CA_{11}x_1(k)$ . As  $x_1(k) = 0$ , it follows that  $x_2(k) = 0$  as well, since  $CA_{12}$  is assumed to be non-singular. Hence, all the states become zero at the same instant as the output hits zero. This happens in finite time for any appropriately designed reaching law, which can bring the nominal system to the sliding surface in finite time. Thus, one can conclude that the system states become finite-time stable with the choice of relative degree two output.

Note that,  $\text{Ker}(C) = 0$  is only a sufficient condition and not a necessary one in order to achieve finite-time stability of system states. The above theorem points out an important achievement in the closed-loop reduced order dynamics compared to the choice of the relative degree one output. Of course, if there is a disturbance, then the finite-time stability would be changed to finite time-bounded stability, i.e. the system states will only enter an ultimate band in a finite time and stay there.

**Remark 1.** *In simulations, the parameter  $C$  is chosen the same for both relative degree one and two outputs for comparison purposes. However, selection of the parameter  $C$  for relative degree two output*

does not in any way require a priori design of the same parameter for a relative degree one output. The property of finite-time stability is inherent to the relative degree two output systems provided  $C$  is selected as per the conditions in Theorem 1, which are easy to satisfy.

### 3.3. Non-switching reaching law

In Ref. [3], a reaching law for discrete-time systems is introduced as

$$s(k+1) = s_d(k+1) + d(k)$$

$$s_d(k) = \begin{cases} \frac{k^* - k}{k^*} s(0) & \text{for } k < k^* \\ 0 & \text{for } k \geq k^* \end{cases} \quad (11)$$

and  $d(k)$  is an uncertainty derived from the system uncertainty  $f(k)$ . It is evident that this reaching law makes the sliding variable  $|s(k)| \leq d_m \forall k \geq k^*$ , i.e.  $d_m$  is the ultimate band for the sliding variable  $s(k)$ , where the uncertainty  $d(k)$  is bounded as  $|d(k)| \leq d_m$ .

#### 3.3.1. Ultimate band for relative degree one output

It is evident that

$$s_1(k+1) = C_1^T x(k+1) = C_1^T A x(k) + C_1^T B (u_1(k) + f(k)) \quad (12)$$

which requires  $d(k) = d_1(k) = C_1^T B f(k)$  in Eq. (11) so that the control

$$u_1(k) = -(C_1^T B)^{-1} \left[ (C_1^T A) x(k) - s_d(k+1) \right] \quad (13)$$

does not contain any uncertain terms. This makes the bound of  $d_1(k)$  for relative degree one outputs as

$$d_{1m} = \|C_1^T B\| f_m = \|B_2\| f_m \quad (14)$$

which is the ultimate band  $\delta_1$  as well.

#### 3.3.2. Ultimate band for relative degree two output

It is already shown that  $s_2(k+1)$  does not contain the control input as well as the matched disturbance, being a relative degree two output. Hence, we obtain

$$s_2(k+2) = C_2^T x(k+2) = C_2^T A^2 x(k) + C_2^T A B (u(k) + f(k)) \quad (15)$$

containing the control input and this requires to extend the reaching law in Eq. (11) to find  $s_2(k+2)$ . It is done by taking the nominal part of the reaching law (without  $d(k)$ ) and adding an unit delay to find  $s_2(k+2)$ . Then we include  $d_2(k)$  to take care of the matched disturbance. This gives the extended reaching law for relative degree two outputs as

$$s_2(k+2) = s_d(k+2) + d_2(k)$$

$$s_d(k) = \begin{cases} \frac{k^* - k}{k^*} s(0) & \text{for } k < k^* \\ 0 & \text{for } k \geq k^* \end{cases} \quad (16)$$

With  $d_2(k) = C_2^T ABf(k)$  in Eq. (16), the control input

$$u_2(k) = -(C_2^T AB)^{-1} [(C_2^T A^2 x(k) - s_d(k+2))] \quad (17)$$

does not contain any uncertain terms. The bound of  $d_2(k)$  in this case is

$$d_{2m} = \|C_2^T AB\| f_m \leq \|CA_{12}\| \|B_2\| f_m = \|CA_{12}\| d_{1m} \quad (18)$$

which is the ultimate band  $\delta_2$  as well.

**Theorem 2.** *If in addition to the conditions in Theorem 1, C also satisfies  $\lambda_{\max}(CA_{12}) < 1$ , then the ultimate band  $\delta_2$  for the relative degree two output with reaching law in Eq. (16) is lesser than the ultimate band  $\delta_1$  for the relative degree one output with reaching law in Eq. (11), irrespective of whether the parameter C is chosen same for both relative degree cases.*

*Proof.* The property is straightforward to see from Eq. (18).

### 3.4. Switching reaching law

In Ref. [17], Gao et al. proposed a switching reaching law for discrete time SMC systems, which has the form

$$s_1(k+1) = \alpha s_1(k) - \beta_1 \text{sign}(s_1(k)) + d_1(k) \quad (19)$$

where  $\alpha \in (0, 1)$  and  $\beta_1 > d_{1m}$  are real constants,  $d_1(k)$  is the uncertainty derived from the system uncertainty  $f(k)$  and bounded as  $|d_1(k)| < d_{1m}$ . At present there are two ways to analyse Gao's reaching law, one provided in Ref. [4] and the other in Ref. [12]. In this work, the well-known analysis established in Ref. [4] is followed.

#### 3.4.1. Ultimate band for relative degree one output

It is already shown that

$$s_1(k+1) = C_1^T x(k+1) = C_1^T Ax(k) + C_1^T B(u_1(k) + f(k)) \quad (20)$$

which requires  $d_1(k) = C_1^T Bf(k)$  in Eq. (19) so that the control input

$$u_1(k) = -(C_1^T B)^{-1} [C_1^T Ax(k) - \alpha C_1^T x(k) + \beta_1 \text{sign}(C_1^T x(k))] \quad (21)$$

does not contain uncertain terms. This makes the bound of  $d_1(k)$  for relative degree one outputs as

$$d_{1m} = \|C_1^T B\| f_m = \|B_2\| f_m \quad (22)$$

which is the same as Eq. (14) in Section 3.3.1.

As per the analysis in Ref. [4] of the reaching law in Eq. (19), we need  $\beta_1 > \frac{(1+\alpha)}{(1-\alpha)} d_{1m}$  for crossing-recrossing  $s_1(k) = 0$  at each successive step after crossing it for the first time. The ultimate band is then calculated as

$$\delta_1 = \beta_1 + d_{1m} > \frac{2d_{1m}}{1-\alpha} \quad (23)$$

### 3.4.2. Ultimate band for relative degree two output

It is already shown that  $s_2(k+1)$  does not contain the input. Hence, we calculate

$$s_2(k+2) = C_2^T x(k+2) = C_2^T A^2 x(k) + C_2^T AB(u_2(k) + f(k)) \quad (24)$$

where the control input appears. This requires one to also extend the reaching law in Eq. (11) to find  $s_2(k+2)$ . This is done by taking the nominal part of the reaching law (i.e. with  $d(k) = 0$ ) and adding another unit delay to find  $s_2(k+2)$ . Then we include  $d_2(k)$  to take care of the matched disturbance. This gives the extended reaching law as

$$s_2(k+2) = \alpha^2 s_2(k) - \alpha\beta_2 \text{sign}(s_2(k)) - \beta_2 \text{sign}(s_2(k+1)) + d_2(k) \quad (25)$$

With  $d_2(k) = C_2^T ABf(k)$  in Eq. (25), the control

$$u_2(k) = -(C_2^T AB)^{-1} \left[ (C_2^T A^2 - \alpha^2 C_2^T) x(k) + \alpha\beta_2 \text{sign}(C_2^T x(k)) + \beta_2 \text{sign}(C_2^T Ax(k)) \right] \quad (26)$$

becomes devoid of any uncertain terms. The bound of  $d_2(k)$  in this case is

$$d_{2m} = \|C_2^T AB\| f_m \leq \|CA_{12}\| \|B_2\| f_m = \|CA_{12}\| d_{1m} \quad (27)$$

which is same as Eq. (18) in Section 3.3.2. The task now is to determine the ultimate band  $\delta_2$  and the conditions on  $\beta_2$  that needs to be satisfied. These are evaluated keeping in mind the property of crossing-recrossing about  $s_2(k) = 0$  as imposed in the original work in Ref. [17] for relative degree one output. For simplicity, we perform the analysis assuming  $s_2(k) \in R$ . For a higher-dimensional output  $s_2(k)$ , the same analysis shall hold for each element of the vector.

Let us consider the sliding variable  $s_2(k)$  at two consecutive time instants. In other words, we take into account the values of both  $s_2(k)$  and  $s_2(k+1)$ , where  $k$  is any non-negative integer. Then, one can either have  $\text{sign}(s_2(k+1)) = \text{sign}(s_2(k))$  or  $\text{sign}(s_2(k+1)) = -\text{sign}(s_2(k))$ .

**Lemma 1.** *If  $\beta_2 > \frac{d_{2m}}{1+\alpha}$  and  $\text{sign}(s_2(k+1)) = \text{sign}(s_2(k))$ , then  $|s_2(k+2)|$  is strictly smaller than  $|s_2(k)|$  or crosses the hyperplane  $s_2(k) = 0$ .*

*Proof.* For  $\text{sign}(s_2(k+1)) = \text{sign}(s_2(k)) = 1$ , from Eq. (25) we get

$$s_2(k+2) \leq \alpha^2 s_2(k) - (1+\alpha)\beta_2 + d_{2m} < s_2(k) \quad (28)$$

since  $\beta_2 > \frac{d_{2m}}{1+\alpha}$ .

For  $\text{sign}(s_2(k+1)) = \text{sign}(s_2(k)) = -1$ , from Eq. (25) we get

$$s_2(k+2) \geq \alpha^2 s_2(k) + (1+\alpha)\beta_2 - d_{2m} > s_2(k) \quad (29)$$

It is straightforward to conclude from the above two inequalities that  $|s_2(k+2)| < |s_2(k)|$  or  $\text{sign}(s_2(k+2)) = -\text{sign}(s_2(k+1)) = -\text{sign}(s_2(k))$ .

Lemma 1 can be geometrically interpreted as follows: if the states  $x(k)$  and  $x(k+1)$  are on the same side of the sliding hyperplane, then either  $x(k+2)$  is at the same side of the hyperplane and closer to it than  $x(k)$  or  $x(k+2)$  is on the other side of the hyperplane.

As  $k$  is an arbitrary non-negative integer, the above lemma demonstrates that there exists such a finite  $k_0 > 0$  that  $\forall i < k_0$ , we have  $\text{sign}[s_2(i)] = \text{sign}[s_2(0)]$  and  $\text{sign}[s_2(k_0)] = -\text{sign}[s_2(0)]$ . That is, there exists a finite time instant  $k_0$ , at which the sliding variable  $s_2(k)$  changes its sign. In other words, the system crosses the sliding surface in finite time.

**Lemma 2.** If  $\beta_2 > \frac{d_{2m}}{1-\alpha}$  and  $\text{sign}(s_2(k+1)) = -\text{sign}(s_2(k))$ , then  $\text{sign}(s_2(k+2)) = \text{sign}(s_2(k))$ .

*Proof.* With  $\text{sign}(s_2(k+1)) = -\text{sign}(s_2(k))$ , from Eq. (25) we get

$$\begin{aligned} s_2(k+2) &= \alpha^2 s_2(k) - \alpha\beta_2 \text{sign}(s_2(k)) - \beta_2 \text{sign}(s_2(k+1)) + d_2(k) \\ &= \alpha^2 s_2(k) - \alpha\beta_2 \text{sign}(s_2(k)) + \beta_2 \text{sign}(s_2(k)) + d_2(k) \\ &= \alpha^2 s_2(k) + (1-\alpha)\beta_2 \text{sign}(s_2(k)) + d_2(k) \end{aligned} \quad (30)$$

Since  $\beta_2 > \frac{d_{2m}}{1-\alpha}$ , then for any  $|d_2(k)| < d_{2m}$ , we get  $\text{sign}(s_2(k+2)) = \text{sign}(s_2(k))$ .

As  $k$  is an arbitrary non-negative integer, the above lemma implies that  $\beta_2 > \frac{d_{2m}}{1-\alpha}$  is both a necessary and sufficient condition for crossing-recrossing the sliding hyperplane  $s_2(k) = 0$  at each successive step after crossing it for the first time. Furthermore, the condition on  $\beta_2$  in Lemma 2 automatically guarantees that the condition on  $\beta_2$  in Lemma 1 holds. This concludes that the former is a necessary and sufficient condition for generating the quasi-sliding mode in the sense of Gao [17]. Indeed, when  $\beta_2 > \frac{d_{2m}}{1-\alpha}$  is satisfied, then the system crosses the sliding hyperplane in a finite time and then recrosses it again in every consecutive step. However, the sequence  $\{|s(k)|\}$  may not necessarily approach zero monotonically, but the sequence of every alternate sample of  $\{|s(k)|\}$  does. Ultimately, the quasi-sliding mode is achieved when  $\{|s(k)|\}$  starts crossing-recrossing about  $s(k) = 0$  at each time step.

With the help of these ideas, the ultimate band  $\delta_2$  for the sliding variable  $s_2(k)$  can be found out, which gives a measure of the robustness of the system concerned. The ultimate band must be equal to the largest steady-state value of the sliding variable for the maximum disturbance  $|d_2(k)| = d_{2m}$ . This is obtained from Eq. (25) putting  $s_2(k) = \delta_2$ , which also gives the value of  $s_2(k+2) = \delta_2$ . Thus,

$$\delta_2 = \alpha^2 \delta_2 - \alpha \beta_2 + \beta_2 + d_{2m} \quad (31)$$

which gives

$$\delta_2 = \frac{(1 - \alpha)\beta_2 + d_{2m}}{(1 - \alpha^2)} > \frac{2d_{2m}}{(1 - \alpha^2)} \quad (32)$$

since  $\beta_2 > \frac{d_{2m}}{(1 - \alpha)}$ .

**Theorem 3.** *If in addition to the conditions as in Theorem 1,  $C$  also satisfies  $\sigma_{\max}(CA_{12}) < 1 + \alpha$ , then the ultimate band  $\delta_2$  for the relative degree two output with reaching law in Eq. (25) is lesser than the ultimate band  $\delta_1$  for the relative degree one output with reaching law in Eq. (19), irrespective of the parameter  $C$  chosen same for both relative degree cases.*

*Proof.* Let us consider  $\rho > 1$ . Then the inequalities in Eqs. (23) and (32) can be written as equalities multiplying the RHS with this  $\rho$ . This gives us

$$\delta_1 = \rho \frac{2d_{1m}}{(1 - \alpha)} \quad \delta_2 = \rho \frac{2d_{2m}}{(1 - \alpha^2)} \quad (33)$$

Taking into account the fact that  $d_{2m} \leq \|CA_{12}\|d_{1m}$ , we get

$$\frac{\delta_2}{\delta_1} = \frac{2d_{2m}/2d_{1m}}{(1 + \alpha)} \leq \frac{\|CA_{12}\|}{(1 + \alpha)} \quad (34)$$

Hence,  $\delta_2 < \delta_1$ , if the condition  $\lambda_{\max}(CA_{12}) < 1 + \alpha$  is satisfied.

Here,  $\rho$  is selected the same for both the ultimate bands  $\delta_1$  and  $\delta_2$ . It can be considered as a selection parameter for  $\delta_1$  which is kept same for the selection of  $\delta_2$  for fair comparison between the two ultimate bands.

**Remark 2.** *Compared to Theorem 2, the condition on  $C$  in Theorem 3 is more relaxed. Hence, with the switching reaching law in Eq. (25), we can decrease the ultimate band for relative degree two output with a less strict condition than required with the non-switching reaching law in Eq. (11).*

### 3.5. Simulation example

Simulation examples are shown for a second-order discrete LTI system with outputs of both relative degree one and two to compare performance.

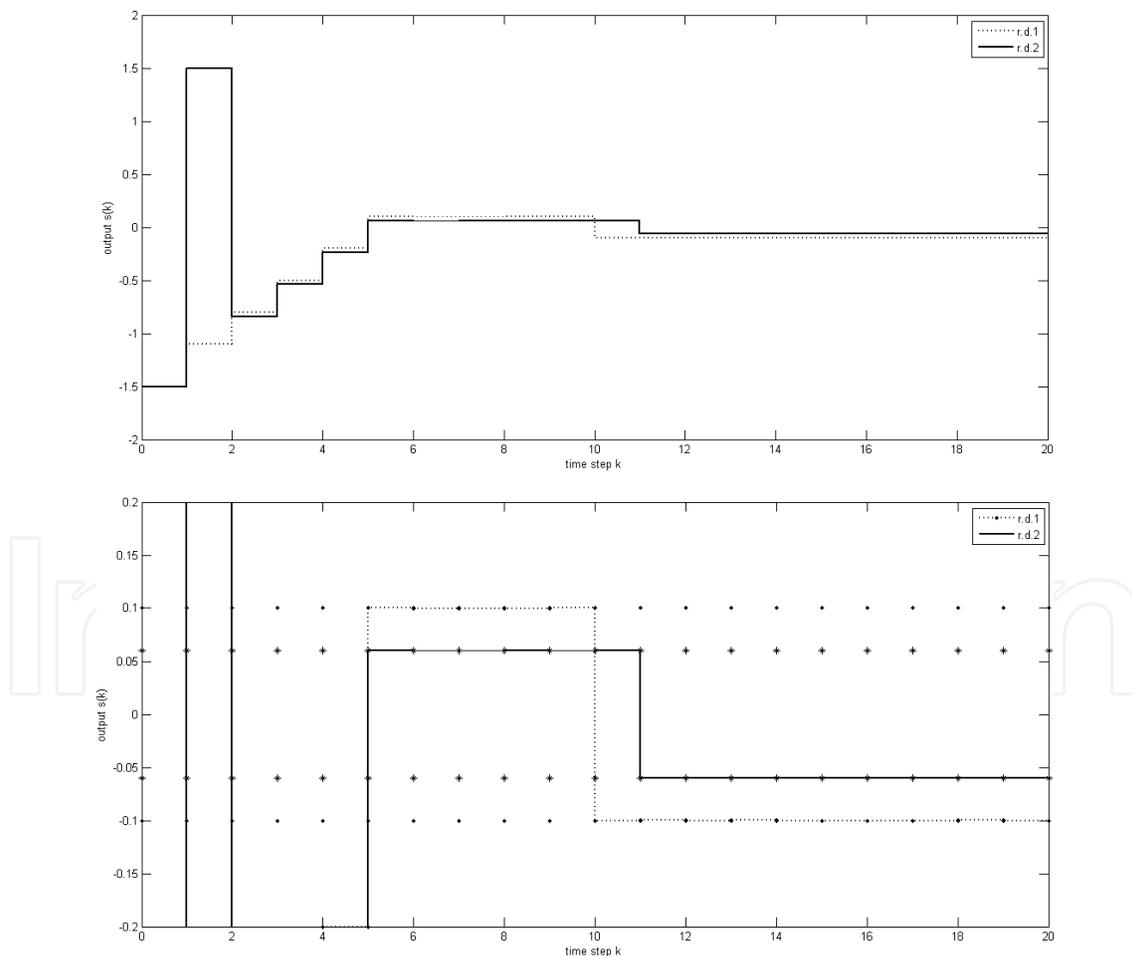
We consider an inherently unstable dynamical system

$$x(k+1) = \begin{bmatrix} 1 & 1.2 \\ 5 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(k) + f(k)) \quad (35)$$

where  $f(k)$  is a disturbance assuming value  $+0.1$  for the first half of the simulation cycle and  $-0.1$  for the last half. The disturbance is chosen at these extremities to bring out the worst behaviour of the system. The comparison between choices of relative degree one and two outputs can be considered fair under such a scenario.

### 3.5.1. Non-switching reaching law

The reaching law of [3] with  $k^* = 5$  is used for simulations. The surface parameter is selected as  $C = 0.5$ , which satisfies the conditions required in Theorem 2. The ultimate bands for the relative degree one and two outputs are calculated to be  $\delta_1 = 0.1$  and  $\delta_2 = 0.06$ , respectively. **Figure 1** shows the plots of the output  $s(k)$  along with a zoomed view to show the ultimate bands. The plots of the state variables and control input are given in **Figure 2**. The plots corresponding to relative degree one output are shown with a dotted line whereas those with relative degree two output are shown with a smooth line. It can be easily seen from **Figure 2**



**Figure 1.** Sliding variable for non-switching reaching law.

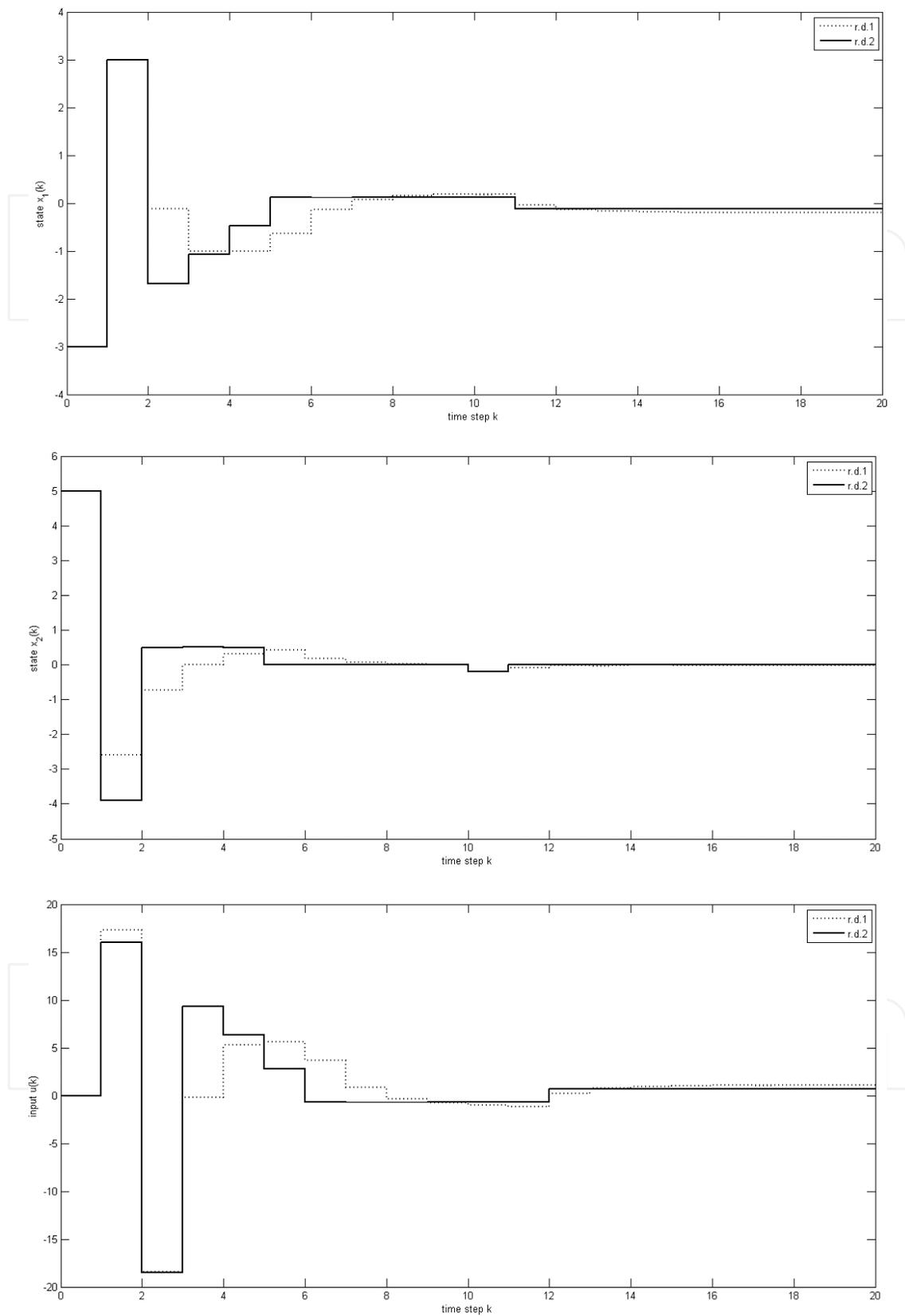
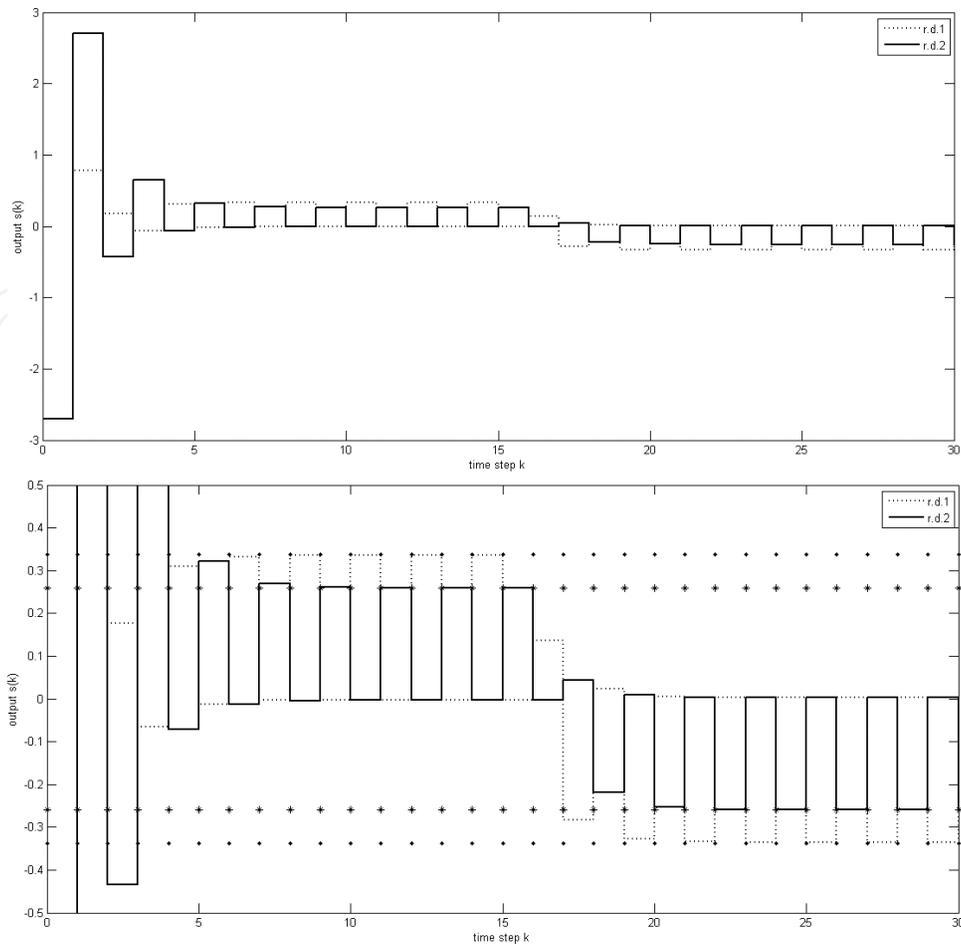


Figure 2. State variables and control input for non-switching reaching law.



**Figure 3.** Sliding variable for switching reaching law.

that both the state errors as well as the control effort are also reduced for relative degree two output compared to relative degree one output.

### 3.5.2. Switching reaching law

The reaching law of Ref. [17] is used for simulations. The surface parameter is chosen as  $C = 0.9$  which satisfies the conditions of Theorem 3 with  $\alpha = 0.4$ . For the purpose of simulations,  $\rho = 1.01$  is selected which gives the ultimate bands as  $\delta_1 = 0.3367$  and  $\delta_2 = 0.2597$ . For these values of the ultimate bands,  $\beta_1 = 0.2367$  and  $\beta_2 = 0.1836$  are calculated. **Figure 3** shows the plots of the output  $s(k)$  along with a zoomed view to show the ultimate bands. The plots of the state variables and control input are given in **Figure 2**. The plots corresponding to relative degree one output are shown with a dotted line whereas those with relative degree two output are shown with a smooth line. It can be easily seen from **Figure 4** that both the state errors as well as the control effort are also reduced for relative degree two output compared to relative degree one output.

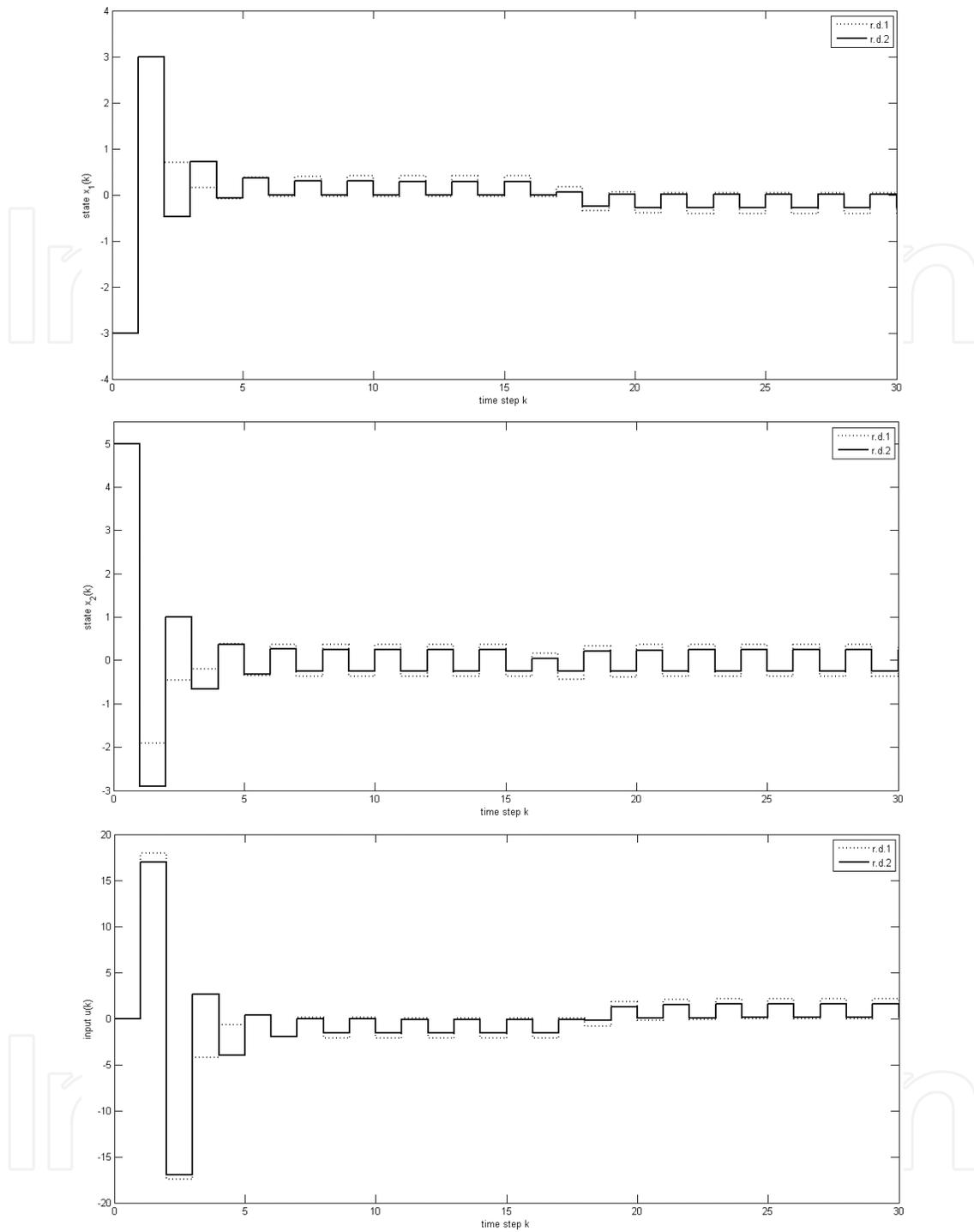


Figure 4. State variables and control input for switching reaching law.

#### 4. Systems with arbitrary relative degree outputs

In Section 3, the system order  $n$  was arbitrary but the relative degree of the output was fixed to two. In this section, the relative degree is extended to arbitrary  $r > 1$  where  $r \in \mathbb{N}_+$ . For the

purpose of the theoretical development presented in this chapter,  $r = n$  is considered, i.e. the relative degree of the output matches the system order. For such an assumption, the system structure can generally take a canonical form, called the lower Hessenberg form, whenever  $r > 2$ .

Consider a chain of  $n$  unit delays with the system output defined as  $y(k) = cx_1(k)$ , where  $x_1(k)$  is the output of the last unit delay in the chain. Such a system structure is the popular controller canonical form for LTI systems, which can be obtained from any LTI system model by a simple linear transformation. However, with  $r = n$ , a model  $(A_n, B_n, C_n)$  of increased complexity can be considered, which is the lower Hessenberg form. This can be described by the system matrices  $A_n = [a_{ij}]$ ,  $i, j = 1$  to  $n$ , where  $a_{ij} = 0 \forall i = 1$  to  $(n - 2)$ ,  $j = (i + 2)$  to  $n$ ,  $B_n = [0_{n-1} \ b]^T$  and  $C_n = [c \ 0_{n-1}]$ . Below is the general structure of the system matrix  $A_n$ :

$$A_n = \begin{bmatrix} & a_{11} & a_{12} & 0 & 0 & & & 0 & 0 \\ & a_{21} & a_{22} & a_{23} & 0 & & \dots & 0 & 0 \\ & & \vdots & & & & \ddots & & \vdots \\ a_{(n-2)1} & a_{(n-2)2} & a_{(n-2)3} & a_{(n-2)4} & & & a_{(n-2)(n-1)} & & 0 \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & a_{(n-1)4} & \dots & & a_{(n-1)(n-1)} & & a_{(n-1)n} \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & & & a_{n(n-1)} & & a_{nn} \end{bmatrix}$$

Of course,  $y(k)$ ,  $u(k)$  and  $f(k)$  are all scalar functions and the structure ensures that  $y(k)$  is of relative degree  $r = n$  as per the definition given in Section 2.

#### 4.1. Finite-time stability of all states

Let us consider the system

$$x(k + 1) = A_n x(k) + B_n (u(k) + f(k)) \quad y(k + 1) = C_n x(k) \tag{36}$$

with  $f(k) = 0$ . Assuming this nominal system reaches sliding mode, the following proposition can be made.

**Theorem 4.** *If the output of the system in Eq. (36) is of relative degree  $r = n$ , then  $x_1(k) = x_2(k) = \dots = x_n(k) = 0 \forall k \geq K$ , where  $K$  is the time step at which the output  $y(k)$  starts sliding, i.e.  $y(k) = 0 \forall k \geq K$ .*

*Proof.* During sliding,  $y(k) = cx_1(k) = 0 \forall k \geq K$  implying  $x_1(k) = 0 \forall k \geq K$  since  $|c| \in (0, \infty)$ . Now, obviously  $y(k + 1) = 0 \forall k \geq K$ . This means

$$0 = ca_{11}x_1(k) + ca_{12}x_2(k) \quad \forall k \geq K \tag{37}$$

implying  $x_2(k) = 0 \forall k \geq K$  as  $x_1(k) = 0 \forall k \geq K$  and  $|a_{11}|, |a_{12}| \in (0, \infty)$  as per the system structure.

Similarly,  $y(k + 2) = y(k + 3) = \dots = y(k + n - 1) = 0 \forall k \geq K$  and proceeding in the same line of argument, it can be shown that  $x_3(k) = x_4(k) = \dots = x_n(k) = 0 \forall k \geq K$ . This implies that every state hits zero in finite time, which is the same as the time instant when the output hits zero, and stays there for all future times.

It is obvious that in the presence of uncertainty  $f(k)$ , the states will not reach zero but remain inside some ultimate band  $\forall k \geq K$ .

## 4.2. Improved robustness of the system

With relative degree of the output equal to the order of the system, better robustness can be obtained when compared to usual outputs of relative degree one, by satisfying certain sufficient conditions. The robustness is measured by the width of the ultimate band of the output or the sliding variable. For this, systems with outputs of relative degree two and three are first discussed and then the result is generalized for arbitrary relative degree outputs.

For a relative degree one output of an  $n$ -order system in Eq. (36),  $C_n B_n = b$  if the sliding surface is linear, i.e.  $C_n = [c \ c_2 \dots 1]$ . Hence, the control can always be computed from Utkin's reaching law [6]

$$y(k+1) = d_1(k) \quad (38)$$

with  $|d_1(k)| \leq d_{1m} = C_n B_n f_m = b f_m$ . This gives the control as

$$u(k) = -(C_n B_n)^{-1} C_n A_n x(k) \quad (39)$$

devoid of any uncertain terms, for any system dimension  $n$ .

### 4.2.1. Relative degree two outputs

With system order  $n = 2$ , the LTI system becomes

$$x(k+1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ b \end{bmatrix} (u(k) + f(k)) \quad (40)$$

The output

$$y(k) = c x_1(k) = C_2 x(k) \quad (41)$$

is clearly of relative degree two, since  $C_2 B_2 = 0$  and  $C_2 A_2 B_2 \neq 0$ . Hence, one needs

$$y(k+2) = C_2 A_2^2 x(k) + C_2 A_2 B_2 (u_2(k) + f(k)) \quad (42)$$

to obtain the equivalent control from the extended Utkin's reaching law for relative degree two outputs, which is easily obtained from Eq. (38) as

$$y(k+2) = d_2(k) \quad (43)$$

with  $|d_2(k)| \leq d_{2m} = C_2 A_2 B_2 f_m = c a_{12} b f_m$ . This makes the control

$$u_2(k) = -(C_2 A_2 B_2)^{-1} C_2 A_2^2 x(k) \quad (44)$$

devoid of any uncertain terms.

Obviously, the output  $y(k)$  will be bounded inside the ultimate band  $\delta_2 = d_{2m} \forall k \geq 2$ . For the output with relative degree one, the ultimate band is simply  $\delta_1 = d_{1m} = bf_m$ . From the above, it is straightforward to put down the below theorem.

**Theorem 5.** *For the same LTI system in Eq. (40), the equivalent control will lead to a decrease in the width of the ultimate band with an output of relative degree two compared to an output of relative degree one if  $ca_{12} < 1$ .*

#### 4.2.2. Relative degree three systems

With system order  $n = 3$ , the LTI system becomes

$$x(k+1) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} (u(k) + f(k)) \quad (45)$$

The output

$$y(k) = cx_1(k) = C_3x(k) \quad (46)$$

is clearly of relative degree three, since  $C_3B_3 = C_3A_3B_3 = 0$  and  $C_3A_3^2B_3 \neq 0$ . Hence, one needs

$$y(k+3) = C_3A_3^3x(k) + C_3A_3^2B_3(u_3(k) + f(k)) \quad (47)$$

to obtain the control from the extended Utkin's reaching law for relative degree three outputs. This is easily obtained from Eq. (38) as

$$y(k+3) = d_3(k) \quad (48)$$

with  $|d_3(k)| \leq d_{3m} = C_3A_3^2B_3f_m = ca_{12}a_{23}bf_m$ . This makes the control

$$u_3(k) = -(C_3A_3^2B_3)^{-1}C_3A_3^3x(k) \quad (49)$$

devoid of any uncertain terms.

Obviously, the output  $y(k)$  will be bounded inside the ultimate band  $\delta_3 = d_{3m} \forall k \geq 3$ . For the output with relative degree one, the ultimate band is simply  $\delta_1 = d_{1m} = bf_m$ . From the above, it is straightforward to put down the below theorem.

**Theorem 6.** *For the same LTI system in Eq. (40), the equivalent control will lead to a decrease in the width of the ultimate band with an output of relative degree three compared to an output of relative degree one if  $ca_{12}a_{23} < 1$ .*

#### 4.2.3. Systems with outputs of arbitrary relative degree

With relative degree of the output equal to the order of the system for an arbitrary  $r = n$ , the system is as given in Eq. (36) and  $y(k+r)$  needs to be calculated from the output equation

$$y(k) = cx_1(k) = C_r x(k) \tag{50}$$

In the same way as in previous subsections, the control devoid of any uncertainty can be derived as

$$u_r(k) = -(C_r A_r^{r-1} B_r)^{-1} C_r A_r^r x(k) \tag{51}$$

from the extended Utkin's reaching law

$$y(k+r) = d_r(k) \tag{52}$$

where  $|d_r(k)| \leq d_{rm} = c \prod_{i=2}^r a(i-1) b f_m$ .

Obviously, the output will be bounded inside an ultimate band  $\delta_r = d_{rm} \forall k \geq r$ . From the above, it is straightforward to put down the following theorem.

**Theorem 7.** For the same LTI system in Eq. (36), the equivalent control will lead to a decrease in the width of the ultimate band with an output of relative degree  $r = n$  compared to an output of relative degree one if  $c \prod_{i=2}^r a(i-1) < 1$ .

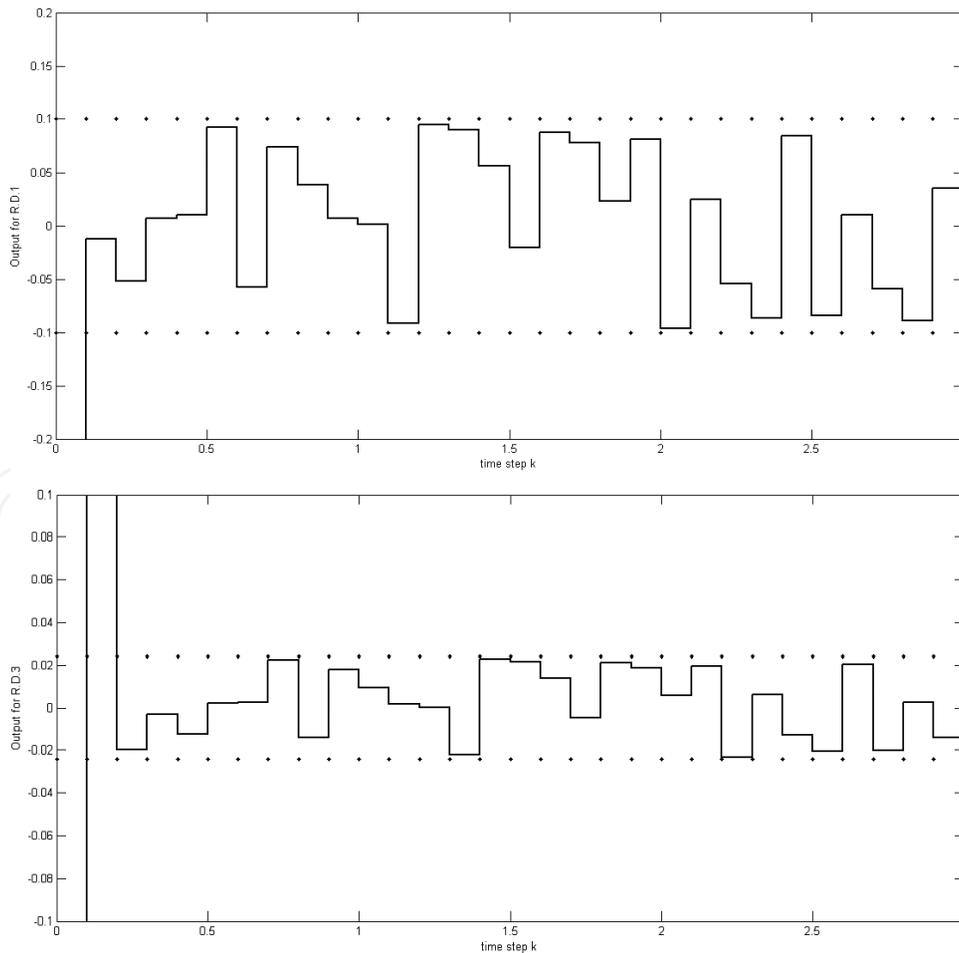
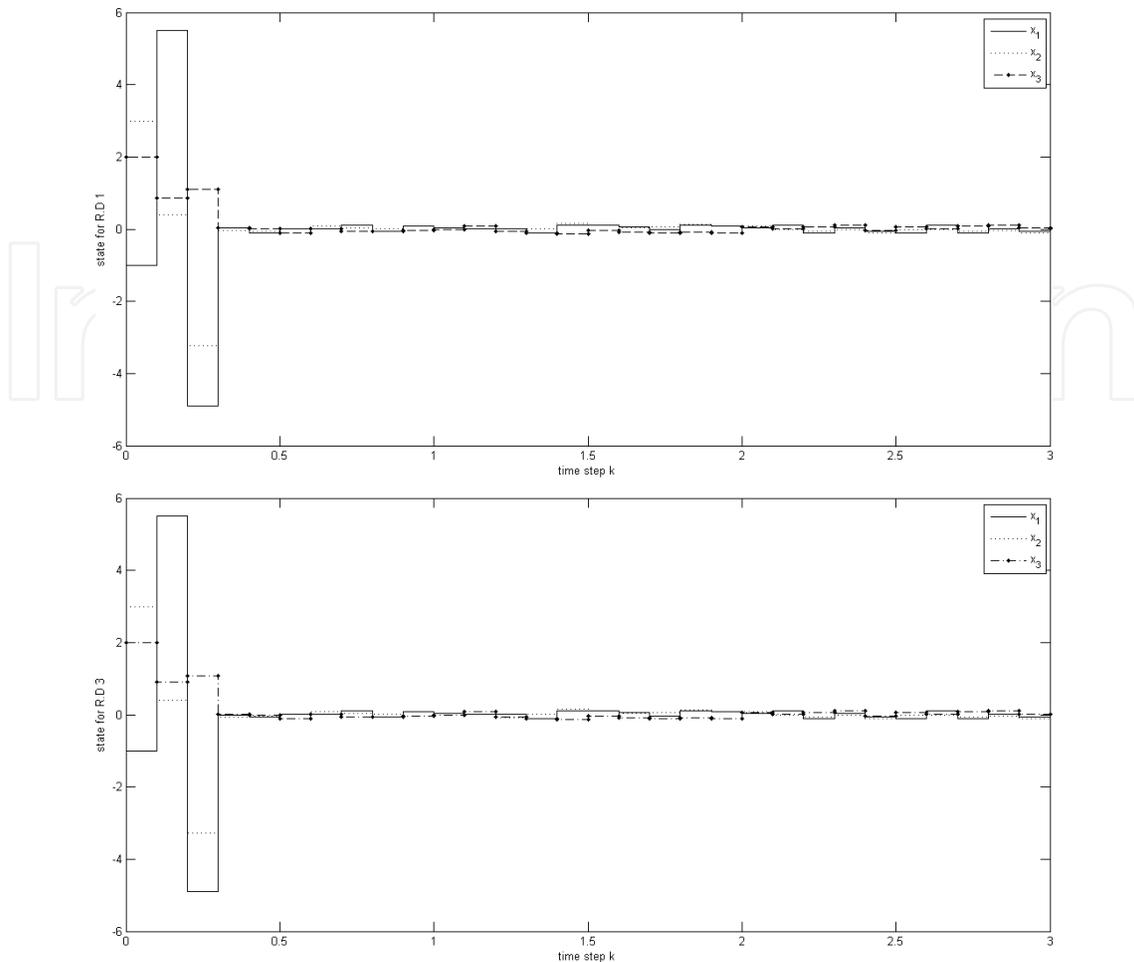


Figure 5. Comparing robustness of outputs with relative degree one and relative degree three.



**Figure 6.** State dynamics for relative degree one and relative degree three with disturbance.

**Remark 3.** In case of outputs with relative degree more than one, the scaling  $c$  can be dropped and simply  $y(k) = x_1(k)$ . Hence, the robustness entirely depends on the system parameters. It is thus possible that for some systems for which the parameters do not satisfy the condition in Theorem 7, the robustness worsens with choice of relative degree  $r = n$  with Utkin's equivalent control law.

### 4.3. Simulation result

A third-order discrete-time LTI system is considered with output of relative degree three for simulation. For comparison, the results for the output designed to be of relative degree one are also shown. It can be readily observed that with design parameters kept same for both, the system with relative degree three output shows better robustness in presence of disturbance and also achieves finite-time stability of all states in the absence of disturbance.

Let the system be

$$x(k+1) = \begin{bmatrix} -1 & 1.5 & 0 \\ -0.5 & 0.5 & -0.8 \\ -3 & 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (u(k) + f(k)) \quad (53)$$

where  $f(k)$  is a random number bounded by  $\pm 0.1$ . The initial states are assumed to be  $[-1 \ 3 \ -2]^T$ .

An output of relative degree one is designed as

$$y_1(k) = [0.2 \ -0.625 \ -1]x(k) \tag{54}$$

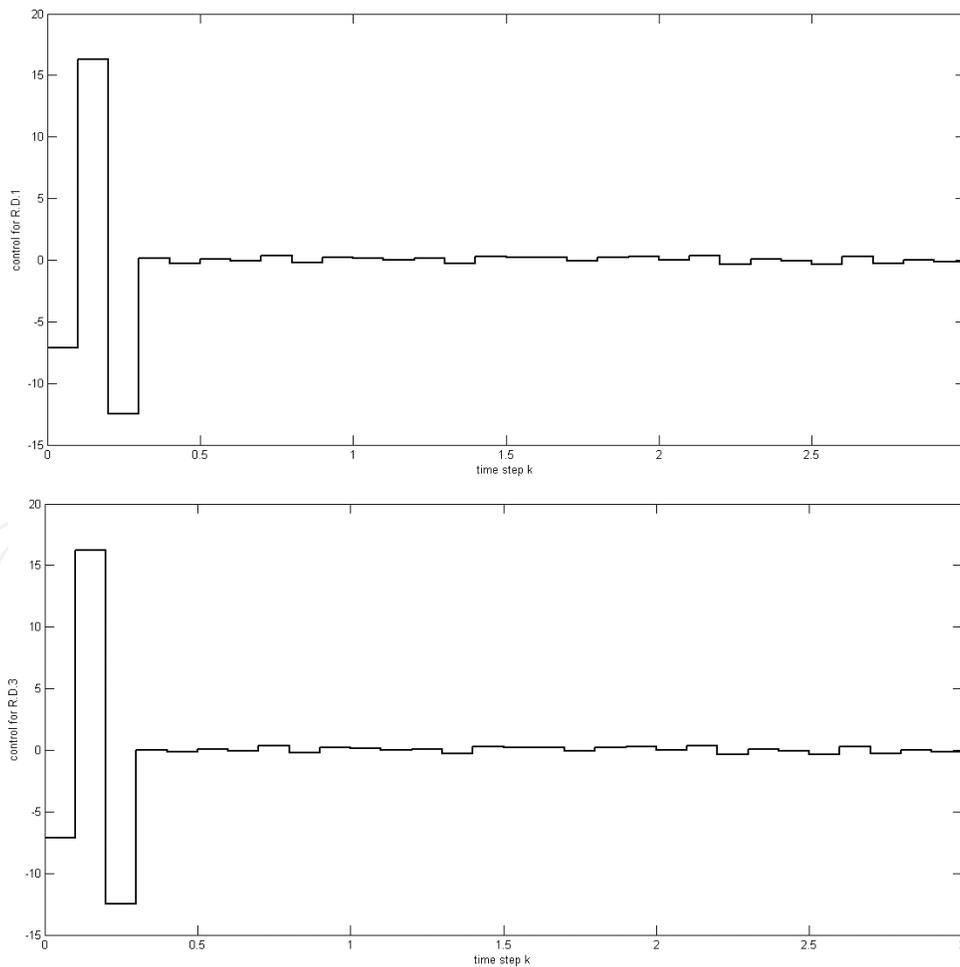
which makes the poles of the reduced-order system in the sliding mode as 0.1 and  $-0.1$ , which are sufficiently nice pole placement to obtain asymptotic stability of the states fast enough.

The output of relative degree three is designed as

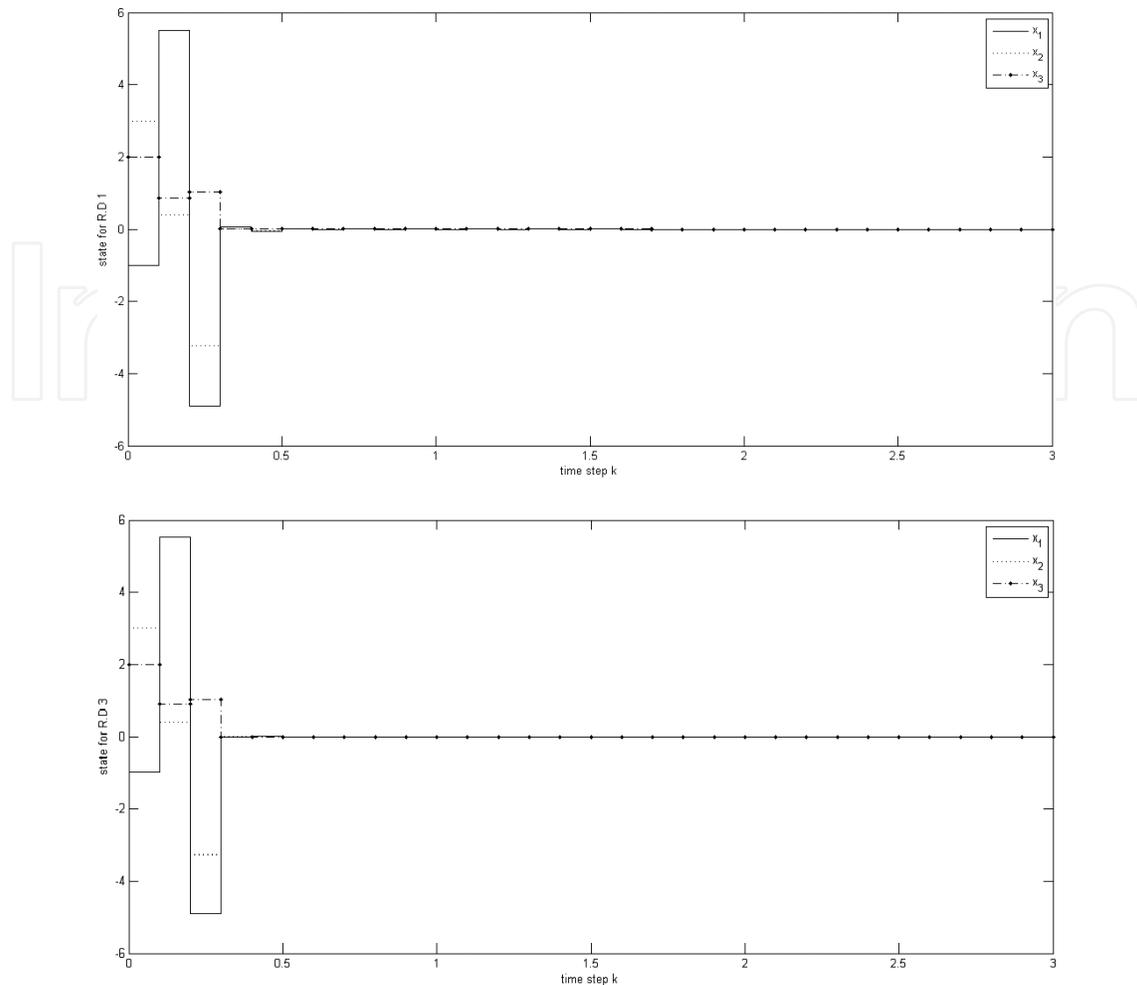
$$y_3(k) = [0.2 \ 0 \ 0]x(k) \tag{55}$$

by keeping the first entry of the output matrix same as in Eq. (54). The ultimate bands calculated for the relative degree one and three outputs are  $\delta_1 = 0.1$  and  $\delta_3 = 0.024$ , respectively. The zoomed views of the outputs for the two cases are shown in **Figure 5**, with the ultimate band superimposed on each plot.

**Figures 6** and **7** show the states and the control input for the two cases when the system is affected by the disturbance  $f(k)$ . Not much visible difference can be found between the



**Figure 7.** Control input for relative degree one and relative degree three with disturbance.



**Figure 8.** State dynamics for relative degree one and relative degree three without disturbance.

simulations of the states in **Figure 6** because of the presence of disturbance. However, in **Figure 8**, it is clear that the states of the system in absence of disturbance become finite-time stable for relative degree three output, whereas for relative degree one output, only asymptotic stability is achieved.

## 5. Conclusion

In this chapter, an important advancement in the direction of discrete-time sliding mode control is presented. As opposed to the traditional consideration of outputs of relative degree one, it is shown that with higher relative degree outputs, improved robustness and performance of the system can be guaranteed under certain conditions. New reaching laws are proposed for these higher relative degree outputs, which are extensions of existing reaching laws proposed in Refs. [2, 3, 17] for relative degree one outputs. These reaching laws are analysed to find out conditions for increased robustness of the system. Along with such increased robustness attributed to a reduction in the ultimate band of the sliding variable or

output, the system states are also proved to be finite-time stable in absence of disturbance. In presence of disturbance, they are finite time ultimately bounded. Moreover, this finite time step is same as the time step at which the output hits the sliding surface.

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