

# We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

5,500

Open access books available

135,000

International authors and editors

165M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index  
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?  
Contact [book.department@intechopen.com](mailto:book.department@intechopen.com)

Numbers displayed above are based on latest data collected.  
For more information visit [www.intechopen.com](http://www.intechopen.com)



# Optimal Sliding Mode Control for a Class of Uncertain Nonlinear Systems Based on Feedback Linearization

Hai-Ping Pang and Qing Yang  
*Qingdao University of Science and Technology  
China*

## 1. Introduction

Optimal control is one of the most important branches in modern control theory, and linear quadratic regulator (LQR) has been well used and developed in linear control systems. However, there would be several problems in employing LQR to uncertain nonlinear systems. The optimal LQR problem for nonlinear systems often leads to solving a nonlinear two-point boundary-value (TPBV) problem (Tang et al. 2008; Pang et al. 2009) and an analytical solution generally does not exist except some simplest cases (Tang & Gao, 2005). Additionally, the optimal controller design is usually based on the precise mathematical models. While if the controlled system is subject to some uncertainties, such as parameter variations, unmodeled dynamics and external disturbances, the performance criterion which is optimized based on the nominal system would deviate from the optimal value, even the system becomes unstable (Gao & Hung, 1993 ; Pang & Wang, 2009).

The main control strategies to deal with the optimal control problems of nonlinear systems are as follows. First, obtain approximate solution of optimal control problems by iteration or recursion, such as successive approximate approach (Tang, 2005), SDRE (Shamma & Cloutier, 2001), ASRE (Cimen & Banks, 2004). These methods could have direct results but usually complex and difficult to be realized. Second, transform the nonlinear system into a linear one by the approximate linearization (i.e. Jacobian linearization), then optimal control can be realized easily based on the transformed system. But the main problem of this method is that the transformation is only applicable to those systems with less nonlinearity and operating in a very small neighborhood of equilibrium points. Third, transform the nonlinear system into a linear one by the exact linearization technique (Mokhtari et al. 2006; Pang & Chen, 2009). This differs entirely from approximate linearization in that the approximate linearization is often done simply by neglecting any term of order higher than 1 in the dynamics while exact linearization is achieved by exact state transformations and feedback.

As a precise and robust algorithm, the sliding mode control (SMC) (Yang & Özgüner, 1997; Choi et al. 1993; Choi et al. 1994) has attracted a great deal of attention to the uncertain nonlinear system control problems. Its outstanding advantage is that the sliding motion exhibits complete robustness to system uncertainties. In this chapter, combining LQR and SMC, the design of global robust optimal sliding mode controller (GROSMC) is concerned. Firstly, the GROSMC is designed for a class of uncertain linear systems. And then, a class of

affine nonlinear systems is considered. The exact linearization technique is adopted to transform the nonlinear system into an equivalent linear one and a GROSMC is designed based on the transformed system. Lastly, the global robust optimal sliding mode tracking controller is studied for a class of uncertain affine nonlinear systems. Simulation results illustrate the effectiveness of the proposed methods.

## 2. Optimal sliding mode control for uncertain linear system

In this section, the problem of robustify LQR for a class of uncertain linear systems is considered. An optimal controller is designed for the nominal system and an integral sliding surface (Lee, 2006; Laghrouche et al. 2007) is constructed. The ideal sliding motion can minimize a given quadratic performance index, and the reaching phase, which is inherent in conventional sliding mode control, is completely eliminated (Basin et al. 2007). Then the sliding mode control law is synthesized to guarantee the reachability of the specified sliding surface. The system dynamics is global robust to uncertainties which satisfy matching conditions. A GROSMC is realized. To verify the effectiveness of the proposed scheme, a robust optimal sliding mode controller is developed for rotor position control of an electrical servo drive system.

### 2.1 System description and problem formulation

Consider an uncertain linear system described by

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + \delta(x, t) \quad (1)$$

where  $x(t) \in R^n$  and  $u(t) \in R^m$  are the state and the control vectors, respectively.  $\Delta A$  and  $\Delta B$  are unknown time-varying matrices representing system parameter uncertainties.  $\delta(x, t)$  is an uncertain extraneous disturbance and/or unknown nonlinearity of the system.

**Assumption 1.** The pair  $(A, B)$  is controllable and  $\text{rank}(B) = m$ .

**Assumption 2.**  $\Delta A$ ,  $\Delta B$  and  $\delta(x, t)$  are continuously differentiable in  $x$ , and piecewise continuous in  $t$ .

**Assumption 3.** There exist unknown continuous functions of appropriate dimension  $\tilde{\Delta A}$ ,  $\tilde{\Delta B}$  and  $\tilde{\delta}(x, t)$ , such that

$$\Delta A = B\tilde{\Delta A}, \quad \Delta B = B\tilde{\Delta B}, \quad \delta(x, t) = B\tilde{\delta}(x, t).$$

These conditions are the so-called matching conditions.

From these assumptions, the state equation of the uncertain dynamic system (1) can be rewritten as

$$\dot{x}(t) = Ax(t) + Bu(t) + B\tilde{\delta}(x, t), \quad (2)$$

where

**Assumption 4.** There exist unknown positive constants  $\gamma_0$  and  $\gamma_1$  such that

$$\|\tilde{\delta}(x, t)\| \leq \gamma_0 + \gamma_1 \|x(t)\|.$$

where  $\|\bullet\|$  denotes the Euclidean norm.

By setting the uncertainty to zero, we can obtain the dynamic equation of the original system of (1), as

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{3}$$

For the nominal system (3), let's define a quadratic performance index as follows:

$$J_0 = \frac{1}{2} \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt, \tag{4}$$

where  $Q \in R^{n \times n}$  is a semi-positive definite matrix, the weighting function of states;  $R \in R^{m \times m}$  is a positive definite matrix, the weighting function of control variables.

According to optimal control theory and considering Assumption1, there exists an optimal feedback control law that minimizes the index (4). The optimal control law can be written as

$$u^*(t) = -R^{-1}B^T Px(t), \tag{5}$$

where  $P \in R^{n \times n}$  is a positive definite matrix solution of Riccati matrix equation:

$$-PA - A^T P + PBR^{-1}B^T P - Q = 0. \tag{6}$$

So the dynamic equation of the closed-loop system is

$$\dot{x}(t) = (A - BR^{-1}B^T P)x(t). \tag{7}$$

Obviously, according to optimal control theory, the closed-loop system is asymptotically stable. However, when the system is subjected to uncertainties such as external disturbances and parameter variations, the optimal system behavior could be deteriorated, even unstable. In the next part, we will utilize sliding mode control strategy to robustify the optimal control law.

## 2.2 Design of optimal sliding mode controller

### 2.2.1 Design of optimal sliding mode surface

Considering the uncertain system (2), we chose the integral sliding surface as follows:

$$s(x, t) = G[x(t) - x(0)] - G \int_0^t (A - BR^{-1}B^T P)x(\tau) d\tau = 0. \tag{8}$$

where  $G \in R^{m \times n}$ , which satisfies that  $GB$  is nonsingular,  $x(0)$  is the initial state vector. In sliding mode, we have  $s(x, t) = 0$  and  $\dot{s}(x, t) = 0$ . Differentiating (8) with respect to  $t$  and considering (1), we obtain

$$\begin{aligned} \dot{s} &= G[(A + \Delta A)x + (B + \Delta B)u + \delta] - G(A - BR^{-1}B^T P)x \\ &= G\Delta Ax + G(B + \Delta B)u + G\delta + GBR^{-1}B^T Px \\ &= G(\Delta Ax + BR^{-1}B^T Px) + G\delta + G(B + \Delta B)u \end{aligned} \tag{9}$$

the equivalent control becomes

$$u_{eq} = -[G(B + \Delta B)]^{-1}[G(\Delta A + BR^{-1}B^T P)x + G\delta]. \tag{10}$$

Substituting (10) into (1) and considering Assumption3, the ideal sliding mode dynamics becomes

$$\begin{aligned}
\dot{x} &= (A + \Delta A)x - (B + \Delta B)[G(B + \Delta B)]^{-1}[G(\Delta A + BR^{-1}B^T P)x + G\delta] + \delta \\
&= (A + B\Delta\tilde{A})x - (B + B\Delta\tilde{B})[G(B + B\Delta\tilde{B})]^{-1}[GB(\Delta\tilde{A}x + \tilde{\delta}) + GBR^{-1}B^T Px] + B\tilde{\delta} \\
&= (A - BR^{-1}B^T P)x
\end{aligned} \tag{11}$$

Comparing equation (11) with equation (7), we can see that they have the same form. So the sliding mode is asymptotically stable. Furthermore, it can be seen from (11) that the sliding mode is robust to uncertainties which satisfying matching conditions. So we call (8) a robust optimal sliding surface.

### 2.2.2 Design of sliding mode control law

To ensure the reachability of sliding mode in finite time, we chose the sliding mode control law as follows:

$$\begin{aligned}
u(t) &= u_c(t) + u_d(t), \\
u_c(t) &= -R^{-1}B^T Px(t), \\
u_d(t) &= -(GB)^{-1}(\eta + \gamma_0 \|GB\| + \gamma_1 \|GB\| \|x(t)\|) \text{sgn}(s).
\end{aligned} \tag{12}$$

Where  $\eta > 0$ ,  $u_c(t)$  is the continuous part, used to stabilize and optimize the nominal system;  $u_d(t)$  is the discontinuous part, which provides complete compensation for uncertainties of system (1). Let's select a quadratic performance index as follows:

$$J(t) = \frac{1}{2} \int_0^\infty [x^T(t)Qx(t) + u_c^T(t)Ru_c(t)]dt. \tag{13}$$

where the meanings of  $Q$  and  $R$  are as the same as that in (4).

**Theorem 1.** Consider uncertain linear system (1) with Assumptions 1-4. Let  $u$  and sliding surface be given by (12) and (8), respectively. The control law (12) can force the system trajectories with arbitrarily given initial conditions to reach the sliding surface in finite time and maintain on it thereafter.

**Proof.** Choosing  $V = (1/2)s^T s$  as a lyapunov function, and differentiating this function with respect to  $t$  and considering Assumptions 1-4, we have

$$\begin{aligned}
\dot{V} &= s^T \dot{s} \\
&= s^T \{G[(A + \Delta A)x + (B + \Delta B)u + \delta] - G(A - BR^{-1}B^T P)x\} \\
&= s^T \{GBu + GB\tilde{\delta} + GBR^{-1}B^T Px\} \\
&= s^T \{-GBR^{-1}B^T Px - (\eta + \gamma_0 \|GB\| + \gamma_1 \|GB\| \|x\|) \text{sgn}(s) + GB\tilde{\delta} + GBR^{-1}B^T Px\} \\
&= s^T \{-(\eta + \gamma_0 \|GB\| + \gamma_1 \|GB\| \|x\|) \text{sgn}(s) + GB\tilde{\delta}\} \\
&= -\eta \|s\|_1 - (\gamma_0 \|GB\| + \gamma_1 \|GB\| \|x\|) \|s\|_1 + s^T GB\tilde{\delta} \\
&\leq -\eta \|s\|_1 - (\gamma_0 \|GB\| + \gamma_1 \|GB\| \|x\|) \|s\|_1 + (\gamma_0 \|GB\| + \gamma_1 \|GB\| \|x\|) \|s\|
\end{aligned}$$

where  $\|\bullet\|_1$  denotes 1-norm. Noting the fact that  $\|s\|_1 \geq \|s\|$ , we get

$$\dot{V} = s^T \dot{s} \leq -\eta \|s\| \tag{14}$$

This implies that the sliding mode control law we have chosen according to (12) could ensure the trajectories which start from arbitrarily given points be driven onto the sliding

surface (8) in finite time and would not leave it thereafter despite uncertainties. The proof is complete.

**Conclusion 1.** The uncertain system (1) with the integral sliding surface (8) and the control law (12) achieves global sliding mode, and the performance index (13) is minimized. So the system designed is global robust and optimal.

### 2.3 Application to electrical servo drive

The speed and position electrical servo drive systems are widely used in engineering systems, such as CNC machines, industrial robots, winding machines and etc. The main properties required for servo systems include high tracking behavior, no overshoot, no oscillation, quick response and good robustness.

In general, with the implementation of field-oriented control, the mechanical equation of an induction motor drive or a permanent synchronous motor drive can be described as

$$J_m \ddot{\theta}(t) + B_m \dot{\theta}(t) + T_d = T_e \tag{15}$$

where  $\theta$  is the rotor position;  $J_m$  is the moment of inertia;  $B_m$  is the damping coefficient;  $T_d$  denotes the external load disturbance, nonlinear friction and unpredicted uncertainties;  $T_e$  represents the electric torque which defined as

$$T_e = K_t i \tag{16}$$

where  $K_t$  is the torque constant and  $i$  is the torque current command.

Define the position tracking error  $e(t) = \theta_d(t) - \theta(t)$ , where  $\theta_d(t)$  denotes the desired position, and let  $x_1(t) = e(t)$ ,  $x_2(t) = \dot{x}_1(t)$ ,  $u = i$ , then the error state equation of an electrical servo drive can be described as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{B_m}{J_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{B_m}{J_m} \end{bmatrix} \dot{\theta}_d + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ddot{\theta}_d + \begin{bmatrix} 0 \\ -\frac{K_t}{J_m} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{J_m} \end{bmatrix} T_d. \tag{17}$$

Supposing the desired position is a step signal, the error state equation can be simplified as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{B_m}{J_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{K_t}{J_m} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{J_m} \end{bmatrix} T_d. \tag{18}$$

The parameters of the servo drive model in the nominal condition with  $T_d = 0\text{Nm}$  are (Lin & Chou, 2003):

$$\begin{aligned} \hat{J} &= 5.77 \times 10^{-2} \text{ Nms}^2, \\ \hat{B} &= 8.8 \times 10^{-3} \text{ Nms/rad}, \\ \hat{K}_t &= 0.667 \text{ Nm/A}. \end{aligned}$$

The initial condition is  $x(0) = [1 \ 0]^T$ . To investigate the effectiveness of the proposed controller, two cases with parameter variations in the electrical servo drive and load torque disturbance are considered here.

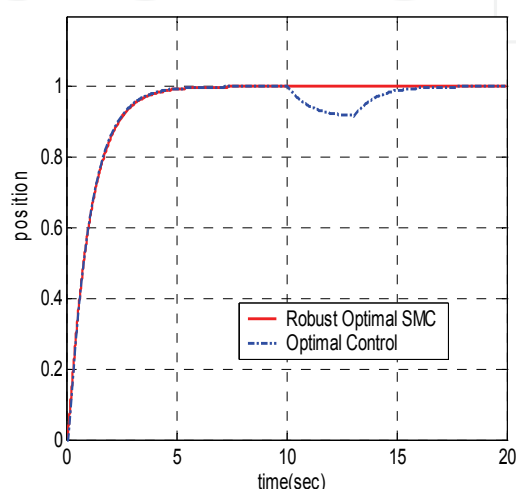
Case 1:  $J_m = \hat{J}_m$ ,  $B_m = \hat{B}_m$ ,  $T_d = 1(t - 10)\text{Nm} - 1(t - 13)\text{Nm}$ .

Case 2:  $J_m = 3\hat{J}_m$ ,  $B_m = \hat{B}_m$ ,  $T_d = 0$ .

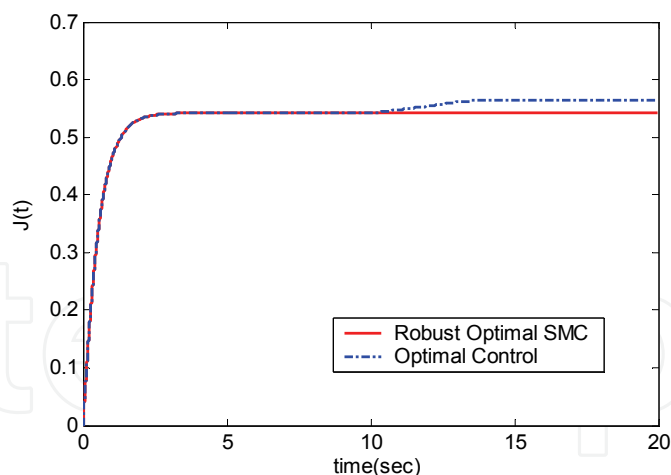
The optimal controller and the optimal robust SMC are designed, respectively, for both cases. The optimal controller is based on the nominal system with a quadratic performance index (4). Here

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1$$

In Case 1, the simulation results by different controllers are shown in Fig. 1. It is seen that when there is no disturbance ( $t < 10s$ ), both systems have almost the same performance.



(a) Position responses



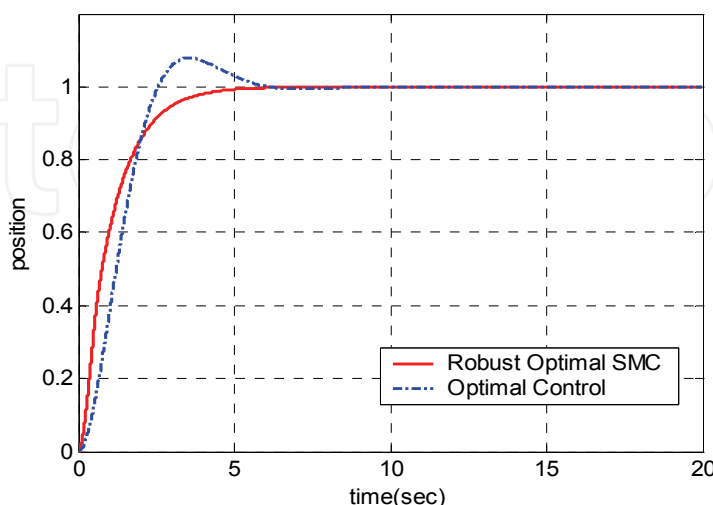
(b) Performance indexes

Fig. 1. Simulation results in Case 1

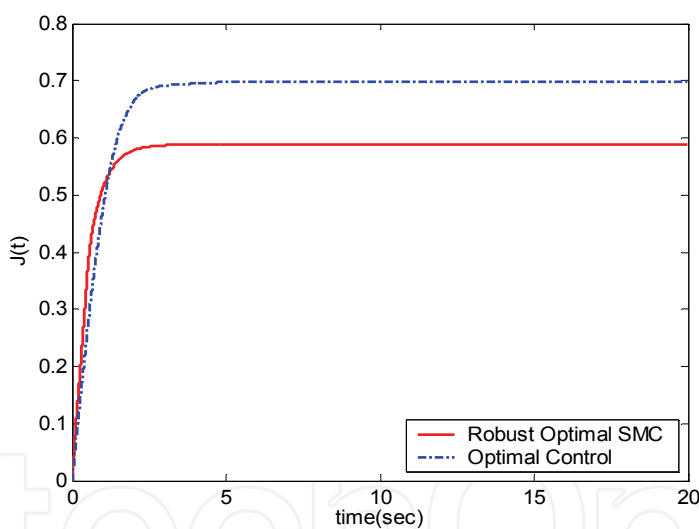
But when a load torque disturbance occurs at  $t = (10 \sim 13)s$ , the position trajectory of optimal control system deviates from the desired value, nevertheless the position trajectory of the robust optimal SMC system is almost not affected.

In Case 2, the simulation results by different controllers are given in Fig.2. It is seen that the robust optimal SMC system is insensitive to the parameter uncertainty, the position trajectory is almost as the same as that of the nominal system. However, the optimal control

system is affected by the parameter variation. Compared with the nominal system, the position trajectory is different, bigger overshoot and the relative stability degrades. In summary, the robust optimal SMC system owns the optimal performance and global robustness to uncertainties.



(a) Position responses



(b) Performance indexes

Fig. 2. Simulation results in Case 2

## 2.4 Conclusion

In this section, the integral sliding mode control strategy is applied to robustifying the optimal controller. An optimal robust sliding surface is designed so that the initial condition is on the surface and reaching phase is eliminated. The system is global robust to uncertainties which satisfy matching conditions and the sliding motion minimizes the given quadratic performance index. This method has been adopted to control the rotor position of an electrical servo drive. Simulation results show that the robust optimal SMCs are superior to optimal LQR controllers in the robustness to parameter variations and external disturbances.



### 3. Optimal sliding mode control for uncertain nonlinear system

In the section above, the robust optimal SMC design problem for a class of uncertain linear systems is studied. However, nearly all practical systems contain nonlinearities, there would exist some difficulties if optimal control is applied to handling nonlinear problems (Chiou & Huang, 2005; Ho, 2007, Cimen & Banks, 2004; Tang et al., 2007). In this section, the global robust optimal sliding mode controller (GROSMC) is designed based on feedback linearization for a class of MIMO uncertain nonlinear system.

#### 3.1 Problem formulation

Consider an uncertain affine nonlinear system in the form of

$$\begin{aligned}\dot{x} &= f(x) + g(x)u + d(t, x), \\ y &= H(x),\end{aligned}\tag{19}$$

where  $x \in R^n$  is the state,  $u \in R^m$  is the control input, and  $f(x)$  and  $g(x)$  are sufficiently smooth vector fields on a domain  $D \subset R^n$ . Moreover, state vector  $x$  is assumed available,  $H(x)$  is a measured sufficiently smooth output function and  $H(x) = (h_1, \dots, h_m)^T$ .  $d(t, x)$  is an unknown function vector, which represents the system uncertainties, including system parameter variations, unmodeled dynamics and external disturbances.

**Assumption 5.** There exists an unknown continuous function vector  $\delta(t, x)$  such that  $d(t, x)$  can be written as

$$d(t, x) = g(x)\delta(t, x).$$

This is called matching condition.

**Assumption 6.** There exist positive constants  $\gamma_0$  and  $\gamma_1$ , such that

$$\|\delta(t, x)\| \leq \gamma_0 + \gamma_1 \|x\|$$

where the notation  $\|\cdot\|$  denotes the usual Euclidean norm.

By setting all the uncertainties to zero, the nominal system of the uncertain system (19) can be described as

$$\begin{aligned}\dot{x} &= f(x) + g(x)u, \\ y &= H(x).\end{aligned}\tag{20}$$

The objective of this paper is to synthesize a robust sliding mode optimal controller so that the uncertain affine nonlinear system has not only the optimal performance of the nominal system but also robustness to the system uncertainties. However, the nominal system is nonlinear. To avoid the nonlinear TPBV problem and approximate linearization problem, we adopt the feedback linearization to transform the uncertain nonlinear system (19) into an equivalent linear one and an optimal controller is designed on it, then a GROSMC is proposed.

#### 3.2 Feedback linearization

Feedback linearization is an important approach to nonlinear control design. The central idea of this approach is to find a state transformation  $z = T(x)$  and an input transformation

$u = u(x, v)$  so that the nonlinear system dynamics is transformed into an equivalent linear time-variant dynamics, in the familiar form  $\dot{z} = Az + Bv$ , then linear control techniques can be applied.

Assume that system (20) has the vector relative degree  $\{r_1, \dots, r_m\}$  and  $r_1 + \dots + r_m = n$ . According to relative degree definition, we have

$$\begin{aligned} y_i^{(k)} &= L_f^k h_i, \quad 0 \leq k \leq r_i - 1 \\ y_i^{(r_i)} &= L_f^{r_i} h_i + \sum_{j=1}^m g_j (L_f^{r_i-1} h_i) u_j, \end{aligned} \tag{21}$$

and the decoupled matrix

$$E(x) = (e_{ij})_{m \times m} = \begin{bmatrix} L_{g_1} (L_f^{r_1-1} h_1) & \dots & L_{g_m} (L_f^{r_1-1} h_1) \\ \vdots & \dots & \vdots \\ L_{g_1} (L_f^{r_m-1} h_m) & \dots & L_{g_m} (L_f^{r_m-1} h_m) \end{bmatrix}$$

is nonsingular in some domain  $\forall x \in X_0$ .

Choose state and input transformations as follows:

$$z_i^j = T_i^j(x) = L_f^j h_i, \quad i = 1, \dots, m; j = 0, 1, \dots, r_i - 1 \tag{22}$$

$$u = E^{-1}(x)[v - K(x)], \tag{23}$$

where  $K(x) = (L_f^{r_1} h_1, \dots, L_f^{r_m} h_m)^T$ ,  $v$  is an equivalent input to be designed later. The uncertain nonlinear system (19) can be transformed into  $m$  subsystems; each one is in the form of

$$\begin{bmatrix} \dot{z}_i^0 \\ \dot{z}_i^1 \\ \vdots \\ \dot{z}_i^{r_i-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} z_i^0 \\ z_i^1 \\ \vdots \\ z_i^{r_i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v_i + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L_d L_f^{r_i-1} h_i \end{bmatrix}. \tag{24}$$

So system (19) can be transformed into the following equivalent model of a simple linear form:

$$\dot{z}(t) = Az(t) + Bv(t) + \omega(t, z), \tag{25}$$

where  $z \in R^n$ ,  $v \in R^m$  are new state vector and input, respectively.  $A \in R^{n \times n}$  and  $B \in R^{n \times m}$  are constant matrixes, and  $(A, B)$  are controllable.  $\omega(t, z) \in R^n$  is the uncertainties of the equivalent linear system. As we can see,  $\omega(t, z)$  also satisfies the matching condition, in other words there exists an unknown continuous vector function  $\tilde{\omega}(t, z)$  such that  $\omega(t, z) = B\tilde{\omega}(t, z)$ .

### 3.3 Design of GROSMC

#### 3.3.1 Optimal control for nominal system

The nominal system of (25) is

$$\dot{z}(t) = Az(t) + Bv(t). \quad (26)$$

For (26), let  $v = v_0$  and  $v_0$  can minimize a quadratic performance index as follows:

$$J = \frac{1}{2} \int_0^{\infty} [z^T(t)Qz(t) + v_0^T(t)Rv_0(t)]dt \quad (27)$$

where  $Q \in R^{n \times n}$  is a symmetric positive definite matrix,  $R \in R^{m \times m}$  is a positive definite matrix. According to optimal control theory, the optimal feedback control law can be described as

$$v_0(t) = -R^{-1}B^T Pz(t) \quad (28)$$

with  $P$  the solution of the matrix Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0. \quad (29)$$

So the closed-loop dynamics is

$$\dot{z}(t) = (A - BR^{-1}B^T P)z(t). \quad (30)$$

The closed-loop system is asymptotically stable.

The solution to equation (30) is the optimal trajectory  $z^*(t)$  of the nominal system with optimal control law (28). However, if the control law (28) is applied to uncertain system (25), the system state trajectory will deviate from the optimal trajectory and even the system becomes unstable. Next we will introduce integral sliding mode control technique to robustify the optimal control law, to achieve the goal that the state trajectory of uncertain system (25) is the same as that of the optimal trajectory of the nominal system (26).

#### 3.3.2 The optimal sliding surface

Considering the uncertain system (25) and the optimal control law (28), we define an integral sliding surface in the form of

$$s(t) = G[z(t) - z(0)] - G \int_0^t (A - BR^{-1}B^T P)z(\tau) d\tau \quad (31)$$

where  $G \in R^{m \times n}$ , which satisfies that  $GB$  is nonsingular,  $z(0)$  is the initial state vector. Differentiating (31) with respect to  $t$  and considering (25), we obtain

$$\begin{aligned} \dot{s}(t) &= G\dot{z}(t) - G(A - BR^{-1}B^T P)z(t) \\ &= G[Az(t) + Bv(t) + \omega(t, z)] - G(A - BR^{-1}B^T P)z(t) \\ &= GBv(t) + GBR^{-1}B^T Pz(t) + G\omega(t, z) \end{aligned} \quad (32)$$

Let  $\dot{s}(t) = 0$ , the equivalent control becomes

$$v_{eq}(t) = -(GB)^{-1} [GBR^{-1}B^T Pz(t) + G\omega(t, z)] \quad (33)$$

Substituting (33) into (25), the sliding mode dynamics becomes

$$\begin{aligned} \dot{z} &= Az - B(GB)^{-1}(GBR^{-1}B^T Pz + G\omega) + \omega \\ &= Az - BR^{-1}B^T Pz - B(GB)^{-1}G\omega + \omega \\ &= Az - BR^{-1}B^T Pz - B(GB)^{-1}GB\tilde{\omega} + B\tilde{\omega} \\ &= (A - BR^{-1}B^T P)z \end{aligned} \quad (34)$$

Comparing (34) with (30), we can see that the sliding mode of uncertain linear system (25) is the same as optimal dynamics of (26), thus the sliding mode is also asymptotically stable, and the sliding motion guarantees the controlled system global robustness to the uncertainties which satisfy the matching condition. We call (31) a global robust optimal sliding surface. Substituting state transformation  $z = T(x)$  into (31), we can get the optimal switching function  $s(x, t)$  in the  $x$ -coordinates.

### 3.3.3 The control law

After designing the optimal sliding surface, the next step is to select a control law to ensure the reachability of sliding mode in finite time.

Differentiating  $s(x, t)$  with respect to  $t$  and considering system (20), we have

$$\dot{s} = \frac{\partial s}{\partial x} \dot{x} + \frac{\partial s}{\partial t} = \frac{\partial s}{\partial x} (f(x) + g(x)u) + \frac{\partial s}{\partial t}. \quad (35)$$

Let  $\dot{s} = 0$ , the equivalent control of nonlinear nominal system (20) is obtained

$$u_{eq}(t) = - \left[ \frac{\partial s}{\partial x} g(x) \right]^{-1} \left[ \frac{\partial s}{\partial x} f(x) + \frac{\partial s}{\partial t} \right]. \quad (36)$$

Considering equation (23), we have  $u_{eq} = E^{-1}(x)[v_0 - K(x)]$ .

Now, we select the control law in the form of

$$\begin{aligned} u(t) &= u_{con}(t) + u_{dis}(t), \\ u_{con}(t) &= - \left[ \frac{\partial s}{\partial x} g(x) \right]^{-1} \left[ \frac{\partial s}{\partial x} f(x) + \frac{\partial s}{\partial t} \right], \\ u_{dis}(t) &= - \left[ \frac{\partial s}{\partial x} g(x) \right]^{-1} (\eta + (\gamma_0 + \gamma_1 \|x\|) \left\| \frac{\partial s}{\partial x} g(x) \right\|) \text{sgn}(s), \end{aligned} \quad (37)$$

where  $\text{sgn}(s) = [\text{sgn}(s_1) \text{sgn}(s_2) \dots \text{sgn}(s_m)]^T$  and  $\eta > 0$ .  $u_{con}(t)$  and  $u_{dis}(t)$  denote continuous part and discontinuous part of  $u(t)$ , respectively.

The continuous part  $u_{con}(t)$ , which is equal to the equivalent control of nominal system (20), is used to stabilize and optimize the nominal system. The discontinuous part  $u_{dis}(t)$  provides the complete compensation of uncertainties for the uncertain system (19).

**Theorem 2.** Consider uncertain affine nonlinear system (19) with Assumptions 5-6. Let  $u$  and sliding surface be given by (37) and (31), respectively. The control law can force the system trajectories to reach the sliding surface in finite time and maintain on it thereafter.

**Proof.** Utilizing  $V = (1/2)s^T s$  as a Lyapunov function candidate, and taking the Assumption 5 and Assumption 6, we have

$$\begin{aligned}
 \dot{V} &= s^T \dot{s} = s^T \left( \frac{\partial s}{\partial x} (f + gu + d) + \frac{\partial s}{\partial t} \right) = \\
 &= s^T \left\{ \frac{\partial s}{\partial x} f - \left[ \frac{\partial s}{\partial x} f + \frac{\partial s}{\partial t} + \left( \eta + (\gamma_0 + \gamma_1 \|x\|) \left\| \frac{\partial s}{\partial x} g \right\| \right) \text{sgn}(s) \right] + \frac{\partial s}{\partial x} d + \frac{\partial s}{\partial t} \right\} = \\
 &= s^T \left\{ - \left[ \eta + (\gamma_0 + \gamma_1 \|x\|) \left\| \frac{\partial s}{\partial x} g \right\| \right] \text{sgn}(s) \right\} + s^T \frac{\partial s}{\partial x} d = -\eta \|s\|_1 - (\gamma_0 + \gamma_1 \|x\|) \left\| \frac{\partial s}{\partial x} g \right\| \|s\|_1 + s^T \frac{\partial s}{\partial x} g \delta \quad (38) \\
 &\leq -\eta \|s\|_1 - (\gamma_0 + \gamma_1 \|x\|) \left\| \frac{\partial s}{\partial x} g \right\| \|s\|_1 + \left\| \frac{\partial s}{\partial x} g \right\| \|s\| \|\delta\| \leq \\
 &\leq -\eta \|s\|_1 - (\gamma_0 + \gamma_1 \|x\|) \left\| \frac{\partial s}{\partial x} g \right\| \|s\|_1 + (\gamma_0 + \gamma_1 \|x\|) \left\| \frac{\partial s}{\partial x} g \right\| \|s\|
 \end{aligned}$$

where  $\|\cdot\|_1$  denotes the 1-norm. Noting the fact that  $\|s\|_1 \geq \|s\|$ , we get

$$\dot{V} = s^T \dot{s} \leq -\eta \|s\| < 0, \text{ for } \|s\| \neq 0. \quad (39)$$

This implies that the trajectories of the uncertain nonlinear system (19) will be globally driven onto the specified sliding surface  $s = 0$  despite the uncertainties in finite time. The proof is complete.

From (31), we have  $s(0) = 0$ , that is the initial condition is on the sliding surface. According to Theorem 2, we know that the uncertain system (19) with the integral sliding surface (31) and the control law (37) can achieve global sliding mode. So the system designed is global robust and optimal.

### 3.4 A simulation example

Inverted pendulum is widely used for testing control algorithms. In many existing literatures, the inverted pendulum is customarily modeled by nonlinear system, and the approximate linearization is adopted to transform the nonlinear systems into a linear one, then a LQR is designed for the linear system.

To verify the effectiveness and superiority of the proposed GROSMC, we apply it to a single inverted pendulum in comparison with conventional LQR.

The nonlinear differential equation of the single inverted pendulum is

$$\begin{aligned}
 \dot{x}_1 &= x_2, \\
 \dot{x}_2 &= \frac{g \sin x_1 - amLx_2^2 \sin x_1 \cos x_1 + au \cos x_1}{L(4/3 - am \cos^2 x_1)} + d(t), \quad (40)
 \end{aligned}$$

where  $x_1$  is the angular position of the pendulum (rad),  $x_2$  is the angular speed (rad/s),  $M$  is the mass of the cart,  $m$  and  $L$  are the mass and half length of the pendulum, respectively.  $u$  denotes the control input,  $g$  is the gravity acceleration,  $d(t)$  represents the external disturbances, and the coefficient  $a = m / (M + m)$ . The simulation parameters are as follows:  $M = 1 \text{ kg}$ ,  $m = 0.2 \text{ kg}$ ,  $L = 0.5 \text{ m}$ ,  $g = 9.8 \text{ m/s}^2$ , and the initial state vector is  $x(0) = [-\pi / 18 \quad 0]^T$ .

Two cases with parameter variations in the inverted pendulum and external disturbance are considered here.

Case 1: The  $m$  and  $L$  are 4 times the parameters given above, respectively. Fig. 3 shows the robustness to parameter variations by the suggested GROSMC and conventional LQR.

Case 2: Apply an external disturbance  $d(t) = 0.01 \sin 2t$  to the inverted pendulum system at  $t = 9 \text{ s}$ . Fig. 4 depicts the different responses of these two controllers to external disturbance.

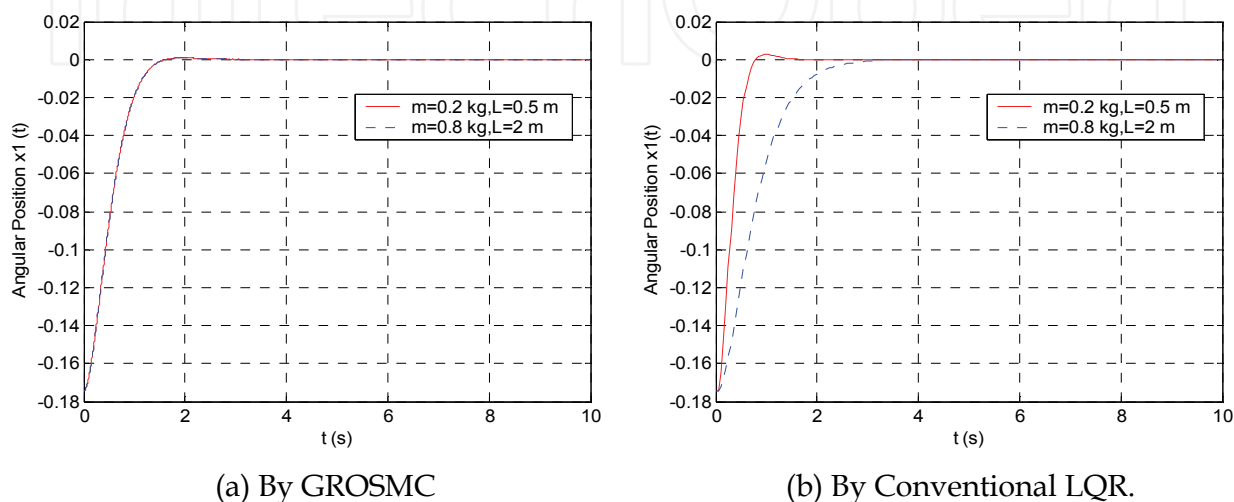


Fig. 3. Angular position responses of the inverted pendulum with parameter variation

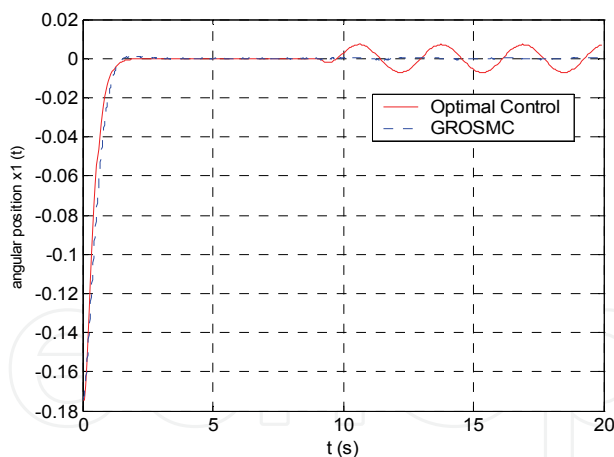


Fig. 4. Angular position responses of the inverted pendulum with external disturbance.

From Fig. 3 we can see that the angular position responses of inverted pendulum with and without parameter variations are exactly same by the proposed GROSMC, but the responses are obviously sensitive to parameter variations by the conventional LQR. It shows that the proposed GROSMC guarantees the controlled system complete robustness to parameter variation. As depicted in Fig. 4, without external disturbance, the controlled system could be driven to the equilibrium point by both of the controllers at about  $t = 2 \text{ s}$ . However, when the external disturbance is applied to the controlled system at  $t = 9 \text{ s}$ , the inverted pendulum system could still maintain the equilibrium state by GROSMC while the LQR not.

The switching function curve is shown in Fig. 5. The sliding motion occurs from the beginning without any reaching phase as can be seen. Thus, the GROSMC provides better features than conventional LQR in terms of robustness to system uncertainties.

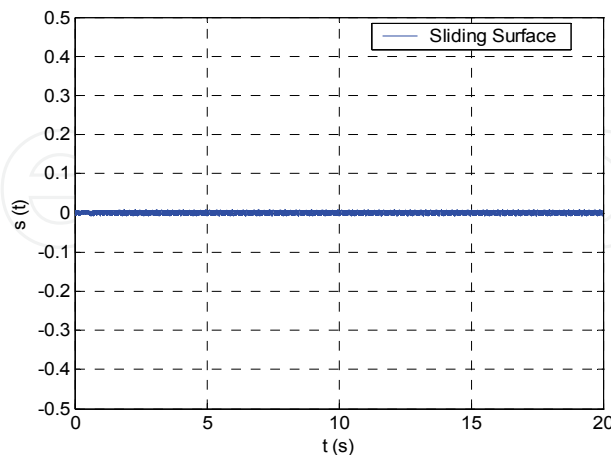


Fig. 5. The switching function  $s(t)$

### 3.5 Conclusion

In this section, the exact linearization technique is firstly adopted to transform an uncertain affine nonlinear system into a linear one. An optimal controller is designed to the linear nominal system, which not only simplifies the optimal controller design, but also makes the optimal control applicable to the entire transformation region. The sliding mode control is employed to robustify the optimal regulator. The uncertain system with the proposed integral sliding surface and the control law achieves global sliding mode, and the ideal sliding dynamics can minimized the given quadratic performance index. In summary, the system designed is global robust and optimal.

## 4. Optimal sliding mode tracking control for uncertain nonlinear system

With the industrial development, there are more and more control objectives about the system tracking problem (Ouyang et al., 2006; Mauder, 2008; Smolders et al., 2008), which is very important in control theory synthesis. Taking the robot as an example, it is often required to follow some special trajectories quickly as well as provide robustness to system uncertainties, including unmodeled dynamics, internal parameter variations and external disturbances. So the main tracking control problem becomes how to design the controller, which can not only get good tracking performance but also reject the uncertainties effectively to ensure the system better dynamic performance. In this section, a robust LQR tracking control based on intergral sliding mode is proposed for a class of nonlinear uncertain systems.

### 4.1 Problem formulation and assumption

Consider a class of uncertain affine nonlinear systems as follows:

$$\begin{cases} \dot{x} = f(x) + \Delta f(x) + g(x)[u + \delta(x, t, u)] \\ y = h(x) \end{cases} \quad (41)$$

where  $x \in R^n$  is the state vector,  $u \in R^m$  is the control input with  $m=1$ , and  $y \in R$  is the system output.  $f(x), g(x), \Delta f(x)$  and  $h(x)$  are sufficiently smooth in domain  $D \subset R^n$ .  $\delta(x, t, u)$  is continuous with respect to  $t$  and smooth in  $(x, u)$ .  $\Delta f(x)$  and  $\delta(x, t, u)$  represent the system uncertainties, including unmodelled dynamics, parameter variations and external disturbances.

Our goal is to design an optimal LQR such that the output  $y$  can track a reference trajectory  $y_r(t)$  asymptotically, some given performance criterion can be minimized, and the system can exhibit robustness to uncertainties.

**Assumption 7.** The nominal system of uncertain affine nonlinear system (41), that is

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (42)$$

has the relative degree  $\rho$  in domain  $D$  and  $\rho = n$ .

**Assumption 8.** The reference trajectory  $y_r(t)$  and its derivations  $y_r^{(i)}(t)$  ( $i = 1, \dots, n$ ) can be obtained online, and they are limited to all  $t \geq 0$ .

While as we know, if the optimal LQR is applied to nonlinear systems, it often leads to nonlinear TPBV problem and an analytical solution generally does not exist. In order to simplify the design of this tracking problem, the input-output linearization technique is adopted firstly.

Considering system (41) and differentiating  $y$ , we have

$$y^{(k)} = L_f^k h(x), \quad 0 \leq k \leq n-1$$

$$y^{(n)} = L_f^n h(x) + L_{\Delta f} L_f^{n-1} h(x) + L_g L_f^{n-1} h(x)[u + \delta(x, t, u)].$$

According to the input-output linearization, choose the following nonlinear state transformation

$$z = T(x) = \begin{bmatrix} h(x) & \dots & L_f^{n-1} h(x) \end{bmatrix}^T. \quad (43)$$

So the uncertain affine nonlinear system (40) can be written as

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, \dots, n-1 \\ \dot{z}_n &= L_f^n h(x) + L_{\Delta f} L_f^{n-1} h(x) + L_g L_f^{n-1} h(x)[u + \delta(x, t, u)]. \end{aligned}$$

Define an error state vector in the form of

$$e = \begin{bmatrix} z_1 - y_r \\ \vdots \\ z_n - y_r^{(n-1)} \end{bmatrix} = z - \mathfrak{R},$$

where  $\mathfrak{R} = \begin{bmatrix} y_r & \dots & y_r^{(n-1)} \end{bmatrix}^T$ . By this variable substitution  $e = z - \mathfrak{R}$ , the error state equation can be described as follows:

$$\begin{aligned} \dot{e}_i &= e_{i+1}, \quad i = 1, \dots, n-1 \\ \dot{e}_n &= L_f^n h(x) + L_{\Delta f} L_f^{n-1} h(x) + L_g L_f^{n-1} h(x)u(t) + L_g L_f^{n-1} h(x)\delta(x, t, u) - y_r^{(n)}(t). \end{aligned}$$



Let the feedback control law be selected as

$$u(t) = \frac{-L_f^n h(x) + v(t) + y_r^{(n)}(t)}{L_g L_f^{n-1} h(x)} \quad (44)$$

The error equation of system (40) can be given in the following forms:

$$\dot{e}(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} e(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ L_{\Delta f} L_f^{n-1} h(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} v(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ L_g L_f^{n-1} h(x) \delta(x, t, u) \end{bmatrix}. \quad (45)$$

Therefore, equation (45) can be rewritten as

$$\dot{e}(t) = Ae(t) + \Delta A + Bv(t) + \Delta \delta. \quad (46)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ L_{\Delta f} L_f^{n-1} h(x) \end{bmatrix}, \quad \Delta \delta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ L_g L_f^{n-1} h(x) \delta(x, t, u) \end{bmatrix}.$$

As can be seen,  $e \in R^n$  is the system error vector,  $v \in R$  is a new control input of the transformed system.  $A \in R^{n \times n}$  and  $B \in R^{n \times m}$  are corresponding constant matrixes.  $\Delta A$  and  $\Delta \delta$  represent uncertainties of the transformed system.

**Assumption 9.** There exist unknown continuous function vectors of appropriate dimensions  $\Delta \tilde{A}$  and  $\Delta \tilde{\delta}$ , such that

$$\Delta A = B \Delta \tilde{A}, \quad \Delta \delta = B \Delta \tilde{\delta}$$

**Assumption 10.** There exist known constants  $a_m, b_m$  such that

$$\|\Delta \tilde{A}\| \leq a_m, \quad \|\Delta \tilde{\delta}\| \leq b_m$$

Now, the tracking problem becomes to design a state feedback control law  $v$  such that  $e \rightarrow 0$  asymptotically. If there is no uncertainty, i.e.  $\delta(t, e) = 0$ , we can select the new input as  $v = -Ke$  to achieve the control objective and obtain the closed loop dynamics  $\dot{e} = (A - BK)e$ . Good tracking performance can be achieved by choosing  $K$  using optimal

control theory so that the closed loop dynamics is asymptotically stable. However, in presence of the uncertainties, the closed loop performance may be deteriorated. In the next section, the integral sliding mode control is adopted to robustify the optimal control law.

## 4.2 Design of optimal sliding mode tracking controller

### 4.2.1 Optimal tracking control of nominal system.

Ignoring the uncertainties of system (46), the corresponding nominal system is

$$\dot{e}(t) = Ae(t) + Bv(t). \quad (47)$$

For the nominal system (47), let  $v = v_0$  and  $v_0$  can minimize the quadratic performance index as follows:

$$I = \frac{1}{2} \int_0^{\infty} [e^T(t)Qe(t) + v_0^T(t)Rv_0(t)]dt \quad (48)$$

where  $Q \in R^{n \times n}$  is a symmetric positive definite matrix,  $R \in R^{m \times m}$  (here  $m = 1$ ) is a positive definite matrix.

According to optimal control theory, an optimal feedback control law can be obtained as:

$$v_0(t) = -R^{-1}B^T P e(t) \quad (49)$$

with  $P$  the solution of matrix Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0.$$

So the closed-loop system dynamics is

$$\dot{e}(t) = (A - BR^{-1}B^T P)e(t). \quad (50)$$

The designed optimal controller for system (47) is sensitive to system uncertainties including parameter variations and external disturbances. The performance index (48) may deviate from the optimal value. In the next part, we will use integral sliding mode control technique to robustify the optimal control law so that the uncertain system trajectory could be same as nominal system.

### 4.2.2 The robust optimal sliding surface.

To get better tracking performance, an integral sliding surface is defined as

$$s(e, t) = Ge(t) - G \int_0^t (A - BR^{-1}B^T P)e(\tau) d\tau - Ge(0), \quad (51)$$

where  $G \in R^{m \times n}$  is a constant matrix which is designed so that  $GB$  is nonsingular. And  $e(0)$  is the initial error state vector.

Differentiating (51) with respect to  $t$  and considering system (46), we obtain

$$\begin{aligned} \dot{s}(e, t) &= G\dot{e}(t) - G(A - BR^{-1}B^T P)e(t) \\ &= G[Ae(t) + \Delta A + Bv(t) + \Delta \delta] - G(A - BR^{-1}B^T P)e(t) \\ &= GBv(t) + GBR^{-1}B^T P e(t) + G(\Delta A + \Delta \delta). \end{aligned} \quad (52)$$

Let  $\dot{s}(e, t) = 0$ , the equivalent control can be obtained by

$$v_{\text{eq}}(t) = -(GB)^{-1}[GBR^{-1}B^T Pe(t) + G(\Delta A + \Delta \delta)]. \quad (53)$$

Substituting (53) into (46), and considering Assumption 10, the ideal sliding mode dynamics becomes

$$\begin{aligned} \dot{e}(t) &= Ae(t) + \Delta A + Bv_{\text{eq}}(t) + \Delta \delta \\ &= Ae(t) + \Delta A - B(GB)^{-1}[GBR^{-1}B^T Pe(t) + G(\Delta A + \Delta \delta)] + \Delta \delta \\ &= (A - BR^{-1}B^T P)e(t) - B(GB)^{-1}G[\Delta A + \Delta \delta] + \Delta A + \Delta \delta \\ &= (A - BR^{-1}B^T P)e(t) - B(GB)^{-1}GB(\Delta \tilde{A} + \Delta \tilde{\delta}) + B(\Delta \tilde{A} + \Delta \tilde{\delta}) \\ &= (A - BR^{-1}B^T P)e(t). \end{aligned} \quad (54)$$

It can be seen from equation (50) and (54) that the ideal sliding motion of uncertain system and the optimal dynamics of the nominal system are uniform, thus the sliding mode is also asymptotically stable, and the sliding mode guarantees system (46) complete robustness to uncertainties. Therefore, (51) is called a robust optimal sliding surface.

#### 4.2.3 The control law.

For uncertain system (46), we propose a control law in the form of

$$\begin{aligned} v(t) &= v_c(t) + v_d(t), \\ v_c(t) &= -R^{-1}B^T Pe(t), \\ v_d(t) &= -(GB)^{-1}[ks + \varepsilon \text{sgn}(s)]. \end{aligned} \quad (55)$$

where  $v_c$  is the continuous part, which is used to stabilize and optimize the nominal system. And  $v_d$  is the discontinuous part, which provides complete compensation for system uncertainties.  $\text{sgn}(s) = [\text{sgn}(s_1) \ \cdots \ \text{sgn}(s_m)]^T$ .  $k$  and  $\varepsilon$  are appropriate positive constants, respectively.

**Theorem 3.** Consider uncertain system (46) with Assumption 9-10. Let the input  $v$  and the sliding surface be given as (55) and (51), respectively. The control law can force system trajectories to reach the sliding surface in finite time and maintain on it thereafter if  $\varepsilon \geq (a_m + d_m)\|GB\|$ .

**Proof:** Utilizing  $V = (1/2)s^T s$  as a Lyapunov function candidate, and considering Assumption 9-10, we obtain

$$\begin{aligned} \dot{V} &= s^T \dot{s} = s^T [G\dot{e}(t) - G(A - BR^{-1}B^T P)e(t)] \\ &= s^T \{G[Ae(t) + \Delta A + Bv(t) + \Delta \delta] - G(A - BR^{-1}B^T P)e(t)\} \\ &= s^T [G\Delta A - GBR^{-1}B^T Pe(t) - (ks + \varepsilon \text{sgn}(s)) + G\Delta \delta + GBR^{-1}B^T Pe(t)] \\ &= s^T \{-[ks + \varepsilon \text{sgn}(s)] + G\Delta A + G\Delta \delta\} = -k\|s\|_1 - \varepsilon\|s\| + s^T(G\Delta A + G\Delta \delta) \\ &\leq -k\|s\|_1 - \varepsilon\|s\| + (a_m + d_m)\|GB\|\|s\| \leq -k\|s\| - [\varepsilon - (a_m + d_m)\|GB\|]\|s\| \end{aligned}$$

where  $\|\cdot\|_1$  denotes the 1-norm. Note the fact that for any  $\|s\| \neq 0$ , we have  $\|s\|_1 \geq \|s\|$ . If  $\varepsilon \geq (a_m + d_m)\|GB\|$ , then

$$\dot{V} = s^T \dot{s} \leq -\varepsilon \|s\|_1 + \|G\| \|\delta\| \|s\| \leq -(\varepsilon - \|G\| \|\delta\|) \|s\| < 0. \tag{56}$$

This implies that the trajectories of uncertain system (46) will be globally driven onto the specified sliding surface  $s(e,t)=0$  in finite time and maintain on it thereafter. The proof is completed.

From (51), we have  $s(0)=0$ , that is to say, the initial condition is on the sliding surface. According to Theorem 3, uncertain system (46) achieves global sliding mode with the integral sliding surface (51) and the control law (55). So the system designed is global robust and optimal, good tracking performance can be obtained with this proposed algorithm.

### 4.3 Application to robots.

In the recent decades, the tracking control of robot manipulators has received a great of attention. To obtain high-precision control performance, the controller is designed which can make each joint track a desired trajectory as close as possible. It is rather difficult to control robots due to their highly nonlinear, time-varying dynamic behavior and uncertainties such as parameter variations, external disturbances and unmodeled dynamics. In this section, the robot model is investigated to verify the effectiveness of the proposed method.

A 1-DOF robot mathematical model is described by the following nonlinear dynamics:

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{C(q,\dot{q})}{M(q)} \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{G(q)}{M(q)} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M(q)} \end{bmatrix} \tau - \begin{bmatrix} 0 \\ \frac{1}{M(q)} \end{bmatrix} d(t), \tag{57}$$

where  $q, \dot{q}$  denote the robot joint position and velocity, respectively.  $\tau$  is the control vector of torque by the joint actuators.  $m$  and  $l$  are the mass and length of the manipulator arm, respectively.  $d(t)$  is the system uncertainties.  $C(q,\dot{q})=0.03 \cos(q)$ ,  $G(q)=mgl \cos(q)$ ,  $M(q)=0.1+0.06 \sin(q)$ . The reference trajectory is  $y_r(t)=\sin \pi t$ .

According to input-output linearization technique, choose a state vector as follows:

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}.$$

Define an error state vector of system (57) as  $e = [e_1 \ e_2]^T = [q - y_r \ \dot{q} - \dot{y}_r]^T$ , and let the control law  $\tau = (v + \ddot{y}_r)M(q) + C(q,\dot{q})\dot{q} + G(q)$ .

So the error state dynamic of the robot can be written as:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v - \begin{bmatrix} 0 \\ 1/M(q) \end{bmatrix} d(t) \tag{58}$$

Choose the sliding mode surface and the control law in the form of (51) and (55), respectively, and the quadratic performance index in the form of (48). The simulation parameters are as follows:  $m=0.02$ ,  $g=9.8$ ,  $l=0.5$ ,  $d(t)=0.5 \sin 2\pi t$ ,  $k=18$ ,  $\varepsilon=6$ ,

$G = [0 \ 1]$ ,  $Q = \begin{bmatrix} 10 & 2 \\ 2 & 1 \end{bmatrix}$ ,  $R = 1$ . The initial error state vector is  $e = [0.5 \ 0]^T$ .

The tracking responses of the joint position  $q$  and its velocity are shown in Fig. 6 and Fig. 7, respectively. The control input  $\tau$  is displayed in Fig. 8. From Fig. 6 and Fig. 7 it can be seen that the position error can reach the equilibrium point quickly and the position track the

reference sine signal  $y_r$  well. Simulation results show that the proposed scheme manifest good tracking performance and the robustness to parameter variations and the load disturbance.

#### 4.4 Conclusions

In order to achieve good tracking performance for a class of nonlinear uncertain systems, a sliding mode LQR tracking control is developed. The input-output linearization is used to transform the nonlinear system into an equivalent linear one so that the system can be handled easily. With the proposed control law and the robust optimal sliding surface, the system output is forced to follow the given trajectory and the tracking error can minimize the given performance index even if there are uncertainties. The proposed algorithm is applied to a robot described by a nonlinear model with uncertainties. Simulation results illustrate the feasibility of the proposed controller for trajectory tracking and its capability of rejecting system uncertainties.

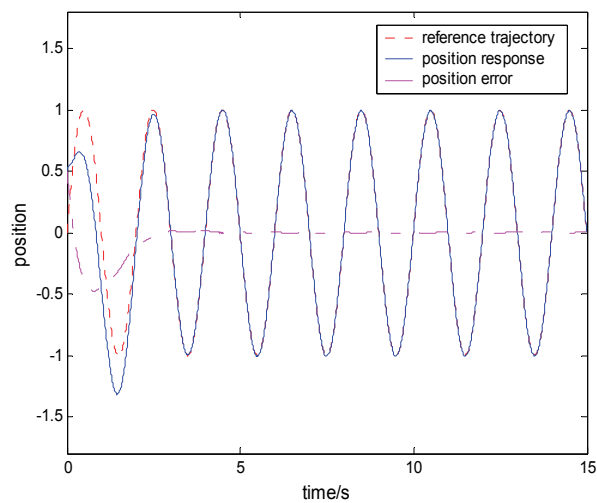


Fig. 6. The tracking response of  $q$

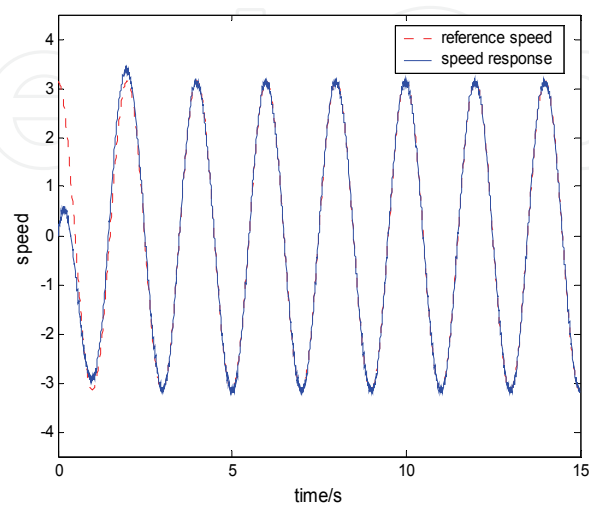


Fig. 7. The tracking response of  $\dot{q}$

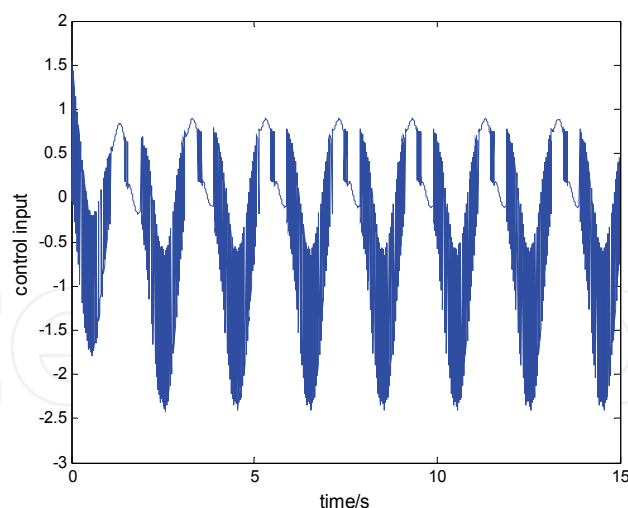


Fig. 8. The control input  $\tau$

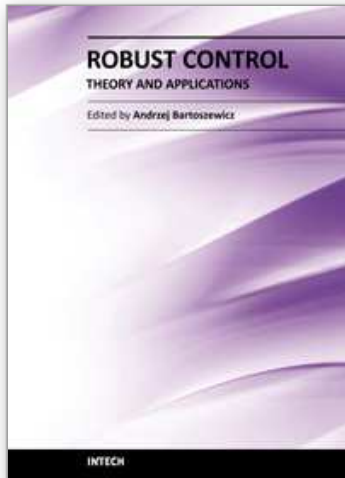
## 5. Acknowledgements

This work is supported by National Nature Science Foundation under Grant No. 60940018.

## 6. References

- Basin, M.; Rodriguez-Gonzalez, J.; Fridman, L. (2007). Optimal and robust control for linear state-delay systems. *Journal of the Franklin Institute*. Vol.344, pp.830–845.
- Chen, W. D.; Tang, D. Z.; Wang, H. T. (2004). Robust Tracking Control Of Robot Manipulators Using Backstepping. *Journal of System Simulation*. Vol. 16, No. 4, pp. 837-837, 841.
- Choi, S. B.; Cheong, C. C.; Park, D. W. (1993). Moving switch surfaces for robust control of second-order variable structure systems. *International Journal of Control*. Vol.58, No.1, pp. 229-245.
- Choi, S. B.; Park, D. W.; Jayasuriya, S. (1994). A time-varying sliding surface for fast and robust tracking control of second-order uncertain systems. *Automatica*. Vol.30, No.5, pp. 899-904.
- Chiou, K. C.; Huang, S. J. (2005). An adaptive fuzzy controller for robot manipulators. *Mechatronics*. Vol.15, pp.151-177.
- Cimen, T.; Banks, S. P. (2004). Global optimal feedback control for general nonlinear systems with nonquadratic performance criteria. *System & Control Letters*. Vol.53, pp.327-346.
- Cimen, T.; Banks, S. P. (2004). Nonlinear optimal tracking control with application to super-tankers for autopilot design. *Automatica*. Vol.40, No.11, pp.1845 - 1863.
- Gao, W. B.; Hung, J. C. (1993). Variable structure control of nonlinear systems: A new approach. *IEEE Transactions on Industrial Electronics*. Vol.40, No.1, pp. 45-55.
- Ho, H. F.; Wong, Y. K.; Rad, A. B. (2007). Robust fuzzy tracking control for robotic manipulators. *Simulation Modelling Practice and Theory*. Vol.15, pp.801-816.
- Laghrouche, S.; Plestan, F.; Glumineau, A. (2007). Higher order sliding mode control based on integral sliding mode. *Automatica*. Vol. 43, pp.531-537.

- Lee, J. H. (2006). Highly robust position control of BLDDSM using an improved integral variable structure system. *Automatica*. Vol.42, pp.929-935.
- Lin, F. J.; Chou, W. D. (2003). An induction motor servo drive using sliding mode controller with genetic algorithm. *Electric Power Systems Research*. Vol.64, pp.93-108.
- Mokhtari, A.; Benallegue, A.; Orlov, Y. (2006). Exact linearization and sliding mode observer for a quadrotor unmanned aerial vehicle. *International Journal of Robotics and Automation*. Vol. 21, No.1, pp.39-49.
- Mauder, M. Robust tracking control of nonholonomic dynamic systems with application to the bi-steerable mobile robot. (2008). *Automatica*. Vol.44, No.10, pp.2588-2592.
- Ouyang, P. R.; Zhang, W. J.; Madan M. Gupta. (2006). An adaptive switching learning control method for trajectory tracking of robot manipulators. *Mechatronics*. Vol.16, No.1, pp.51-61.
- Tang, G. Y.; Sun, H. Y.; Pang, H. P. (2008). Approximately optimal tracking control for discrete time-delay systems with disturbances. *Progress in Natural Science*. Vol.18, pp. 225-231.
- Pang, H. P.; Chen, X. (2009). Global robust optimal sliding mode control for uncertain affine nonlinear systems. *Journal of Systems Engineering and Electronics*. Vol.20, No.4, pp. 838-843.
- Pang H. P.; Wang L.P. (2009). Global Robust Optimal Sliding Mode Control for a class of Affine Nonlinear Systems with Uncertainties Based on SDRE. *Proceeding of 2009 Second International Workshop on Computer Science and Engineering*. Vol. 2, pp. 276-280.
- Pang, H. P.; Tang, G. Y.; Sun, H.Y. (2009). Optimal Sliding Mode Design for a Class of Uncertain Systems with Time-delay. *Information and Control*. Vol.38, No.1, pp.87-92.
- Tang, G. Y.; Gao, D. X. (2005). Feedforward and feedback optimal control for nonlinear systems with persistent disturbances. *Control and Decision*. Vol.20, No.4, pp. 366-371.
- Tang, G. Y. (2005). Suboptimal control for nonlinear systems: a successive approximation approach. *Systems and Control Letters*. Vol. 54, No.5, pp.429-434.
- Tang, G. Y.; Zhao, Y. D.; Zhang, B. L. (2007). Optimal output tracking control for nonlinear systems via successive approximation approach. *Nonlinear Analysis*. Vol.66, No.6, pp.1365-1377.
- Shamma, J. S.; Cloutier, J. R. (2001). Existence of SDRE stabilizing feedback. *Proceedings of the American Control Conference*. pp.4253-4257, Arlington VA.
- Smolders, K.; Volckaert, M. Swevers, J. (2008). Tracking control of nonlinear lumped mechanical continuous-time systems: A model-based iterative learning approach. *Mechanical Systems and Signal Processing*. Vol.22, No.8, pp.1896-1916.
- Yang K. D.; Özgüner. Ü. (1997). Sliding-mode design for robust linear optimal control. *Automatica*. Vol. 33, No. 7, pp. 1313-1323.



## **Robust Control, Theory and Applications**

Edited by Prof. Andrzej Bartoszewicz

ISBN 978-953-307-229-6

Hard cover, 678 pages

**Publisher** InTech

**Published online** 11, April, 2011

**Published in print edition** April, 2011

The main objective of this monograph is to present a broad range of well worked out, recent theoretical and application studies in the field of robust control system analysis and design. The contributions presented here include but are not limited to robust PID, H-infinity, sliding mode, fault tolerant, fuzzy and QFT based control systems. They advance the current progress in the field, and motivate and encourage new ideas and solutions in the robust control area.

### **How to reference**

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Hai-Ping Pang and Qing Yang (2011). Optimal Sliding Mode Control for a Class of Uncertain Nonlinear Systems Based on Feedback Linearization, Robust Control, Theory and Applications, Prof. Andrzej Bartoszewicz (Ed.), ISBN: 978-953-307-229-6, InTech, Available from:  
<http://www.intechopen.com/books/robust-control-theory-and-applications/optimal-sliding-mode-control-for-a-class-of-uncertain-nonlinear-systems-based-on-feedback-linearizat>

**INTECH**  
open science | open minds

### **InTech Europe**

University Campus STeP Ri  
Slavka Krautzeka 83/A  
51000 Rijeka, Croatia  
Phone: +385 (51) 770 447  
Fax: +385 (51) 686 166  
[www.intechopen.com](http://www.intechopen.com)

### **InTech China**

Unit 405, Office Block, Hotel Equatorial Shanghai  
No.65, Yan An Road (West), Shanghai, 200040, China  
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元  
Phone: +86-21-62489820  
Fax: +86-21-62489821



© 2011 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the [Creative Commons Attribution-NonCommercial-ShareAlike-3.0 License](https://creativecommons.org/licenses/by-nc-sa/3.0/), which permits use, distribution and reproduction for non-commercial purposes, provided the original is properly cited and derivative works building on this content are distributed under the same license.

IntechOpen

IntechOpen