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Chapter

The Paradigm of Complex Probability and Quantum Mechanics: The Infinite Potential Well Problem – The Position Wave Function

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Abstract

The system of axioms for probability theory laid in 1933 by Andrey Nikolaevich Kolmogorov can be extended to encompass the imaginary set of numbers and this by adding to his original five axioms an additional three axioms. Therefore, we create the complex probability set \( C \), which is the sum of the real set \( R \) with its corresponding real probability, and the imaginary set \( M \) with its corresponding imaginary probability. Hence, all stochastic experiments are performed now in the complex set \( C \) instead of the real set \( R \). The objective is then to evaluate the complex probabilities by considering supplementary new imaginary dimensions to the event occurring in the “real” laboratory. Consequently, the corresponding probability in the whole set \( C \) is always equal to one and the outcome of the random experiments that follow any probability distribution in \( R \) is now predicted totally in \( C \). Subsequently, it follows that chance and luck in \( R \) is replaced by total determinism in \( C \). Consequently, by subtracting the chaotic factor from the degree of our knowledge of the stochastic system, we evaluate the probability of any random phenomenon in \( C \). My innovative complex probability paradigm (CPP) will be applied to the established theory of quantum mechanics in order to express it completely deterministically in the universe \( C = R + M \).

Keywords: chaotic factor, degree of our knowledge, complex random vector, probability norm, complex probability set \( C \), position wave function

“Nothing in nature is by chance ... Something appears to be chance only because of our lack of knowledge.”

Baruch Spinoza.

“You believe in the God who plays dice, and I in complete law and order.”

Albert Einstein, Letter to Max Born

“Without mathematics, we cannot penetrate deeply into philosophy. Without philosophy, we cannot penetrate deeply into mathematics. Without both, we cannot penetrate deeply into anything ...”

Gottfried Wilhelm von Leibniz.
1. Introduction

There are several names for this idealized and highly artificial potential, prominent among them: The infinite square-well potential, the infinite potential [1–3]. However, it is the phrase particle in an escape-proof box that is more likely to be intuitively appealing. Very simply it tells us about a particle moving inside the box as a free particle except at the box walls, which are postulated to be impenetrable by definition. The particle in an escape proof box is a sleek, easy-on-the-mathematics model for initiating students into quantum mechanics, with the added advantage that it is one of the few sectors within quantum mechanics where the Schrödinger equation can be solved analytically without resorting to approximation techniques. In the context of this simple potential, students typically find their first very intuitive understanding of the meaning of bound states, boundary conditions, stationary states, and energy-momentum quantization. It is even an introduction to quantum tunneling by emphasizing by contrast why a particle in a box cannot tunnel out of the box! The infinite square well is then an easy introduction to a more general understanding of the time independent Schrödinger equation for bound states in more sophisticated potentials, where the quantum tunneling phenomenon is exhibited. It is easy stepping-stones away from this first potential to the more complicated structures, such as the simple harmonic oscillator, which plays a seminal role in quantum field theory. In a clear and present sense, the quantum adventure can fairly be said to begin with this humble but very remarkable particle in an escape-proof box conception. However, genius in simplicity is another watchword for this potential. Remarkably, from such simplicity, one is also able to extract an enormous amount of excellent physics. Never mind that there are no actual confining forces in the world that are infinitely strong, physicists successfully deploy the square well potential to model complicated physics all the time, witness the infinite square well potential, which was used by physicist Sommerfeld to model his electron gas theory, where he construed the moving electrons as free particles confined to an escape-proof box. And again, the particle in a box is also deployed to model and investigate a myriad of other complex physical systems – the Hexatriene molecule, among others, as well as in fabricated semiconductor layers. As Cartwright notes, “Of course, this is not a true description of the potentials that are actually produced by the walls and the environment. But it is not exactly false either. It is just the way to achieve the results in the model that the walls and environment are supposed to achieve in reality. The infinite potential is a good piece of stage setting.” [3] True, the particle in the escape proof-box is by definition a highly contrived and idealized model. Consequently, this important and well-known problem in quantum mechanics will be related to my complex probability paradigm (CPP) in order to express it totally deterministically.

In the end, and to conclude, this research work’s first chapter is organized as follows: After the introduction in section 1, the purpose and the advantages of the present work are presented in section 2. Afterward, in section 3, the extended Kolmogorov’s axioms, and hence, the complex probability paradigm with their original parameters and interpretation, will be explained and summarized. Moreover, in section 4, we will explain briefly the one-dimensional case of the infinite square well problem considered in this work. Additionally, in section 5, the new paradigm will be
related to the particle in a box problem after applying CPP to the position wave function; hence, some corresponding simulations will be done, and afterward, the characteristics of this stochastic distribution will be computed in the probabilities sets \( R \), \( M \), and \( C \). Finally, we conclude the work by doing a comprehensive summary in section 6, and then present in section 7 the list of references cited in the current chapter. Furthermore, in the following second chapter, the new paradigm will be related to the particle in a box problem after applying CPP to the momentum wave function of the problem; hence, some corresponding simulations will be done, and afterward, the characteristics of this stochastic distribution will be computed in the probabilities sets \( R \), \( M \), and \( C \). Also, in the following chapter, CPP will be used to extend and verify Heisenberg uncertainty principle in \( R \), \( M \), and \( C \). In addition, we will calculate and determine the position and the momentum wave functions entropies in \( R \), \( M \), and \( C \).

2. The purpose and the advantages of the current publication

All our work in classical probability theory is to compute probabilities. The original idea in this research work is to add new dimensions to our random experiment, which will make the work deterministic. In fact, the probability theory is a nondeterministic theory by nature, that means that the outcome of the events is due to chance and luck. By adding new dimensions to the event in \( R \), we make the work deterministic, and hence, a random experiment will have a certain outcome in the complex set of probabilities \( C \). It is of great importance that the stochastic system, like the problem in quantum mechanics considered here, becomes totally predictable since we will be totally knowledgeable to foretell the outcome of chaotic and random events that occur in nature, for example, in statistical mechanics or in all stochastic processes. Therefore, the work that should be done is to add to the real set of probabilities \( R \), the contributions of \( M \), which is the imaginary set of probabilities that will make the event in \( C = R + M \) deterministic. If this is found to be fruitful, then a new theory in statistical sciences and prognostic, and mainly in quantum mechanics is elaborated, this is to understand absolutely deterministically those phenomena that used to be random phenomena in \( R \). This is what I called “The Complex Probability Paradigm (CPP),” which was initiated and elaborated in my 19 previous papers [4–22].

To summarize, the advantages and the purposes of this current work are to:

1. Extend the theory of classical probability to encompass the complex numbers set, hence to bond the theory of probability to the field of complex variables and analysis in mathematics. This mission was elaborated on and initiated in my earlier 19 papers.

2. Apply the novel probability axioms and CPP paradigm to quantum mechanics, specifically to the infinite potential well problem.

3. Show that all nondeterministic phenomena like in the problem considered here can be expressed deterministically in the complex probabilities set \( C \).

4. Compute and quantify both the degree of our knowledge and the chaotic factor of the wave function position and momentum distributions and CPP in the sets \( R \), \( M \), and \( C \).
5. Represent and show the graphs of the functions and parameters of the innovative paradigm related to quantum mechanics.

6. Evaluate all the characteristics of the wave function position and momentum distributions.

7. Demonstrate that the classical concept of probability is permanently equal to one in the set of complex probabilities, hence, no randomness, no chaos, no ignorance, no uncertainty, no nondeterminism, and no unpredictability exist in:

\[ \mathbb{C} (\text{complex set}) = \mathbb{R} (\text{real set}) + \mathbb{M} (\text{imaginary set}) \]

8. Calculate the problem entropies in \( \mathbb{R} \), \( \mathbb{M} \), and \( \mathbb{C} \), and show that there is no disorder and no information loss nor gain in CPP but conservation of information.

9. Verify and extend Heisenberg uncertainty principle in \( \mathbb{R} \) to \( \mathbb{M} \) and \( \mathbb{C} \).

10. Prepare to implement this creative model to other topics and problems in quantum mechanics. These will be the job to be accomplished in my future research publications.

Concerning some applications of the novel founded paradigm and as future work, it can be applied to any nondeterministic phenomenon in quantum mechanics. And compared with existing literature, the major contribution of the current research work is to apply the innovative paradigm of CPP to quantum mechanics and to express it completely deterministically. The next figure displays the major purposes of the complex probability paradigm (CPP) (Figure 1).

3. The complex probability paradigm

3.1 The original Andrey Nikolaevich Kolmogorov system of axioms

The simplicity of Kolmogorov’s system of axioms may be surprising [4–22]. Let \( E \) be a collection of elements \( \{E_1, E_2, \ldots\} \) called elementary events, and let \( F \) be a set of subsets of \( E \) called random events [23–27]. The five axioms for a finite set \( E \) are:
Axiom 1: $F$ is a field of sets.
Axiom 2: $F$ contains the set $E$.
Axiom 3: A nonnegative real number $P_{rob}(A)$ called the probability of $A$, is assigned to each set $A$ in $F$. We have always $0 \leq P_{rob}(A) \leq 1$.
Axiom 4: $P_{rob}(E)$ equals 1.
Axiom 5: If $A$ and $B$ have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that $A$ and $B$ are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both $A$ and $B$ are independent then:

$$P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B).$$

Moreover, we can generalize and say that for $N$ disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \leq j \leq N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}(A_j)$$

And we say also that for $N$ independent events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \leq j \leq N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}(A_j)$$

3.2 Adding the imaginary part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_{m} = i \times (1 - P_{r})$ be the probability of an associated complementary event in $\mathcal{M}$ (the imaginary part or universe) to the event $A$ in $\mathcal{R}$ (the real part or universe). It follows that $P_{r} + P_{m}/i = 1$, where $i$ is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

Axiom 7: We construct the complex number or vector $Z = P_{r} + P_{m} = P_{r} + i(1 - P_{r})$ having a norm $|Z|$ such that:

$$|Z|^2 = P_{r}^2 + (P_{m}/i)^2.$$

Axiom 8: Let $P_c$ denote the probability of an event in the complex probability set and universe $\mathcal{C}$, where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that $P_c$ is the probability of an event $A$ in $\mathcal{R}$ with its associated and complementary event in $\mathcal{M}$ such that:

$$P_c^2 = (P_{r} + P_{m}/i)^2 = |Z|^2 - 2iP_{r}P_{m}$$

and is always equal to 1.

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms...
defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers and realm [28–65].

3.3 A concise interpretation of the original CPP paradigm

To summarize the novel CPP paradigm, we state that in the real probability universe $\mathcal{R}$ the degree of our certain knowledge is undesirably imperfect, and hence, unsatisfactory, thus we extend our analysis to the set of complex numbers $\mathcal{C}$, which incorporates the contributions of both the set of real probabilities, which is $\mathcal{R}$ and the complementary set of imaginary probabilities, which is $\mathcal{M}$. Afterward, this will yield an absolute and perfect degree of our knowledge in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ because $P_c = 1$ constantly and permanently. As a matter of fact, the work in the universe $\mathcal{C}$ of complex probabilities gives way to a sure forecast of any stochastic experiment, since in $\mathcal{C}$ we remove and subtract from the computed degree of our knowledge the measured chaotic factor. This will generate in universe $\mathcal{C}$ a probability equal to 1 ($P_{c}^{2} = DOK - Chf = DOK + MChf = 1 = P_c$). Many applications which take into consideration numerous continuous and discrete probability distributions in my 19 previous research papers confirm this hypothesis and innovative paradigm [4–22]. The Extended Kolmogorov Axioms (EKA for short) or the Complex Probability Paradigm (CPP for short) can be shown and summarized in the next illustration (Figure 2):

4. One-dimensional case of the infinite potential well problem

The simplest form of the particle in a box model considers a one-dimensional system [1, 2]. Here, the particle may only move backward and forwards along a straight line with impenetrable barriers at either end. The walls of a one-dimensional box may be seen as regions of space with an infinitely large potential energy. Conversely, the interior of the box has a constant zero potential energy. This means that
no forces act upon the particle inside the box and it can move freely in that region. However, infinitely large forces repel the particle if it touches the walls of the box, preventing it from escaping. The potential energy in this model is given as:

$$V(x) = \begin{cases} 
0 & x_c - \frac{L}{2} < x < x_c + \frac{L}{2} \\
\infty & \text{otherwise}
\end{cases}$$

where $L$ is the length of the box, $x_c$ is the location of the center of the box and $x$ is the position of the particle within the box. Simple cases include the centered box ($x_c = 0$) and the shifted box ($x_c = \frac{L}{2}$) (Figure 3).

5. The infinite potential well problem in quantum mechanics and the complex probability paradigm (CPP) parameters

In this section, we will relate and link quantum mechanics to the complex probability paradigm with all its parameters by applying it to the infinite potential well problem and by using the four CPP concepts which are: the real probability $P_r$ in the real probability set $\mathcal{R}$, the imaginary probability $P_m$ in the imaginary probability set $\mathcal{M}$, the complex random vector or number $Z$ in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and the deterministic real probability $P_c$ also in the probability set $\mathcal{C}$ [1–22, 66–99].

5.1 The position wave function and CPP: The position wave function solution

In quantum mechanics, the wave function gives the most fundamental description of the behavior of a particle; the measurable properties of the particle (such as its position, momentum, and energy) may all be derived from the wave function. The wave function $\psi(x, t)$ can be found by solving the Schrödinger equation for the system:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x)\psi(x, t)$$

where $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant, $m$ is the mass of the particle, $i$ is the imaginary unit, and $t$ is time.
Inside the box, no forces act upon the particle, which means that the part of the wave function inside the box oscillates through space and time in the same form as a free particle:

$$\psi(x, t) = [A \sin (kx) + B \cos (kx)]e^{-i\omega t}$$

where $A$ and $B$ are arbitrary complex numbers. The frequency of the oscillations through space and time is given by the wave number $k$ and the angular frequency $\omega$, respectively.

$$\Rightarrow \psi_n(x, t) = \begin{cases} A \sin \left(\frac{n\pi x}{L} + L \right) e^{-i\omega t} & \frac{L}{2} < x < x_c + \frac{L}{2} \\ 0 & \text{elsewhere} \end{cases}$$

where $k_n = \frac{n\pi}{L}$.

The unknown constant $A$ may be found by normalizing the wave function, so that the total probability density of finding the particle in the system is 1. It follows that:

$$|A| = \sqrt{\frac{2}{L}}$$

Thus, $A$ may be any complex number with an absolute value $\sqrt{2/L}$; these different values of $A$ yield the same physical state, so $A = \sqrt{2/L}$ can be selected to simplify.

### 5.2 The position wave function probability distribution and CPP

In classical physics, the particle can be detected anywhere in the box with equal probability. In quantum mechanics, however, the probability density for finding a particle at a given position is derived from the wave function as $f(x) = |\psi(x)|^2$. For the particle in a box, the wave function position probability density function (PDF) for finding the particle at a given position depends upon its state and is given by:

$$f(x) = |\psi(x)|^2 = \begin{cases} \frac{2}{L} \sin^2 \left(\frac{k_n}{2} \left(x - x_c + \frac{L}{2}\right)\right) & \frac{L}{2} < x < x_c + \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

Thus, for any value of $n$ greater than one, there are regions within the box for which $f(x) = 0$, indicating that spatial nodes exist at which the particle cannot be found.

Therefore, the wave function position cumulative probability distribution function (CDF), which is equal to $P_r(X)$ in $\mathcal{R}$ is:

$$P_r(X) = F(x_j) = P_{rob} (X \leq x_j) = \int_{-\infty}^{x_j} |\psi(x)|^2 \, dx$$

$$= \begin{cases} \int_{x_c - \frac{L}{2}}^{x_c + \frac{L}{2}} \frac{2}{L} \sin^2 \left(\frac{k_n}{2} \left(x - x_c + \frac{L}{2}\right)\right) \, dx & \frac{L}{2} < x_c - x < x_c + \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$
And the real complementary probability to $P_r(X)$ in $\mathcal{R}$, which is $P_m(X)/i$ is:

$$P_m(X)/i = 1 - P_r(X) = 1 - F(x_j) = 1 - P_{rob}(X \leq x_j) = P_{rob}(X > x_j)$$

$$= 1 - \int_{-\infty}^{x_j} |\psi(x)|^2 dx = \int_{x_j}^{\infty} |\psi(x)|^2 dx$$

$$= \left\{ \begin{array}{ll}
1 - \int_{x_j}^{x_c} \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx & \text{if } x_c - \frac{L}{2} < x_j < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array} \right.$$

$$= \left\{ \begin{array}{ll}
\int_{x_j}^{x_c + \frac{L}{2}} \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx & \text{if } x_c - \frac{L}{2} < x_j < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array} \right.$$

Consequently, the imaginary complementary probability to $P_r(X)$ in $\mathcal{M}$, which is $P_m(X)$ is:

$$P_m(X) = i[1 - P_r(X)] = i[1 - F(x_j)] = i[1 - P_{rob}(X \leq x_j)] = iP_{rob}(X > x_j)$$

$$= i \left[ 1 - \int_{-\infty}^{x_j} |\psi(x)|^2 dx \right] = i \int_{x_j}^{\infty} |\psi(x)|^2 dx$$

$$= \left\{ \begin{array}{ll}
i \left[ 1 - \int_{x_c - \frac{L}{2}}^{x_j} \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx \right] & \text{if } x_c - \frac{L}{2} < x_j < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array} \right.$$

$$= \left\{ \begin{array}{ll}
i \int_{x_j}^{x_c + \frac{L}{2}} \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx & \text{if } x_c - \frac{L}{2} < x_j < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array} \right.$$

The Paradigm of Complex Probability and Quantum Mechanics: The Infinite Potential Well...
Furthermore, the complex random number or vector in $\mathbf{C} = \mathcal{R} + \mathcal{M}$, which is $Z(X)$ is:

$$Z(X) = P_i(X) + P_m(X) = P_i(X) + i[1 - P_i(X)] = F(x_i) + i[1 - F(x_i)]$$

$$= P_{rob}(X \leq x_i) + i[1 - P_{rob}(X \leq x_i)] = P_{rob}(X \leq x_i) + iP_{rob}(X > x_i)$$

$$= \int_{-\infty}^{x_i} |\psi(x)|^2 dx + i \int_{x_i}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{x_i} |\psi(x)|^2 dx + i \int_{x_i}^{\infty} |\psi(x)|^2 dx$$

$$= \begin{cases} \int_{-\infty}^{x_i} \frac{2}{L} \sin^2 k_e \left(x - x_r + \frac{L}{2}\right) dx + i \int_{x_i}^{\infty} \frac{2}{L} \sin^2 k_e \left(x - x_r + \frac{L}{2}\right) dx & \text{for } x_i - \frac{L}{2} < x_i < x_i + \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \int_{-\infty}^{x_i} \frac{2}{L} \sin^2 k_e \left(x - x_r + \frac{L}{2}\right) dx + i \int_{x_i}^{\infty} \frac{2}{L} \sin^2 k_e \left(x - x_r + \frac{L}{2}\right) dx & \text{for } x_i - \frac{L}{2} < x_i < x_i - \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

Additionally, the degree of our knowledge, which is $DOK(X)$ is:

$$DOK(X) = |P_i(X)|^2 + |P_m(X)/i|^2 = |P_i(X)|^2 + [1 - P_i(X)]^2$$

$$= [P_i(x_i)]^2 + [1 - P_i(x_i)]^2 = [P_{rob}(X \leq x_i)]^2 + [1 - P_{rob}(X \leq x_i)]^2$$

$$= [P_{rob}(X \leq x_i)]^2 + [P_{rob}(X > x_i)]^2$$

$$= \begin{cases} \int_{-\infty}^{x_i} |\psi(x)|^2 dx \right)^2 + \left[1 - \int_{x_i}^{\infty} |\psi(x)|^2 dx \right]^2 & \text{for } x_i - \frac{L}{2} < x_i < x_i + \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \int_{-\infty}^{x_i} \frac{2}{L} \sin^2 k_e \left(x - x_r + \frac{L}{2}\right) dx \right)^2 + \left[1 - \int_{x_i}^{\infty} \frac{2}{L} \sin^2 k_e \left(x - x_r + \frac{L}{2}\right) dx \right]^2 & \text{for } x_i - \frac{L}{2} < x_i < x_i - \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$
Moreover, the chaotic factor, which is $\text{Chf}(X)$ is:

$$
\text{Chf}(X) = 2iP_r(X)P_m(X)
$$

$$
= 2iP_r(X) \times i[1 - P_r(X)] = -2P_r(X)(1 - P_r(X)) = -2F(x_y)[1 - F(x_y)]
$$

$$
= -2P_{rob}(X \leq x_y)[1 - P_{rob}(X \leq x_y)] = -2P_{rob}(X \leq x_y)P_{rob}(X > x_y)
$$

$$
= -2 \int_{-\infty}^{x_y} |\psi(x)|^2 dx \times \int_{x_y}^{\infty} |\psi(x)|^2 dx
$$

$$
= -2 \int_{-\infty}^{x_y} |\psi(x)|^2 dx \times \left[ 1 - \int_{-\infty}^{x_y} |\psi(x)|^2 dx \right]
$$

$$
= \left\{
\begin{array}{ll}
-2 \int_{-\infty}^{x_y} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \times \left[ 1 - \int_{-\infty}^{x_y} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \right] & x_y - \frac{L}{2} < x_y < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array}
\right.
$$

In addition, the magnitude of the chaotic factor, which is $M\text{Chf}(X)$ is:

$$
M\text{Chf}(X) = |\text{Chf}(X)| = -2iP_r(X)P_m(X) = -2iP_r(X) \times i[1 - P_r(X)]
$$

$$
= 2P_r(X)[1 - P_r(X)] = 2F(x_y)[1 - F(x_y)]
$$

$$
= 2P_{rob}(X \leq x_y)[1 - P_{rob}(X \leq x_y)] = 2P_{rob}(X \leq x_y)P_{rob}(X > x_y)
$$

$$
= 2 \int_{-\infty}^{x_y} |\psi(x)|^2 dx \times \left[ 1 - \int_{-\infty}^{x_y} |\psi(x)|^2 dx \right]
$$

$$
= 2 \int_{-\infty}^{x_y} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \times \left[ 1 - \int_{-\infty}^{x_y} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \right]
$$

$$
= \left\{
\begin{array}{ll}
2 \int_{-\infty}^{x_y} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \times \left[ 1 - \int_{-\infty}^{x_y} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \right] & x_y - \frac{L}{2} < x_y < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array}
\right.
$$

$$
= \left\{
\begin{array}{ll}
2 \int_{x_y}^{x_y + \frac{L}{2}} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \times \left[ 1 - \int_{x_y}^{x_y + \frac{L}{2}} \frac{2}{L} \sin^2 \left[ k_n (x - x_c + \frac{L}{2}) \right] dx \right] & x_y - \frac{L}{2} < x_y < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array}
\right.
$$
Finally, the real probability in the complex probability universe \( \mathcal{C} = \mathcal{R} \pmb{+} \mathcal{M} \) which is \( P_c(X) \) is:

\[
P_c^2(X) = \{ |P_c(X)| + |P_m(X)/i| \}^2 = \{|P_r(X)| + |1 - P_r(X)|\}^2 = \{|F(x_1) + [1 - F(x_2)]\}^2
\]

\[
= \{ P_{mb}(X \leq x_1) + |1 - P_{mb}(X \leq x_1)| \}^2 = \{ P_{mb}(X \leq x_1) + P_{mb}(X > x_2) \}^2
\]

\[
= \left\{ \int_{x_1}^{x_2} |\psi(x)|^2dx + \left[ 1 - \int_{x_1}^{x_2} |\psi(x)|^2dx \right] \right\}^2
\]

\[
= \left\{ \int_{x_1}^{x_2} |\psi(x)|^2dx + \left[ 1 - \int_{x_1}^{x_2} |\psi(x)|^2dx \right] \right\}^2
\]

\[
= \left\{ \int_{x_1}^{x_2} \left[ 2 \sin^2 k_0 (x - x_c + \frac{L}{2}) dx \right] \right\}^2
\]

\[
= \left\{ \int_{x_1}^{x_2} \left[ 2 \sin^2 k_0 (x - x_c + \frac{L}{2}) dx \right] \right\}^2
\]

And, \( P_c(X) \) can be computed using CPP as follows:

\[
P_c^2(X) = DOK(X) - Chf^2(X) = |P_r(X)|^2 + |P_m(X)/i|^2 - 2iP_r(X)P_m(X)
\]

\[
= |P_r(X)|^2 + |1 - P_r(X)|^2 + 2P_r(X)[1 - P_r(X)] = |P_r(X) + [1 - P_r(X)]|^2
\]

\[
= \left\{ \int_{x_1}^{x_2} |\psi(x)|^2dx + \left[ 1 - \int_{x_1}^{x_2} |\psi(x)|^2dx \right] \right\}^2
\]

\[
= \left\{ \int_{x_1}^{x_2} |\psi(x)|^2dx + \left[ 1 - \int_{x_1}^{x_2} |\psi(x)|^2dx \right] \right\}^2
\]

And, \( P_c(X) \) can be computed using always CPP as follows:

\[
P_c^2(X) = DOK(X) + MChf(X) = |P_r(X)|^2 + |P_m(X)/i|^2 + [-2iP_r(X)P_m(X)]
\]

\[
= |P_r(X)|^2 + |1 - P_r(X)|^2 + 2P_r(X)[1 - P_r(X)] = |P_r(X) + [1 - P_r(X)]|^2
\]

\[
= \left\{ \int_{x_1}^{x_2} |\psi(x)|^2dx + \left[ 1 - \int_{x_1}^{x_2} |\psi(x)|^2dx \right] \right\}^2
\]

\[
= \left\{ \int_{x_1}^{x_2} |\psi(x)|^2dx + \left[ 1 - \int_{x_1}^{x_2} |\psi(x)|^2dx \right] \right\}^2
\]
\[
\begin{align*}
= & \left\{ \begin{array}{ll}
1 & x_c - \frac{L}{2} < x_j < x_c + \frac{L}{2} \\
0 & \text{otherwise}
\end{array} \right. \\
= & P_c(X)
\end{align*}
\]

Hence, the prediction of all the wave function position probabilities of the random infinite potential well problem in the \( C = \mathcal{R} + M \) is permanently certain and perfectly deterministic.

Now, if \( x_c - \frac{L}{2} \leq L_b \) (Lower bound of \( x_j \)), \( U_b \) (Upper bound of \( x_j \)) \( \leq x_c + \frac{L}{2} \)

\[
\int_{L_b}^{U_b} \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx = \frac{2}{L} \int_{L_b}^{U_b} \left\{ 1 - \cos \left[ 2k_n \left( x - x_c + \frac{L}{2} \right) \right] \right\} dx
\]

\[
= \frac{1}{L} \int_{L_b}^{U_b} \left\{ 1 - \cos \left[ 2k_n \left( x - x_c + \frac{L}{2} \right) \right] \right\} dx
\]

\[
= \frac{1}{L} \left\{ x - \frac{\sin \left[ 2k_n \left( x - x_c + \frac{L}{2} \right) \right]}{2k_n} \right\} _{L_b}^{U_b}
\]

\[
= \frac{1}{2k_n L} \left( \left[ 2k_n U_b - \sin \left[ 2k_n \left( U_b - x_c + \frac{L}{2} \right) \right] \right] - \left[ 2k_n L_b - \sin \left[ 2k_n \left( L_b - x_c + \frac{L}{2} \right) \right] \right] \right)
\]

Thus,

\[
\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{-x_c - \frac{L}{2}} f(x)dx + \int_{-x_c - \frac{L}{2}}^{x_c + \frac{L}{2}} f(x)dx + \int_{x_c + \frac{L}{2}}^{+\infty} f(x)dx
\]

\[
= 0 + \int_{L_b = x_c - \frac{L}{2}}^{x_c - \frac{L}{2}} |\psi(x)|^2 dx + 0 = \int \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx
\]

\[
= \frac{1}{2k_n L} \left\{ \left[ 2k_n \left( x_c + \frac{L}{2} \right) - \sin \left[ 2k_n \left( x_c + \frac{L}{2} - x_c + \frac{L}{2} \right) \right] \right] \right.
\]

\[
- \left[ 2k_n \left( x_c - \frac{L}{2} \right) - \sin \left[ 2k_n \left( x_c - \frac{L}{2} - x_c + \frac{L}{2} \right) \right] \right] \right\}
\]

\[
= \frac{1}{2k_n L} \left( \left[ 2k_n x_c + k_n L - \sin \left[ 2k_n L \right] \right] - \left[ 2k_n x_c - k_n L - \sin \left[ 2k_n \left( 0 \right) \right] \right] \right)
\]

\[
= \frac{1}{2k_n L} \left( 2k_n x_c - \sin \left[ 2k_n L \right] \right)
\]
But $k_n = \frac{n \pi}{L}$, so it is equal to:

$$\frac{1}{2k_n L} \left\{ 2k_n L - \sin \left[ \frac{2n \pi L}{L} \right] \right\} = \frac{1}{2k_n L} \left\{ 2k_n L - \sin \left[ 2n \pi \right] \right\} = \frac{1}{2k_n L} (2k_n L - 0),$$

where $n = 1, 2, 3, ...$

$$\frac{2k_n L}{2k_n L} = 1$$

Therefore, $f(x) = \psi(x)^2$ is a probability density function since:

1. $\forall x : 0 \leq |\psi(x)|^2 \leq 1$, as $\forall x : -1 \leq \sin (x) \leq 1 \Leftrightarrow \forall x : 0 \leq \sin^2 (x) \leq 1$

2. $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

Moreover, if $L_b = x_c - \frac{L}{2}$ and $x_c - \frac{L}{2} \leq (U_b = x_j) \leq x_c + \frac{L}{2}$, then:

$$\int_{x_c - \frac{L}{2}}^{x_j} \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx$$

$$= \frac{1}{2k_n L} \left\{ 2k_n \left( x_j - x_c + \frac{L}{2} \right) - \sin \left[ 2k_n \left( x_j - x_c + \frac{L}{2} \right) \right] \right\}$$

$$= \frac{1}{2k_n L} \left\{ 2k_n \left( x_j - x_c + \frac{L}{2} \right) - \sin \left[ 2k_n \left( x_j - x_c + \frac{L}{2} \right) \right] - 2k_n \left( x_j - k_n L \sin \left[ 2k_n \left( x_j - x_c + \frac{L}{2} \right) \right] \right) \right\}$$

$$= \frac{1}{2k_n L} \left\{ 2k_n \left( x_j - x_c + \frac{L}{2} \right) - \sin \left[ 2k_n \left( x_j - x_c + \frac{L}{2} \right) \right] \right\}$$

Additionally, if $x_c - \frac{L}{2} \leq (L_b = x_j) \leq x_c + \frac{L}{2}$ and $U_b = x_c + \frac{L}{2}$, then:

$$\int_{x_c - \frac{L}{2}}^{x_j} \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx$$

$$= \frac{1}{2k_n L} \left\{ 2k_n \left( x_c + \frac{L}{2} \right) - \sin \left[ 2k_n \left( x_c + \frac{L}{2} - x_c + \frac{L}{2} \right) \right] \right\}$$

$$= \frac{1}{2k_n L} \left\{ 2k_n \left( x_c + \frac{L}{2} \right) - \sin \left[ 2k_n \left( x_c + \frac{L}{2} - x_j + \frac{L}{2} \right) \right] \right\}$$

But $k_n = \frac{n \pi}{L}$
So, it is equal to:

\[
\frac{1}{2kaL} \left\{ \left[ 2ka_0x_c + ka_nL - \sin \left( \frac{2n\pi L}{L} \right) \right] - \left[ 2ka_nx_j - \sin \left( \frac{2ka_n(x_j - x_c + \frac{L}{2})}{2ka_nL} \right) \right] \right\}
\]

\[
= \frac{1}{2kaL} \left\{ \left[ 2ka_0x_c + ka_nL - \sin \left[ 2n\pi \right] \right] - \left[ 2ka_nx_j - \sin \left[ 2ka_n(x_j - x_c + \frac{L}{2}) \right] \right] \right\},
\]

where \( n = 1,2,3,... \)

\[
= \frac{1}{2kaL} \left\{ \left[ 2ka_0x_c + ka_nL - 0 \right] - \left[ 2ka_nx_j - \sin \left[ 2ka_n(x_j - x_c + \frac{L}{2}) \right] \right] \right\}
\]

\[
= \frac{1}{2kaL} \left\{ 2ka_n \left( x_c - x_j + \frac{L}{2} \right) + \sin \left[ 2ka_n \left( x_j - x_c + \frac{L}{2} \right) \right] \right\}
\]

5.3 The new model simulations

The following figures (Figures 4–38) illustrate all the calculations done above.

![The Wavefunction Position Probability Density Function](image)

**Figure 4.**
The graph of the PDF of the wave function position probability distribution as a function of the random variable \( X \) for \( n = 1 \).
Figure 5.
The graphs of all the CPP parameters as functions of the random variable $X$ for the wave function position probability distribution for $n = 1$.

Figure 6.
The graphs of DOK and $\text{Chf}_n$, and the deterministic probability $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 1$. 
Figure 7. The graphs of $P_r$, and $P_m/\iota$, and $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 1$.

Figure 8. The graphs of the probabilities $P_r$, and $P_m$, and $Z$ in terms of $X$ for the wave function position probability distribution for $n = 1$. 
Figure 9. The graph of the PDF of the wave function position probability distribution as a function of the random variable $X$ for $n = 2$.

Figure 10. The graphs of all the CPP parameters as functions of the random variable $X$ for the wave function position probability distribution for $n = 2$. 
The graphs of DOK and Chf, and the deterministic probability $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 2$.

The graphs of $P_r$ and $P_{m/i}$, and $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 2$. 

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Figure 13. The graphs of the probabilities $P_r$ and $P_m$ and $Z$ in terms of $X$ for the wave function position probability distribution for $n = 2$.

Figure 14. The graph of the PDF of the wave function position probability distribution as a function of the random variable $X$ for $n = 3$. 
Figure 15. The graphs of all the CPP parameters as functions of the random variable X for the wave function position probability distribution for $n = 3$.

Figure 16. The graphs of DOK and Chf, and the deterministic probability Pc in terms of X and of each other for the wave function position probability distribution for $n = 3$. 

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Figure 17. The graphs of $P_r$ and $P_m/l$, and $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 3$.

Figure 18. The graphs of the probabilities $P_r$ and $P_m$ and $Z$ in terms of $X$ for the wave function position probability distribution for $n = 3$. 
Figure 19. The graph of the PDF of the wave function position probability distribution as a function of the random variable $X$ for $n = 4$.

Figure 20. The graphs of all the CPP parameters as functions of the random variable $X$ for the wave function position probability distribution for $n = 4$. 

23
Figure 21. The graphs of DOK and Chf, and the deterministic probability Pc in terms of X and of each other for the wave function position probability distribution for n = 4.

Figure 22. The graphs of Pr and Pm/i, and Pc in terms of X and of each other for the wave function position probability distribution for n = 4.
Figure 23. The graphs of the probabilities $P_r$, $P_m$, and $Z$ in terms of $X$ for the wave function position probability distribution for $n = 4$.

Figure 24. The graph of the PDF of the wave function position probability distribution as a function of the random variable $X$ for $n = 5$. 
Figure 25. The graphs of all the CPP parameters as functions of the random variable $X$ for the wave function position probability distribution for $n = 5$.

Figure 26. The graphs of DOK and Chf, and the deterministic probability $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 5$. 
Figure 27. The graphs of $P_r$, $P_m/i$, and $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 5$.

Figure 28. The graphs of the probabilities $P_r$ and $P_m$ and $Z$ in terms of $X$ for the wave function position probability distribution for $n = 5$. 

The Paradigm of Complex Probability and Quantum Mechanics: The Infinite Potential Well...

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Figure 29. The graph of the PDF of the wave function position probability distribution as a function of the random variable $X$ for $n = 20$.

Figure 30. The graphs of all the CPP parameters as functions of the random variable $X$ for the wave function position probability distribution for $n = 20$. 

28
Figure 31. The graphs of DOK and Chf, and the deterministic probability Pc in terms of X and of each other for the wave function position probability distribution for n = 20.

Figure 32. The graphs of Pr and Pm/i, and Pc in terms of X and of each other for the wave function position probability distribution for n = 20.
Figure 33. The graphs of the probabilities $P_r$ and $P_m$ and $Z$ in terms of $X$ for the wave function position probability distribution for $n = 20$.

Figure 34. The graph of the PDF of the wave function position probability distribution as a function of the random variable $X$ for $n = 100$. 
Figure 35. The graphs of all the CPP parameters as functions of the random variable $X$ for the wave function position probability distribution for $n = 100$.

Figure 36. The graphs of DOK and Chf, and the deterministic probability $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 100$. 

31
Figure 37. The graphs of $P_r$, $P_m/i$, and $P_c$ in terms of $X$ and of each other for the wave function position probability distribution for $n = 100$.

Figure 38. The graphs of the probabilities $P_r$ and $P_m$ and $Z$ in terms of $X$ for the wave function position probability distribution for $n = 100$. 

32
5.3.1 Simulations interpretation

In Figures 4, 9, 14, 19, 24, 29, and 34, we can see the graphs of the probability density functions (PDF) of the wave function position probability distribution for this problem as functions of the random variable $X : -100 \leq X \leq 100$ for $n = 1, 2, 3, 4, 5, 20, \text{and} 100$.

In Figures 5, 10, 15, 20, 25, 30, and 35, we can see also the graphs and the simulations of all the CPP parameters ($\text{Chf}$, $\text{MChf}$, $\text{DOK}$, $P_r$, $P_m/i$, and $P_c$) as functions of the random variable $X$ for the wave function position probability distribution of the infinite potential well problem for $n = 1, 2, 3, 4, 5, 20, \text{and} 100$. Hence, we can visualize all the new paradigm functions for this problem.

In the cubes (Figures 6, 11, 16, 21, 26, 31, and 36), the simulation of $\text{DOK}$ and $\text{Chf}$ as functions of each other and of the random variable $X$ for the infinite potential well problem wave function position probability distribution can be seen. The thick line in cyan is the projection of the plane $\text{Pc}^2(X) = \text{DOK}(X) - \text{Chf}(X) = 1 = \text{Pc}(X)$ on the plane $X = L_b$ = lower bound of $X = -100$. This thick line starts at the point $(\text{DOK} = 1, \text{Chf} = 0)$ when $X = L_b = -100$, reaches the point $(\text{DOK} = 0.5, \text{Chf} = -0.5)$ when $X = 0$, and returns at the end to $(\text{DOK} = 1, \text{Chf} = 0)$ when $X = U_b$ = upper bound of $X = 100$. The other curves are the graphs of $\text{DOK}(X)$ (red) and $\text{Chf}(X)$ (green, blue, and pink) in different simulation planes. Notice that they all have a minimum at the point $(\text{DOK} = 0.5, \text{Chf} = -0.5, \text{and} X = 0)$. The last simulation point corresponds to $(\text{DOK} = 1, \text{Chf} = 0, \text{and} X = U_b = 100)$.

In the cubes (Figures 7, 12, 17, 22, 27, 32, and 37), we can notice the simulation of the real probability $P_r(X)$ in $\mathcal{R}$ and its complementary real probability $P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable $X$ for the infinite potential well problem wave function position probability distribution. The thick line in cyan is the projection of the plane $\text{Pc}^2(X) = P_r(X) + P_m(X)/i = 1 = \text{Pc}(X)$ on the plane $X = L_b$ = lower bound of $X = -100$. This thick line starts at the point $(P_r = 0, P_m/i = 1)$ and ends at the point $(P_r = 1, P_m/i = 0)$. The red curve represents $P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light gray. This curve starts at the point $(P_r = 0, P_m/i = 1, \text{and} X = L_b$ = lower bound of $X = -100)$, reaches the point $(P_r = 0.5, P_m/i = 0.5, \text{and} X = 0)$, and gets at the end to $(P_r = 1, P_m/i = 0, \text{and} X = U_b$ = upper bound of $X = 100)$. The blue curve represents $P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1 = \text{Pc}(X)$. Notice the importance of the point, which is the intersection of the red and blue curves at $X = 0$, and when $P_r(X) = P_m(X)/i = 0.5$.

In the cubes (Figures 8, 13, 18, 23, 28, 33, and 38), we can notice the simulation of the complex probability $Z(X)$ in $\mathcal{C} = \mathcal{R} \cup \mathcal{M}$ as a function of the real probability $P_r(X) = \text{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary probability $P_m(X) = i \times \text{Im}(Z)$ in $\mathcal{M}$, and this in terms of the random variable $X$ for the infinite potential well problem wave function position probability distribution. The red curve represents $P_r(X)$ in the plane $P_m(X) = 0$, and the blue curve represents $P_m(X)$ in the plane $P_r(X) = 0$. The green curve represents the complex probability $Z(X) = P_r(X) + P_m(X) - \text{Re}(Z) + i \times \text{Im}(Z)$ in the plane $P_r(X) = X = L_b$ = lower bound of $X = -100)$. The thick line in cyan is $P_r(X = L_b = -100) = iP_m(X = L_b = -100) + 1$, and it is the projection of the $Z(X)$ curve on the complex probability plane whose equation is $X = L_b = -100$. This projected thick line starts at the point $(P_r = 0, P_m = i, X = L_b = -100)$ and ends at the point $(P_r = 1, P_m = 0, \text{and} X = L_b = -100)$. Notice the importance of the point corresponding to $X = 0$ and $Z = 0.5 + 0.5i$, when $P_r = 0.5$ and $P_m = 0.5i$. 

33
5.4 The characteristics of the position probability distribution

In quantum mechanics, the average, or expectation value of the position of a particle is given by [10]:

\[ \langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx = \int_{x_c - \frac{L}{2}}^{x_c + \frac{L}{2}} \frac{2}{L} x \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx \]

For the steady state particle in a box, it can be shown that the average position is always \( \langle x \rangle = x_c \), regardless of the state of the particle. For a superposition of states, the expectation value of the position will change based on the cross term, which is proportional to \( \cos(\omega t) \). In the probability set and universe \( \mathcal{R} \), we have:

\[ \langle x \rangle_R = \langle x \rangle = x_c \]

The variance in the position is a measure of the uncertainty in the position of the particle, so in the probability set and universe \( \mathcal{R} \), we have:

\[ \text{Var}_{x,R} = \text{Var}(x) = \langle x^2 \rangle_R - \langle x \rangle_R^2 = \left\{ \int_{-\infty}^{\infty} x^2|\psi(x)|^2 dx \right\} - \left\{ \int_{-\infty}^{\infty} x|\psi(x)|^2 dx \right\}^2 \]

\[ = \left\{ \int_{x_c - \frac{L}{2}}^{x_c + \frac{L}{2}} \frac{2}{L} x^2 \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx \right\} - x_c^2 = \frac{L^2}{12} \left( 1 - \frac{6}{n^2 \pi^2} \right) \]

In the probability set and universe \( \mathcal{M} \), we have:

\[ \langle x \rangle_M = \int_{-\infty}^{\infty} x \left( i \frac{1}{2} |\psi(x)|^2 \right) dx = i \int_{x_c - \frac{L}{2}}^{x_c + \frac{L}{2}} x \left\{ 1 - \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] \right\} dx \]

\[ = i \left\{ \int_{x_c - \frac{L}{2}}^{x_c + \frac{L}{2}} x dx - \int_{x_c - \frac{L}{2}}^{x_c + \frac{L}{2}} \frac{2}{L} x \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx \right\} = i \left\{ \frac{x_c^2}{2} - \frac{(x_c + \frac{L}{2})^2}{2} \right\} = ix_c(L - 1) \]

To simplify, consider here and in what follows that \( x_c = 0 \Rightarrow \langle x \rangle_R = 0 \) and \( \langle x \rangle_M = 0 \).

34
Moreover,

\[ \text{Var}_{x,M} = \langle x^2 \rangle_M - \langle x \rangle_M^2 = \left\{ \int_{-\infty}^{\infty} x^2 \left\{ i \left[ 1 - |\psi(x)|^2 \right] \right\} dx \right\} - \left\{ \int_{-\infty}^{\infty} x \left\{ i \left[ 1 - |\psi(x)|^2 \right] \right\} dx \right\}^2 \]

\[ = i \left\{ \int_{x_c - L/2}^{x_c + L/2} x^2 \left\{ 1 - \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] \right\} dx \right\} - \int_{x_c - L/2}^{x_c + L/2} x^2 dx \]

\[ = i \left\{ \int_{x_c - L/2}^{x_c + L/2} x^2 dx - \int_{x_c - L/2}^{x_c + L/2} x^2 \left\{ \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] \right\} dx \right\} \]

\[ = i \left\{ \int_{L/2}^{L/2} u^2 du - \text{Var}_{x,R} \right\} = i \left\{ \frac{U^3}{3} \right\}^{1/2} - \text{Var}_{x,R} \right\} = i \left\{ \frac{L^3}{12} \frac{L^2}{12} \left( 1 - \frac{6}{n^2 \pi^2} \right) \right\} \]

In the probability set and the universe \( C = R + M \), we have from CPP:

\[ \langle x \rangle_C = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx = \int_{-\infty}^{\infty} x \left\{ |\psi(x)|^2 + i \left[ 1 - |\psi(x)|^2 \right] \right\} dx \]

\[ = \int_{-\infty}^{x_c + L/2} x|\psi(x)|^2 dx + \int_{x_c + L/2}^{x_c - L/2} x \left[ 1 - |\psi(x)|^2 \right] dx \]

\[ = \int_{x_c + L/2}^{x_c} x \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] dx + \int_{x_c + L/2}^{x_c} x \left\{ 1 - \frac{2}{L} \sin^2 \left[ k_n \left( x - x_c + \frac{L}{2} \right) \right] \right\} dx \]

\[ = \langle x \rangle_R + \langle x \rangle_M = x_c + i(L - 1) = x_c + i(L - 1) = 0 \text{ for } x_c = 0 \]

\[ \text{Var}_{x,C} = \langle x^2 \rangle_C - \langle x \rangle_C^2 = \left[ \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right] - [\langle x \rangle_R + \langle x \rangle_M]^2 \]

\[ = \left[ \int_{-\infty}^{\infty} x^2 \left\{ |\psi(x)|^2 + i \left[ 1 - |\psi(x)|^2 \right] \right\} dx \right] - [\langle x \rangle_R + \langle x \rangle_M]^2 \]

\[ = \left[ \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx + \int_{-\infty}^{\infty} x^2 i \left[ 1 - |\psi(x)|^2 \right] dx \right] - [\langle x \rangle_R + \langle x \rangle_M]^2 \]
\[\begin{align*}
&= \left[ \left( \langle x^2 \rangle_R + \langle x^2 \rangle_M \right) - \left( \langle x^2 \rangle_R + \langle x^2 \rangle_M \right) \right] = \left[ \left( \langle x^2 \rangle_R + \langle x^2 \rangle_M \right) - \left( \langle x^2 \rangle_R + \langle x^2 \rangle_M \right) \right] \\
&= \frac{L^2}{12} \left( 1 - \frac{6}{n^2\pi^2} \right) + i \left( \frac{L^2}{12} \left[ L - \left( 1 - \frac{6}{n^2\pi^2} \right) \right] \right) - 2(0)(0) \\
&= \frac{L^2}{12} \left( 1 - \frac{6}{n^2\pi^2} \right) + i \left( \frac{L^2}{12} \left[ L - \left( 1 - \frac{6}{n^2\pi^2} \right) \right] \right) \\
&= \text{Var}_{xR} + \text{Var}_{xM} - 2\langle x \rangle_R \langle x \rangle_M
\end{align*}\]

The following tables (Tables 1–4) compute the position distribution characteristics for \( x_c = 0, L = 200, \) and \( n = 1,2,3,20. \)

<table>
<thead>
<tr>
<th>Position distribution characteristics</th>
<th>( x_c = 0, L = 200, n = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle x \rangle_R )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{Var}_{xR} )</td>
<td>1.3069e+03</td>
</tr>
<tr>
<td>( \langle x \rangle_M )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{Var}_{xM} )</td>
<td>i\times6.6536e+05</td>
</tr>
<tr>
<td>( \langle x \rangle_C )</td>
<td>( \langle x \rangle_R + \langle x \rangle_M )</td>
</tr>
<tr>
<td>( \text{Var}_{xC} )</td>
<td>( \text{Var}<em>{xR} + \text{Var}</em>{xM} - 2\langle x \rangle_R \langle x \rangle_M )</td>
</tr>
</tbody>
</table>

Table 1. The position distribution characteristics for \( x_c = 0, L = 200, \) and \( n = 1. \)

<table>
<thead>
<tr>
<th>Position distribution characteristics</th>
<th>( x_c = 0, L = 200, n = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle x \rangle_R )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{Var}_{xR} )</td>
<td>2.8267e+03</td>
</tr>
<tr>
<td>( \langle x \rangle_M )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{Var}_{xM} )</td>
<td>i\times6.6384e+05</td>
</tr>
<tr>
<td>( \langle x \rangle_C )</td>
<td>( \langle x \rangle_R + \langle x \rangle_M )</td>
</tr>
<tr>
<td>( \text{Var}_{xC} )</td>
<td>( \text{Var}<em>{xR} + \text{Var}</em>{xM} - 2\langle x \rangle_R \langle x \rangle_M )</td>
</tr>
</tbody>
</table>

Table 2. The position distribution characteristics for \( x_c = 0, L = 200, \) and \( n = 2. \)

<table>
<thead>
<tr>
<th>Position distribution characteristics</th>
<th>( x_c = 0, L = 200, n = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle x \rangle_R )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{Var}_{xR} )</td>
<td>3.1082e+03</td>
</tr>
<tr>
<td>( \langle x \rangle_M )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{Var}_{xM} )</td>
<td>i\times6.6356e+05</td>
</tr>
<tr>
<td>( \langle x \rangle_C )</td>
<td>( \langle x \rangle_R + \langle x \rangle_M )</td>
</tr>
<tr>
<td>( \text{Var}_{xC} )</td>
<td>( \text{Var}<em>{xR} + \text{Var}</em>{xM} - 2\langle x \rangle_R \langle x \rangle_M )</td>
</tr>
</tbody>
</table>

Table 3. The position distribution characteristics for \( x_c = 0, L = 200, \) and \( n = 3. \)
For $n \gg 1$ (large $n$) and with $x_c = 0$ we get:

$$\text{Var}_{x,R} \rightarrow \frac{L^2}{12} = 3.333 \ldots e + 03,$$

$$\text{Var}_{x,M} \rightarrow i \left\{ \frac{L^2(L-1)}{12} \right\} = i \times 6.633 \ldots e + 05$$

$$\text{Var}_{x,C} \rightarrow \frac{L^2}{12} + i \left\{ \frac{L^2(L-1)}{12} \right\} - 2(0)(0) = 3.333 \ldots e + 03 + i \times 6.633 \ldots e + 05$$

### 6. Conclusion and perspectives

In the current research work, the original extended model of eight axioms (EKA) of A. N. Kolmogorov was connected and applied to the infinite potential well problem in quantum mechanics theory. Thus, a tight link between quantum mechanics and the novel paradigm (CPP) was achieved. Consequently, the model of “Complex Probability” was more developed beyond the scope of my 19 previous research works on this topic.

Additionally, as it was proved and verified in the novel model, before the beginning of the random phenomenon simulation and at its end we have the chaotic factor ($\text{Chf}$ and $\text{MChf}$) is zero, and the degree of our knowledge ($\text{DOK}$) is one since the stochastic fluctuations and effects have either not started yet or they have terminated and finished their task on the probabilistic phenomenon. During the execution of the nondeterministic phenomenon and experiment, we also have: $0.5 \leq \text{DOK} < 1$, $-0.5 \leq \text{Chf} < 0$, and $0 < \text{MChf} \leq 0.5$. We can see that during this entire process we have incessantly and continually $P_{c}\equiv\text{DOK} - \text{Chf} = \text{DOK} + \text{MChf} = 1 = P_{c}$, which means that the simulation which behaved randomly and stochastically in the real set and universe $R$ is now certain and deterministic in the complex probability set and universe $C = R + M$, and this after adding to the random experiment executed in the real universe $R$, the contributions of the imaginary set and universe $M$, and hence, after eliminating and subtracting the chaotic factor from the degree of our knowledge. Furthermore, the real, imaginary, complex, and deterministic probabilities that correspond to each value of the position random variable $X$ have been determined in the three probabilities sets and universes, which are $R$, $M$, and $C$ by $P_{r}$, $P_{m}$, $Z$ and $P_{c}$, respectively. Consequently, at each value of $X$, the novel quantum mechanics and CPP parameters $P_{r}$, $P_{m}$, $P_{m}/i$, $\text{DOK}$, $\text{Chf}$, $\text{MChf}$, $P_{c}$, and $Z$ are surely
and perfectly predicted in the complex probabilities set and universe $C$ with $P_c$ maintained equal to one permanently and repeatedly.

In addition, referring to all these obtained graphs and executed simulations throughout the whole research work, we are able to quantify and visualize both the system chaos and stochastic effects and influences (expressed and materialized by $Chf$ and $MChf$) and the certain knowledge (expressed and materialized by $DOK$ and $P_c$) of the new paradigm. This is without any doubt very fruitful, wonderful, and fascinating and proves and reveals once again the advantages of extending A. N. Kolmogorov's five axioms of probability, and hence, the novelty and benefits of my inventive and original model in the fields of prognostics, applied mathematics, and quantum mechanics that can be called verily: “The Complex Probability Paradigm.”

As a future and prospective research and challenges, we aim to develop the novel prognostic paradigm conceived and implement it in a large set of random and nondeterministic phenomena in quantum mechanics theory.

**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{R}$</td>
<td>real set of events and probabilities.</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>imaginary set of events and probabilities.</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>complex set of events and probabilities.</td>
</tr>
<tr>
<td>$i$</td>
<td>the imaginary number where $i = \sqrt{-1}$ or $i^2 = -1$</td>
</tr>
<tr>
<td>$EKA$</td>
<td>Extended Kolmogorov’s Axioms.</td>
</tr>
<tr>
<td>$CPP$</td>
<td>complex probability paradigm.</td>
</tr>
<tr>
<td>$P_{rob}$</td>
<td>probability of any event.</td>
</tr>
<tr>
<td>$P_r$</td>
<td>probability in the real set $\mathcal{R}$.</td>
</tr>
<tr>
<td>$P_m$</td>
<td>probability in the imaginary set $\mathcal{M}$ corresponding to the real probability in $\mathcal{R}$.</td>
</tr>
<tr>
<td>$P_c$</td>
<td>probability of an event in $\mathcal{R}$ with its associated complementary event in $\mathcal{M} =$ probability in the complex probability set $\mathcal{C}$.</td>
</tr>
<tr>
<td>$Z$</td>
<td>complex probability number = sum of $P_r$ and $P_m$ = complex random vector</td>
</tr>
<tr>
<td>$DOK =</td>
<td>Z</td>
</tr>
<tr>
<td>$Chf$</td>
<td>the chaotic factor of $Z$.</td>
</tr>
<tr>
<td>$MChf$</td>
<td>magnitude of the chaotic factor of $Z$.</td>
</tr>
<tr>
<td>$</td>
<td>\psi(x)</td>
</tr>
<tr>
<td>$</td>
<td>\phi(p)</td>
</tr>
<tr>
<td>$\langle x \rangle_{\mathcal{R}}, \langle x \rangle_{\mathcal{M}}, \langle x \rangle_{\mathcal{C}}$</td>
<td>means, expectations, or averages of the wave function position probability distribution function in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$, respectively.</td>
</tr>
<tr>
<td>$\text{Var}<em>{x,\mathcal{R}}, \text{Var}</em>{x,\mathcal{M}}, \text{Var}_{x,\mathcal{C}}$</td>
<td>variances of the wave function position probability distribution function in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$, respectively.</td>
</tr>
<tr>
<td>$\langle p \rangle_{\mathcal{R}}, \langle p \rangle_{\mathcal{M}}, \langle p \rangle_{\mathcal{C}}$</td>
<td>means, expectations, or averages of the wave function momentum probability distribution function in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$, respectively.</td>
</tr>
<tr>
<td>$\text{Var}<em>{p,\mathcal{R}}, \text{Var}</em>{p,\mathcal{M}}, \text{Var}_{p,\mathcal{C}}$</td>
<td>variances of the wave function momentum probability distribution function in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$, respectively.</td>
</tr>
</tbody>
</table>
The Paradigm of Complex Probability and Quantum Mechanics: The Infinite Potential Well...
DOI: http://dx.doi.org/10.5772/intechopen.107300

\( H_x^R \) : particle position entropy in the real universe \( \mathcal{R} \).

\( NegH_x^R \) : particle position negative entropy in the real universe \( \mathcal{R} \).

\( \overline{H}_x^R \) : particle position complementary entropy in the real universe \( \mathcal{R} \).

\( H_x^M \) : particle position entropy in the imaginary universe \( \mathcal{M} \).

\( H_x^C \) : particle position entropy in the complex universe \( \mathcal{C} \).

\( H_p^R \) : particle momentum entropy in the real universe \( \mathcal{R} \).

\( NegH_p^R \) : particle momentum negative entropy in the real universe \( \mathcal{R} \).

\( \overline{H}_p^R \) : particle momentum complementary entropy in the real universe \( \mathcal{R} \).

\( H_p^M \) : particle momentum entropy in the imaginary universe \( \mathcal{M} \).

\( H_p^C \) : particle momentum entropy in the complex universe \( \mathcal{C} \).

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