We are IntechOpen, the world’s leading publisher of Open Access books
Built by scientists, for scientists

6,600
Open access books available

177,000
International authors and editors

195M
Downloads

154
Countries delivered to

TOP 1%
Our authors are among the most cited scientists

12.2%
Contributors from top 500 universities

WEB OF SCIENCE™
Selection of our books indexed in the Book Citation Index in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com
Chapter

Some Tauberian Theorems under Triple Statistically Nörlund-Cesáro Summability Method

Carlos Granados

Abstract

In this paper, we extend the notion presented by Braha (2020) in a higher dimension, we introduce the notion of $N_{n,m,g}^{p,q}C_{1,1,1}(1,1,1)$ statistically convergence and show necessity and sufficiency conditions under which the existence of the limit $\lim_{n,m,g \to \infty} x_{n,m,g} = L$ follows from that $\lim_{n,m,g \to \infty} N_{n,m,g}^{p,q}C_{1,1,1}(1,1,1) = L$. These conditions are one-sided or two-sided if $(x_{n,m,g})$ is a sequence of real or complex numbers, respectively.

Keywords: Nörlund-Cesaro summability method, one-sided and two-sided Tauberian conditions, triple statistical convergence

1. Introduction

The concept of statistical convergence was introduced by Fast [1] and Steinhaus [2]. Besides, in this connection, Fridy [3] showed some relation to a Tauberian condition for the statistical convergence of $(x_k)$. Subsequently, many researchers have worked in this area in several settings. For more recent works in this direction, one may refer to [4, 5]. Existing works in this field based on statistical convergence appears to have been restricted to real or complex sequences; however, Parida et al. [6] extended the idea for a locally convex Hausdorff topological linear space. Tauber [7] introduced the first Tauberian theorems for single sequences, that an Abel summable sequence is convergent with some suitable conditions. Later, a huge number of authors such as Landau [8], Hardy and Littlewood [9], and Schmidt [10] obtained some classical Tauberian theorems for Cesáro and Abel summability methods of single sequences. Recently, Braha [11] introduced some notions on statistical convergence by using the Nörlund-Cesáro summability method in a single sequence and proved some Tauberian theorems. In the last year, Canak and Totur [12], and Jena et al. [13] presented and studied several Tauberian theorems for single sequences. On the other hand, Knopp [14] obtained some classical type Tauberian theorems for Abel and $(C, 1, 1)$ summability methods of double sequences and proved that Abel and $(C, 1, 1)$ summability methods hold for the set of bounded sequences. Further, Moricz [15] proved some Tauberian theorems for Cesáro summable double sequences and deduced Tauberian theorems of Landau [16] and Hardy [17] type. Canak and Totur [18]
have proved a Tauberian theorem for Cesàro summability of single integrals and also
the alternative proofs of some classical type Tauberian theorems for the Cesàro
summability of single integrals and later introduced by Parida et al. [6] for double
integrals. Otherwise, the notion of \((C, 1, 1, 1)\) summability of a triple sequence was
originally introduced by Canak and Totur in 2016 [19]. Later, Canak et al. [20] studied
some \((C, 1, 1, 1)\) means of a statistical convergent triple sequence and gave some
classical Tauberian theorems for a triple sequence that \(P\)-convergence follows from
statistically \((C, 1, 1, 1)\) summability under the two-sided boundedness conditions and
slowly oscillating conditions in certain senses. Then, in 2020 Totur and Canak [21]
proved Tauberian conditions under which convergence of triple integrals follows
from \((C, 1, 1, 1)\) summability. For more studies associated to the main topic of this
paper, we refer the reader to [22–24].

Let \(\langle p_{n,m,g} \rangle\) and \(\langle q_{n,m,g} \rangle\) be any two non-negative real sequences with

\[
R_{n,m,g} = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{i,j,k} q_{n-i,m-j-k} \neq 0 \quad ((n, m, g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N})
\]

and \((C, 1, 1, 1)\)-Cesàro summability method. Let \(\langle x_{n,m,g} \rangle\) be a sequence of real of
calculative numbers and set

\[
N_{p,q}^{n,m,g} C_{n,m,g}^{(1,1,1)} = \frac{1}{R_{n,m,g}} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{i,j,k} q_{n-i,m-j-k} \frac{1}{i+1} \frac{1}{j+1} \frac{1}{k+1} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} x_{u,v,y}
\]

for \((n, m, g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}.

In this paper, we show necessary and sufficient conditions under which the
existence of the limit \(\lim_{n,m,g \to \infty} x_{n,m,g} = L\) follows from that of \(\lim_{n,m,g \to \infty} N_{p,q}^{n,m,g} C_{n,m,g}^{(1,1,1)} = L\).

These conditions are one-sided or two-sided if \(\langle x_{n,m,g} \rangle\) is a sequence of real or complex
numbers, respectively.

Given two non-negative sequences \(\langle p_{n,m,g} \rangle\) and \(\langle q_{n,m,g} \rangle\), the convolution \((p \ast q)\) is
defined by

\[
R_{n,m,g} = (p \ast q)_{n,m,g} = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{i,j,k} q_{n-i,m-j-k} = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{n-i,m-j-k} q_{i,j,k}
\]

with \((C, 1, 1, 1)\) we will denote the triple Cesàro summability method. Now, let
\(\langle x_{n,m,g} \rangle\) be a sequence, when \((p \ast q)_{n,m,g} \neq 0\) for all \((n, m, g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\) the general-
zized Nörlund-Cesàro transform of the sequence \(\langle x_{n,m,g} \rangle\) is the sequence \(N_{p,q}^{n,m,g} C_{n,m,g}^{(1,1,1)}\)
obtained by putting

\[
N_{p,q}^{n,m,g} C_{n,m,g}^{(1,1,1)} = \frac{1}{(p \ast q)_{n,m,g}} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{i,j,k} q_{n-i,m-j-k} \frac{1}{i+1} \frac{1}{j+1} \frac{1}{k+1} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} x_{u,v,y}
\]

(1)

We say that the sequence \(\langle x_{n,m,g} \rangle\) is generalized Nörlund-Cesàro summable to \(L\)
determined by the sequences \(\langle p_{n,m,g} \rangle\) and \(\langle q_{n,m,g} \rangle\) (or simply summable \(N_{p,q}^{n,m,g} C_{n,m,g}^{(1,1,1)}\))
to \(L\) if
that there are constants define the following sequence

Some Tauberian Theorems under Triple Statistically Nörlund-Cesáro Summability Method
DOI: http://dx.doi.org/10.5772/intechopen.106141

convergence of sequences

Tauberian conditions are said to be Tauberian theorems for the

Let us consider that

and as we know, \( x = (x_{i,j,k}) \) is not convergent. Notice that (6) can imply (2) under a certain condition, which is called a Tauberian conditions. Any theorem which states that convergence of a sequence follows from its statistical summability and some Tauberian conditions are said to be a Tauberian theorems for the statistical Nörlund-Cesáro summability method.

Next, we will find some conditions under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions statistical convergence of sequences \( (x_{n,m,g}) \), follows from statistically Nörlund-Cesáro summability method.

A sequence \( (x_{n,m,g}) \) is weighted statistically convergent to \( L \) if for every \( \varepsilon > 0 \),

\[
\lim_{n,m,g \to \infty} \frac{1}{(p \ast q)_{n,m,g}} \sum_{i,j,k \leq (p \ast q)_{n,m,g}} \frac{1}{i+j+k+1} \sum_{u=0}^{n} \sum_{v=0}^{m} \sum_{y=0}^{g} x_{u,v,y} - L |\geq \varepsilon| = 0.
\]
And we say that the sequence \( \{x_{n,m,g}\} \) is statistically summable to \( L \) by the weighted summability method \( \mathcal{N}^{n,m,g}C^{(1,1)} \) if \( \lim_{n,m,g} \mathcal{N}^{n,m,g}C^{(1,1)}(x_{n,m,g}) = L \). We will denote by \( \mathcal{N}^{n,m,g}C^{(1,1)}(x_{n,m,g}) \) the set of all sequences which are statistically summable, by the weighted summability method \( \mathcal{N}^{n,m,g}C^{(1,1)} \).

**Theorem 1.2** Let \( R \leq g \) and \( \sum_{i,j,k} x_{i,j,k} \leq g \) for every \( x_{n,m,g} \):

- \( R \leq g \) is not \( \mathcal{N}^{n,m,g}C^{(1,1)} \)-summable to 0.
- \( \sum_{i,j,k} x_{i,j,k} \leq g \) converges statistically to 0.

**Proof:** From the fact that \( \sum_{i,j,k} x_{i,j,k} \leq g \) converges statistically to 0, we have

\[
\lim_{n,m,g} \mathcal{N}^{n,m,g}C^{(1,1)}(x_{n,m,g}) = L.
\]

We will denote \( E \) and \( \overline{E} \) as follows:

\[
E = \{i,j,k \leq n, m, g : |x_{i,j,k} - L| \geq \varepsilon\}, \quad \overline{E} = \{i,j,k \leq n, m, g : |x_{i,j,k} - L| \leq \varepsilon\}.
\]

Then,

\[
\mathcal{N}^{n,m,g}C^{(1,1)}(x_{n,m,g}) = \lim_{n,m,g} \frac{\sum_{i,j,k} x_{i,j,k} - L}{R_{n,m,g}}.
\]
Some Tauberian Theorems under Triple Statistically Nörlund-Cesáro Summability Method

DOI: http://dx.doi.org/10.5772/intechopen.106141

Let (13) satisfies, then for every \( \in \mathbb{R} \),

\[
P_{i,j,k} a_{i,m-j-g-k} \leq \frac{1}{R_{n,m,g}} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{g} \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} |a_{u,v,y} - L|
\]

\[
+ \frac{1}{R_{n,m,g}} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{g} \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} |a_{u,v,y} - L|
\]

\[
\leq \frac{1}{R_{n,m,g}} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{g} \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} |a_{u,v,y} - L|
\]

\[
1 + \epsilon \leq M \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{g} \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} |a_{u,v,y} - L|
\]

as \( n, m, g \to \infty \),

for some constant \( C_2 \).

Converse of Theorem 1.2 is not true as can be seen in the following example.

Consider that \( p_{n,m,g} = (n+1)(m+1)(g+1) \), \( q_{n,m,g} = 1 \) for some \( (n, m, g) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \) and define the sequence \( x = (x_{n,m,g}) \) as follows:

\[
x_{i,j,k} = \begin{cases} 1, & \text{for } i = p^2 - p, \ldots, p^2 - j = t^2 - t, \ldots, t^2 - 1 \text{ and } k = o^2 - o, \ldots, o^2 - 1; \\ \frac{1}{p^0}, & \text{for } i = p^2, p = 2, \ldots, j = t^2, t = 2, \ldots, \text{ and } k = o^2, o = 2, \ldots \\ 0, & \text{otherwise} \end{cases}
\]

Under these conditions, after some basic calculations we get that \( x = (x_{n,m,g}) \) is \( N^m_{p,q} C_{n,m,g}^{(1,1,1)} \)-summable to 1. Therefore, by Theorem 1.2, \( x = (x_{n,m,g}) \) is \( N^m_{p,q} C_{n,m,g}^{(1,1,1)} \)-statistically convergent. On the other hand, the sequences \( p^2; p = 2, 3, \ldots, t^2; t = 2, 3, \ldots \) have natural density zero and it is clear that \( s^{-}\lim \inf_{n,m,g} x_{n,m,g} = 0 \) and \( s^{-}\lim \sup_{n,m,g} x_{n,m,g} = 1 \). Hence, \( (x_{i,j,k}) \) is not statistically convergent.

2. Tauberian theorems under \( N^m_{p,q} C_{n,m,g}^{(1,1,1)} \)-statistically convergence

In this section, we show the results that we obtained. Throughout this paper, \( R_{i,j,k} \) and \( R_{i,j,k}^{\epsilon} \) will have the same meaning.

Consider that \( s^{-}\lim i,j,kx_{i,j,k} = L; (x_{n,m,g}) \in N^m_{p,q} C_{n,m,g}^{(1,1,1)} \)-statistically convergent and (13) satisfies, then for every \( t > 1 \), is valid the following relation

\[
s^{-}\lim_{i,j,k} \frac{1}{R_{i,j,k}^{\epsilon} - R_{i,j,k}^{\epsilon}} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} p_{u,v,y} q_{u,v,y} x_{u,v,y} - x_{i,j,k} = 0
\]

and in case where \( 0 < t < 1 \),
which is enough. Under this condition, we obtain

\[ \frac{1}{(w+1)(e+1)(r+1)} \sum_{i=0}^{j} \sum_{j=0}^{k} (x_{i,j,k} - x_{w,e,r}) = 0. \]

The condition given by relation (13) is equivalent to the condition

\[ \lim_{n,m,g \to \infty} R_{n,m,g} > 1, \quad 0 < \lambda < 1. \]  (9)

**Proof:** Suppose that relation (13) is valid, \( 0 < \lambda < 1, w = \lambda n, e = \lambda m, r = \lambda g \) and \( r = \lambda g = [\lambda g], (n, m, g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \). Then, it follows that

\[ \frac{1}{\lambda} > 1 \Rightarrow \frac{w}{\lambda} = \frac{\lambda n}{\lambda} \leq n, \quad \frac{1}{\lambda} > 1 \Rightarrow \frac{e}{\lambda} = \frac{\lambda m}{\lambda} \leq m \quad \text{and} \quad \frac{1}{\lambda} > 1 \Rightarrow \frac{r}{\lambda} = \frac{\lambda g}{\lambda} \leq g. \]

From above relation and definition of sequences \( (p_{n,m,g}) \) and \( (q_{n,m,g}) \), we have

\[ \frac{R_{n,m,g}}{R_{n,m,g}} \geq \liminf_{n,m,g \to \infty} \frac{R_{n,m,g}}{R_{n,m,g}} \geq \liminf_{n,m,g \to \infty} \frac{R_{n,m,g}}{R_{n,m,g}} > 1. \]

Conversely, suppose that (9) is valid. Now, let \( \lambda > 1 \) be given and let \( \lambda_1, \lambda_2, \lambda_3 \) be chosen such that \( 1 < \lambda_1, \lambda_2, \lambda_3 < \lambda \). Set \( w = \lambda n, e = \lambda m, r = \lambda g \) and \( r = \lambda g = [\lambda g] \). From \( 0 < \frac{1}{\lambda} < \lambda_1, \lambda_2, \lambda_3 < 1, \) it follows that

\[ n \leq \lambda n - 1 < \frac{\lambda n}{\lambda_1} = \frac{w}{\lambda_1}, \quad m \leq \lambda m - 1 < \frac{\lambda m}{\lambda_2} = \frac{e}{\lambda_2} \quad \text{and} \quad g \leq \lambda g - 1 < \frac{\lambda g}{\lambda_3} = \frac{r}{\lambda_3} \]

provided \( \lambda_1, \lambda_2, \lambda_3 \leq 1 - \frac{1}{\lambda}, \lambda - \frac{1}{\lambda}, \lambda - \frac{1}{\lambda} \), which is a case where if \( n, m, g \) are large enough. Under this condition, we obtain

\[ \frac{R_{n,m,g}}{R_{n,m,g}} \geq \liminf_{n,m,g \to \infty} \frac{R_{n,m,g}}{R_{n,m,g}} \geq \liminf_{n,m,g \to \infty} \frac{R_{n,m,g}}{R_{n,m,g}} > 1. \]

Consider that (13) is satisfied and let \( x = (x_{i,j,k}) \) be a sequence of complex numbers which is \( N_{-\lambda,g} C_{1,1,1}^{(1,1,1)} \) statistically convergent to \( L \). Then,

\[ \lim_{n,m,g \to \infty} \frac{1}{R_{n,m,g}} \sum_{i=0}^{j} \sum_{j=m+1}^{k} \sum_{k=g+1}^{j} P_{i,j,k} \lambda_{i-j,m-j,k-g} \]

\[ \frac{1}{(i+1)(j+1)(k+1)} \sum_{i=0}^{j} \sum_{j=0}^{k} x_{i,j,k} = L \quad \text{for} \quad \lambda > 1 \]

(10)
Some Tauberian Theorems under Triple Statistically Nörlund–Césaro Summability Method
DOI: http://dx.doi.org/10.5772/intechopen.106141

and

$$\text{ST} = \lim_{n,m,g \to \infty} \frac{1}{R_{n,m,g} - R_{n,m,g}} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} P_{i,j,k} q_{n-i,m-j,g-k}$$
$$\frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{w=0}^{k} x_{u,v,w} = L \text{ for } 0 < \lambda < 1.$$ (11)

**Proof:** We begin proving the case (10), i.e. when $\lambda > 1$. Then, we have

$$\frac{1}{R_{n,m,g} - R_{n,m,g}} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} P_{i,j,k} q_{n-i,m-j,g-k} (x_{u,v,w} - L)$$
$$= \frac{R_{n,m,g}}{R_{n,m,g} - R_{n,m,g}} \frac{1}{i+1 \lambda \mu \nu} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} P_{i,j,k} q_{n-i,m-j,g-k} (x_{u,v,w} - L)$$
$$\frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{w=0}^{k} x_{u,v,w} = L$$
$$= R_{n,m,g} \frac{1}{R_{n,m,g} - R_{n,m,g}} \frac{1}{i+1 \lambda \mu \nu} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} P_{i,j,k} q_{n-i,m-j,g-k} (x_{u,v,w} - L)$$
$$= R_{n,m,g} \frac{1}{R_{n,m,g} - R_{n,m,g}} \frac{1}{i+1 \lambda \mu \nu} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} P_{i,j,k} q_{n-i,m-j,g-k} (x_{u,v,w} - L)$$
$$1 \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{w=0}^{k} x_{u,v,w} = L$$
$$= R_{n,m,g} \frac{1}{R_{n,m,g} - R_{n,m,g}} \frac{1}{i+1 \lambda \mu \nu} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} P_{i,j,k} q_{n-i,m-j,g-k} (x_{u,v,w} - L)$$
$$1 \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{w=0}^{k} x_{u,v,w} = L$$
$$= R_{n,m,g} \frac{1}{R_{n,m,g} - R_{n,m,g}} \frac{1}{i+1 \lambda \mu \nu} \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{g} P_{i,j,k} q_{n-i,m-j,g-k} (x_{u,v,w} - L)$$

(12)
From (12), definition of the sequence \((q_{n,m,g})\) and relation \(\limsup_{n,m,g} R_{n,m,g} < \infty\), we get (10).

Prove of (11) is made similarly to the prove of (10).

In the following theorem, we characterize the converse implication when the statistically convergence follows from its \(N_{p,q}^m C_{n,m,g}^{1,1,1}\) statistically convergence.

**Theorem 1.3** Let \((p_{n,m,g})\) and \((q_{n,m,g})\) be two non-negative real sequences and

\[
\liminf_{n,m,g} \frac{R_{n,m,g}}{R_{n,m,g}} > 1 \quad \text{for every } \lambda > 1,
\]

(13)

where \(\lambda_{n,m,g} = \lambda_{n,m,g}^{\lambda} = [\lambda n]\lambda m|\lambda g|\) denotes the integral part of \(\lambda n\lambda m\lambda g\) for every \((n,m,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\), and let \((x_{n,m,g})\) be a sequence of real numbers which is \(N_{p,q}^m C_{n,m,g}^{1,1,1}\)-statistically convergent to a finite number \(L\). Then, \((x_{n,m,g})\) is st-convergent to the same number \(L\) if and only if the following two conditions hold

\[
\inf_{\lambda > 1} \limsup_{n,m,g} \frac{1}{R_{n,m,g}} \left\{ i,j,k \leq R_{n,m,g} : \frac{1}{R_{i,j,k}} \sum_{i=1}^{\lambda_n} \sum_{j=1}^{\lambda_m} \sum_{k=1}^{\lambda_g} p_{w,x,r} q_{i,w,j-r-q_{i,j,k}} \right\} = 0,
\]

(14)

and

\[
\inf_{0 < i < 1} \limsup_{n,m,g} \frac{1}{R_{n,m,g}} \left\{ i,j,k \leq R_{n,m,g} : \frac{1}{R_{i,j,k}} \sum_{i=1}^{\lambda_n} \sum_{j=1}^{\lambda_m} \sum_{k=1}^{\lambda_g} p_{w,x,r} q_{i-w,j-r-q_{i,j,k}} \right\} = 0.
\]

(15)

**Proof: Necessity**: Suppose that \(\lim_{n,m,g \to \infty} x_{n,m,g} = L\) and (13) holds. By Proposition 2, we have

\[
\lim_{n,m,g \to \infty} R_{n,m,g} = \frac{\sum_{i=1}^{\lambda_n} \sum_{j=1}^{\lambda_m} \sum_{k=1}^{\lambda_g} p_{i,j,k} q_{i-w,j-r-q_{i,j,k}}}{(i+1)(j+1)(k+1)} \sum_{i=0}^{\lambda_n} \sum_{j=0}^{\lambda_m} x_{n,m,g}
\]

\[
= \lim_{n,m,g \to \infty} \left\{ \frac{1}{R_{n,m,g}} \sum_{i=1}^{\lambda_n} \sum_{j=1}^{\lambda_m} \sum_{k=1}^{\lambda_g} p_{i,j,k} q_{i-w,j-r-q_{i,j,k}} \right\} = 0,
\]

8
for every $\lambda > 1$. In case where $0 < \lambda < 1$, we have that

$$
\lim_{n,m,g \to \infty} \frac{1}{R_{n,m,g} - R_{n,m,g}} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{i,j,k} q_{n-i,m-j,g-k} \quad \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{w=0}^{k} (x_{n,m,g} - x_{u,v,w}) = 0.
$$

**Sufficiency:** Consider that $(14)$ and $(15)$ are satisfied. In what follows, we will prove that

$$
\lim_{n,m,g \to \infty} x_{n,m,g} = L. \quad \text{Given any } \epsilon > 0, \text{ by } (14) \text{ we can choose } \lambda_1 > 0 \text{ such that}
$$

$$
\liminf_{n,m,g \to \infty} \frac{1}{R_{n,m,g} - R_{n,m,g}} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{i,j,k} q_{n-i,m-j,g-k} \quad \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{w=0}^{k} (x_{n,m,g} - x_{u,v,w}) \geq - \epsilon,
$$

where $\lambda_m = \lfloor \lambda m \rfloor$, $\lambda_m = \lfloor \lambda m \rfloor$, $\lambda_m = \lfloor \lambda m \rfloor$ and $\lambda_m = \lfloor \lambda m \rfloor$. By the assumed summability $N_{p,q}^{(1,1,1)} C_{n,m,g}^{(1,1,1)}$ of $(x_{n,m,g})$, Proposition 2 and (16), we have

$$
\limsup_{n,m,g \to \infty} x_{n,m,g} \leq L + \epsilon,
$$

for any $\lambda > 1$. On the other hand, if $0 < \lambda < 1$, for every $\epsilon > 0$, we can choose $0 < \lambda_2 < 1$ such that

$$
\liminf_{n,m,g \to \infty} \frac{1}{R_{n,m,g} - R_{n,m,g}} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{g} p_{i,j,k} q_{n-i,m-j,g-k} \quad \frac{1}{(i+1)(j+1)(k+1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{w=0}^{k} (x_{n,m,g} - x_{u,v,w}) \geq - \epsilon,
$$

where $\lambda_2 = \lfloor \lambda m \rfloor$, $\lambda_2 = \lfloor \lambda m \rfloor$, $\lambda_2 = \lfloor \lambda m \rfloor$ and $\lambda_2 = \lfloor \lambda m \rfloor$. By the assumed summability $N_{p,q}^{(1,1,1)} C_{n,m,g}^{(1,1,1)}$ of $(x_{n,m,g})$, Proposition 2 and (18), we have

$$
\liminf_{n,m,g \to \infty} x_{n,m,g} \geq L - \epsilon,
$$

for any $\lambda > 1$. In case where $0 < \lambda < 1$, for every $\epsilon > 0$, we can choose $0 < \lambda_1 < 1$ such that
for any $0 < \lambda < 1$.

Since $\varepsilon > 0$ is arbitrary, combining (17) and (19), we obtain

$$
\lim_{n,m,g \to \infty} x_{n,m,g} = L.
$$

In the following theorem, we will consider the case where $x = (x_{n,m,g})$ is a sequence of complex numbers.

**Theorem 1.4** Let (13) be satisfied and let $(x_{n,m,g})$ be a sequence of complex numbers which is $\lambda_{p,q,g} C^{(1,1,1)}_{n,m,g}$-statistically convergent to a finite number $L$. Then, $(x_{n,m,g})$ is convergent to the same number $L$ if and only if the following two conditions hold

$$
\inf \limsup_{\lambda > 1} \frac{1}{R_{n,m,g}} \left\{ i,j,k \leq R_{n,m,g} : \frac{1}{R_{i,j,k}} - \frac{1}{R_{i,j,k}} \sum_{w=1}^{i} \sum_{v=1}^{j} \sum_{r=1}^{k} p_{w,v,r} l_{w-v,-r}^{e} \right\} = 0,
$$

(20)

and

$$
\inf \limsup_{0 < \lambda < 1} \frac{1}{R_{n,m,g}} \left\{ i,j,k \leq R_{n,m,g} : \frac{1}{R_{i,j,k}} - \frac{1}{R_{i,j,k}} \sum_{w=1}^{i} \sum_{v=1}^{j} \sum_{r=1}^{k} p_{w,v,r} l_{w-v,-r,k} \right\} = 0.
$$

(21)

**Proof:** Necessity: If both (2) and (6) hold, then Proposition 2 yields (20) for every $\lambda > 1$ and (21) for every $0 < \lambda < 1$.

**Sufficiency:** Suppose that (2), (13) and one of the conditions (20) and (21) are satisfied. For any given $\varepsilon > 0$, there exists $\lambda_1 > 0$ such that

$$
\limsup_{n,m,g \to \infty} \left\{ R_{i,j,k} l_{i,j,k} - R_{i,j,k} \sum_{w=1}^{i} \sum_{v=1}^{j} \sum_{r=1}^{k} p_{w,v,r} l_{w-v,-r} \right\} \leq \varepsilon,
$$

where $\lambda_{i} = [\lambda_{i}]$, $\lambda_{m} = [\lambda_{m}]$ and $\lambda_{g} = [\lambda_{g}]$. Taking into account fact that $(x_{n,m,g})$ is $\lambda_{p,q,g} C^{(1,1,1)}_{n,m,g}$ summable to $L$ and Proposition 2, we have the following estimation.
Some Tauberian Theorems under Triple Statistically Nörlund-Cesáro Summability Method

\[ \limsup_{n, m, g \to \infty} |L - x_{n, m, g}| \leq \limsup_{n, m, g \to \infty} \left| \frac{1}{R_{n, m, g}} - \frac{1}{R_{n, m}} \sum_{j=0}^{\lambda_1} \sum_{j=0}^{\lambda_2} \sum_{j=0}^{\lambda_3} P_{ij, k} Q_{n, m, g - i, j, k} \right| \]

\[ \leq \limsup_{n, m, g \to \infty} \left| \frac{1}{(i + 1)(j + 1)(k + 1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} x_{u, v, y} \right| \]

\[ + \limsup_{n, m, g \to \infty} \left| \frac{1}{(i + 1)(j + 1)(k + 1)} \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{y=0}^{k} (x_{n, m, g} - x_{u, v, y}) \right| \]

\[ \leq \varepsilon. \]

For a given \( \varepsilon > 0 \), there exists \( \lambda_2 > 0 \) such that

\[ \limsup_{n, m, g \to \infty} \left| \frac{1}{R_{n, m, g}} - \frac{1}{R_{n, m}} \sum_{j=0}^{\lambda_1} \sum_{j=0}^{\lambda_2} \sum_{j=0}^{\lambda_3} P_{ij, k} Q_{n, m, g - i, j, k} \right| \leq \varepsilon, \]

where \( \lambda_1 = [\lambda_2] \), \( \lambda_2 = [\lambda_m] \) and \( \lambda_3 = [\lambda_g] \). Taking into account fact that \( (x_{n, m, g}) \) is \( N^{n, m, g}_{1,1,1} \) submable to \( L \) and Proposition 2, we obtain the following

\[ \limsup_{n, m, g \to \infty} |L - x_{n, m, g}| \leq \varepsilon. \]

Since \( \varepsilon > 0 \) in either case, we get

\[ \lim_{n, m, g \to \infty} x_{n, m, g} = L. \]
3. Conclusion

In this paper, we have defined and proved new Tauberian theorems under triple statistically Nörlund-Cesáro summability, as a consequence of results showed in 2, some theorems, lemmas and corollaries can be defined and proved similarly by using $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ method of summability. It is well know that Tauberian theorems for single sequences of single variable have been achieved a high degree of development; however, it is still in its infancy for triple sequences. For that reason, the results established in this paper can be extended and studied in some inclusion, Tauberian type theorems and Tauberian convexity type for certain families of generalized Nörlund.

Author details

Carlos Granados
Universidad de Antioquia, Colombia

*Address all correspondence to: carlosgranadosortiz@outlook.es

IntechOpen

© 2022 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
References


[19] Canak I, Totur U. Some classical Tauberian theorems for (C, 1, 1, 1, 1)


