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Chapter

Applications of Fuzzy Set and Fixed Point Theory in Dynamical Systems

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Abstract

This chapter shall discuss various applications of fixed-point theory and fuzzy set theory. Fixed point theory and fuzzy set theory are very useful tools that are applicable in almost all branches of mathematical analysis. There are many problems that cannot be solved by applying the concept of other existing theories but can be solved easily by using the concept of fuzzy set theory and fixed point theory. So here in this chapter, we shall introduce the fuzzy set theory and fixed point theory concerning their applications in existing branches of science, engineering, mathematics, and dynamical systems.

Keywords: fixed point, fuzzy set, dynamical systems, stability, fuzzy differential equations, integral equations

1. Introduction

Fixed point theory is an area of mathematics linked to functional analysis and topology that is still in its infancy. Fixed point theory is an important subject in the fast-growing domains of nonlinear analysis and nonlinear operators. It is a relatively new scientific area that is developing rapidly. In topics as diverse as differential equations, topology, economics, game theory, dynamics, optimal control, and functional analysis, fixed points and fixed point theorems have always been important theoretical tools. Furthermore, with the development of accurate and efficient techniques for computing fixed points, the concept’s relevance for applications has expanded dramatically, making fixed point methods a vital weapon in the arsenal of the applied mathematician.

Set theory, general topology, algebraic topology, and functional analysis are just a few of the major fields of mathematics that give natural settings for fixed point theorems. Approximation theory, potential theory, game theory, mathematical economics, theory of differential equations, and other disciplines use fixed point theorems to solve problems in approximation theory, potential theory, game theory, mathematical economics, and so on. It is possible to evaluate various problems from science and engineering using fixed point approaches when one is concerned with a system of differential/integral/functional equations. This method is particularly beneficial when dealing with control system issues and the idea of elasticity.
Fixed point theorems are the most important tools for proving the existence and uniqueness of solutions to various mathematical models (differential, integral, and partial differential equations, variational inequalities, and so on), which represent phenomena arising in multiple fields such as steady-state temperature distributions, chemical reactions, Neutron transport theories, economic theories, epidemics, and fluid flow. They are also employed to look at the difficulty of determining the best central for these systems.

Let \( F : X \rightarrow X \) represent a function on the set \( X \). A point \( x \in X \) is called a fixed point of \( F \) if \( F(x) = x \), that is, a point, which remains invariant under the transformation \( F \), is called a fixed point, and fixed point theorems are theorems that deal with the attributes and existence of fixed points. If \( F \) is a function defined on the real numbers as \( F(x) = x + 2 \), then it has no fixed points since \( x \) is never equal to \( x + 2 \) for any real number.

Let \( F : [0, 1] \rightarrow [0, 1] \) be defined by \( x/10 \) then \( F(0) = 0 \). Hence 0 is a fixed point of \( F \).

Poincaré [1] was the first to establish the fixed-point theory in 1886. He arrived at the first result on a fixed point using a continuous function.

Browder [2] proved the following useful theorem in 1912. Browder’s fixed point theorems are fundamental in fixed point theory and its applications. Browder’s fixed-point theorem states, “If \( C \) is a unit ball in \( E \) (Euclidean n-dimensional space) and \( T : C \rightarrow C \) a continuous function. Then \( T \) has a fixed point in \( C \), or \( Tx = x \) has a solution.”

The Particular Case of this theorem on the real line can be stated in the following way.

Let \( T : [0, 1] \rightarrow [0, 1] \) be a continuous function. Then \( T \) has a fixed point.

Schauder proved the following theorem for compact maps.

Let \( X \) be a Banach space and let \( C \) be a closed, bounded subset of \( X \). Let \( T : C \rightarrow C \) be a compact map. Then \( T \) has at least one fixed point in \( C \).

This theorem is important in the numerical treatment of equations in analysis.

Banach [3] investigated the concept of contraction type mappings in metric space in 1922. Using the condition of contraction mapping, he established an interesting conclusion in metric space. “Every contraction mapping of a complete metric space into itself has a unique fixed point,” according to the Banach contraction principle.

A contraction mapping is continuous, but a continuous map is not necessarily a contraction.

For example, a translation map \( T : R \rightarrow R \) is defined by \( Tx = x + p, p > 0 \), is continuous but not contraction.

The Banach contraction principle has a lot of uses, but it has one major flaw: It requires the function to be consistent throughout the space. Kannan [4] proved the improved conclusion in fixed point theory to avoid this flaw. The Banach contraction principle has a lot of uses, but it has one major flaw: It requires the function to be consistent throughout the space. Kannan [5] proved the improved conclusion in fixed point theory to avoid this flaw. He proved that “let \( F \) be a self-mapping of complete metric space \( X \) satisfying the following inequality

\[
d(Fx, Fy) \leq \alpha \left[ d(x, Fx) + d(y, Fy) \right] \text{ for all } x, y \in X, 0 < \alpha < 1/2.
\]

Then \( F \) has a unique fixed point.”
Jungck [6] generalized Banach's fixed point theorem by proving a common fixed point theorem for commuting maps. The Banach fixed point theorem has many applications, but it has one flaw: The definition necessitates function continuity.

In 1982, Sessa [7] refined Jungck's result and proposed the concept of weakly commuting mappings in metric space, demonstrating that “two commuting mappings also commute weakly, but two weakly commuting mappings are not certainly commuting.”

Jungck [8] achieved a breakthrough when he declared the new concept of “compatibility” of mappings and demonstrated its utility in achieving a common fixed point of mappings. Every weak commutative pair of mappings is compatible, according to Jungck [9], but the opposite does not have to be true. In his study [10], Singh points out that commutativity does not entail the presence of a series of points that satisfy the compatibility criterion.

Jungck et al. [11] introduced the concept of compatible mappings of type (A) in 1993 and proved some common fixed point theorems.

Common fixed-point theorems for Mann-type iterations are useful for common fixed point theorems and their applications to the best approximation.

In 1999, Popa [12] proved some fixed point theorems for compatible mappings satisfying an implicit relation. Some other results of Popa have been targeted by many authors and common fixed point theorems in different spaces on using implicit relations.

The notion of convex metric spaces was initially introduced by Takahashi [13]. He and others gave some fixed point theorems for non-expansive mappings in convex metric spaces.

In the metric space setting, the strict contractive condition does not ensure the existence of a common fixed point unless the space is assumed compact or the tough conditions are replaced by stronger conditions as in [14, 15]. In 1986, Jungck [8] introduced the notion of compatible mappings. This concept was frequently used to prove the existence of theorems in common fixed point theory. However, the study of common fixed points of non-compatible mappings is also very interesting.

In an attempt to study fixed points of non-self-mappings, Assad and Kirk [16] gave sufficient conditions for such mappings to have a fixed point by proving a result for multivalued mappings in convex metric spaces. Naimpally et al. [17] proved fixed point theorems in convex metric spaces.

Gähler [18] introduced the concept of 2-metric space and further studied 2-metric and other spaces in [19, 20]. He defined 2-metric space as a real-valued function of a point triples on a set X, whose abstract properties were suggested by the area function in Euclidean space. It is natural to expect 3-metric space, which is indicated by the volume function.

In 1998, Pant [21] introduced the notion of R-weakly commuting maps and point-wise R-weakly commuting maps in metric spaces. He has observed that two self-maps on a metric space can fail to be point-wise R-weakly commuting only if they possess a coincidence point at which they do not commute.

The systematic study of fixed points of multivalued mappings had been started with the work of Nadler [22] in 1969, who proved that any multivalued contractive mapping of a complete metric space X into the family of a closed bounded subset of X has a fixed point. Ciric [23] was the first to prove the most general fixed point theorem for a generalized multivalued contraction mapping.
Naimpally et al. [1] obtained some interesting results on fixed point and coincidence point theorems for a hybrid of multivalued and single-valued maps satisfying a contraction condition.

In 1942, Menger [24] suggested associating a distribution function in place of a distance function to any two points in metric space and introduced the concept of probabilistic metric space under statistical metric space.

Sehgal [25] initiated the study of contraction mapping on probabilistic metric space in 1966. Sehgal and Bharucha-Reid [26] proved the Banach contraction principle for probabilistic metric space, stating that “A contraction mapping on a complete probabilistic metric space has a unique fixed point.”

Due to the various paradigmatic changes in science and mathematics, many changes also took place in the concept of uncertainty. One such change is the concept of uncertainty which is at the stage of transition from the traditional view to the modern view and is characterized chiefly by the theories of the uncertainty of probability theory.

An important point in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Zadeh [27], a computer scientist university of California; the U.S.A. was the first who introduce the concept of fuzzy set theory in his seminal paper as a new way to represent vagueness in our everyday life. Zadeh introduced a theory whose objects fuzzy sets are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial but rather a degree. This concept is being used and found to be more appropriate in solving problems of all disciplines.

The concept of fuzzy sets was initially introduced by Zadeh [27] in 1965 and has caused great interest among pure and applied mathematicians. It has also raised enthusiasm among engineers, biologists, psychologists, and economists.

Let X be a set; we define a fuzzy set X as a map $M : X \rightarrow I = [0, 1]$. It is to be remarked that fuzzy sets can be regarded as a generalization of characteristic functions taking values between 0 and 1 (including 0 and 1), and the characteristic function on a set X is the constant mapping.

In classical set theory, a subset $A$ of a set $X$ can be defined by its characteristic function $\chi_A(x) = 0, \text{if } x \notin A$ and $\chi_A(x) = 1, \text{if } x \in A$.

The mapping may be represented as a set of ordered pairs $\{(x, \chi_A(x))\}$ with exactly one ordered pair present for each element of $X$. The first element of the ordered pair is an element of the set $X$, and the second is its value $(0, 1)$. The value 0 is used to represent non-membership, and the value 1 is used to describe the membership of the element $A$. The truth or falsity of the statement, $x$ is in $A$, determined by the ordered pair. The statement is true if the second element of the ordered pair is 1, and the statement is false if it is 0. Similarly, a fuzzy subset $A$ of a set $X$ can be defined as a set of ordered pairs $\{(x, \chi_A(x)) : x \in X\}$, each with the first element from $X$ and the second element the interval $[0, 1]$ with exactly one ordered pair present for each component of $X$. This defines a mapping $\mu_A$ between elements of the set $X$ and the values in the interval $[0, 1]$. That is, $\mu_A : X \rightarrow [0, 1]$.

The value 0 represents complete non-membership, the value 1 represents complete membership, and the values in between represent intermediate degrees of membership.

The fuzzy set theory has an application in neural network theory, robotics, reliability, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences, image processing, control theory, communication, etc.
In the last three decades, fuzzy mathematics’s development and rich growth were tremendous. A large number of authors studied applications of fuzzy set theory in different engineering branches.

Zimmermann and Sebastian [28, 29] defined knowledge base system design and intelligent system design support. Tsourveloudis et al. [30] defined machine flexibility and established a result.

For fuzzy mathematics, we refer to Bedard [31], Butnariu [32], Grabiec [33], and Weiss [34].

2. Main results/discussion/application of fuzzy set and a fixed point in a dynamical system

In this chapter, our main aim is to give an application of fuzzy sets and fixed points in a dynamic system.

A dynamical system is one in which a function explains the time dependency of a point in an ambient space or one in which something evolves with time. For instance, mathematical models that represent the swinging of a clock pendulum, water flow in a conduit, the quantity of fish in a lake each spring, population expansion, and so on.

A dynamic system can be described over either discrete time steps or a continuous timeline.

Discrete-time dynamical system-

\[ x_t = F(x_{t-1}, t) \]  \hspace{1cm} (1)

This type of model is called a difference equation, a recurrence equation, or an iterative map (if the right-hand side is not dependent on \( t \)).

Continuous-time dynamical system-

\[ \frac{dx}{dt} = F(x, t) \]  \hspace{1cm} (2)

This type of model is called a differential equation.

In both cases, \( x_t \) or \( x \) is the system’s state variable at time \( t \), which may take a scalar or vector value. \( F \) is a function that determines the rule by which the system changes its state over time.

Dynamical systems are often modeled by differential equations.

So, here we discuss an application of fuzzy Laplace transforms to solve differential equations/fuzzy differential equations:

Fuzzy differential equations (FDEs) are a natural technique to model dynamic systems under uncertainty. One of the most basic FDEs, first-order linear fuzzy differential equations, can be found in many applications. Chang and Zadeh [35] were the first to present the fuzzy derivative notion. The concept of FDEs was used in the analysis of fuzzy dynamical problems by Kandel and Byatt [4, 36]. FDEs and their fuzzy beginning and boundary value problems are solved using the fuzzy Laplace transform approach. Fuzzy Laplace transforms make it easier to solve an FDE by converting it to an algebraic problem.

Operational calculus is an important area of applied mathematics that involves switching from calculus operations to algebraic operations on transforms. The fuzzy
Laplace transform approach is practically the essential functional method for engineers. The fuzzy Laplace transform also benefits by directly addressing difficulties, fuzzy initial value problems without identifying a general solution, and non-homogeneous differential equations without first solving the corresponding homogeneous equation.

There are a number of mathematicians who studied and developed several approaches to study FIVP [37–41]. They initially used H-Differentiability for fuzzy-valued functions. Using this concept, they looked at the existence and uniqueness of the solution of FIVP [37, 41, 42]. This concept has a drawback: The fuzzy solution behaves quite differently from the crisp solution. Bede B and Gal SG [43] introduced a new idea called the strongly generalized differentiability, and it was studied and used for solving FIVPs in [44–47]. This concept allows us to overcome the drawback mentioned above. So, we use this differentiability concept to find out the solution to FIVP in this chapter.

The fuzzy Laplace transform method solves the problems of FDEs and corresponding fuzzy initial and boundary values. This method solves FIVPs/FDEs directly and gives the complete solution without determining the complementary and particular solution (one can refer to [44, 45, 48–51]. This chapter uses this technique to solve FIVPs/FDEs/ODEs.

We need some definitions and theorems given under the following to solve the fuzzy differential equation by the fuzzy Laplace transform.

**Definition 2.1.** ([46]). Let \( f(x) \) be a continuous fuzzy-value function. Suppose that \( f(x) \odot e^{-px} \) is improper fuzzy Riemann integrable on \([0, \infty)\), then \( \int_0^\infty f(x) \odot e^{-px} \, dx \) is called fuzzy Laplace transforms and is denoted as

\[
L[f(x)] = \int_0^\infty f(x) \odot e^{-px} \, dx = \left( \int_0^\infty f(x, r)e^{-pr} \, dx, \int_0^\infty f(x, \alpha)e^{-p\alpha} \, dx \right)
\]

(Or) \( L[f(x)] = \left( l[f(x, \alpha)], l[f(x, \alpha)] \right) \)

**Theorem 2.2.** Chalco-Cano et al. [46] Let \( f : R \to E \) be a fuzzy valued function and denote \( f(t) = \left( \underline{f}(t, \alpha), \bar{f}(t, \alpha) \right) \), for each \( \alpha \in [0, 1] \). Then

1. If \( f \) is (i)-differentiable, then \( \underline{f}(t, \alpha) \) and \( \bar{f}(t, \alpha) \) are differentiable functions and \( f'(t) = \left( \underline{f}'(t, \alpha), \bar{f}'(t, \alpha) \right) \)

2. If \( f \) is (ii)-differentiable, then \( \underline{f}(t, \alpha) \) and \( \bar{f}(t, \alpha) \) are differentiable functions and \( f'(t) = \left( \bar{f}'(t, \alpha), \underline{f}'(t, \alpha) \right) \)

**Formulae 2.3.** Consider the fuzzy initial value problem

\[
\begin{align*}
y'(t) &= f(t, y(t)) \\
y(0) &= \left( y(0, \alpha), \bar{y}(0, \alpha) \right), \quad 0 < \alpha \leq 1.
\end{align*}
\]
Where \( f : R_+ \times E \rightarrow E \) is a continuous fuzzy mapping. Using the fuzzy Laplace transform method, we have: \( L[y'(t)] = L[f(t, y(t))] \)

Case I—If we consider \( y'(t) \) by using (i)-differentiable, then \( y'(t) = (\dot{y}(t, \alpha), \ddot{y}(t, \alpha)) \) and

\[
L[y'(t)] = (s \circ L[y(t)])^{-1} y(0)
\]

(Or) \( L \int (f(t, y(t), \alpha) = s L \int [y(t, \alpha)] - y(0, \alpha), L \int \int (t, y(t), \alpha)] = s L \int [y(t, \alpha)] - y(0, \alpha) - y(0, \alpha) - y(0, \alpha)
\]

(Or) \( L \int (f(t, y(t), \alpha) = s H_1(s, \alpha) - y(0, \alpha), L \int \int (t, y(t), \alpha)] = s K_1(s, \alpha) - y(0, \alpha)
\)

Where \( L \int [y(t, \alpha)] = H_2(s, \alpha), L \int [y(t, \alpha)] = K_2(s, \alpha)
\)

Case II—If we consider \( y'(t) \) by using (ii)-differentiable, then \( y'(t) = (\dot{y}(t, \alpha), \ddot{y}(t, \alpha)) \) and

\[
L[y'(t)] = y(0)^{-1} (s \circ L[y(t)])
\]

(Or) \( L \int (f(t, y(t), \alpha) = s L \int [y(t, \alpha)] - y(0, \alpha), L \int \int (t, y(t), \alpha)] = s L \int [y(t, \alpha)] - y(0, \alpha) - y(0, \alpha) - y(0, \alpha)
\]

(Or) \( L \int (f(t, y(t), \alpha) = s H_1(s, \alpha) - y(0, \alpha), L \int \int (t, y(t), \alpha)] = s K_1(s, \alpha) - y(0, \alpha)
\)

Where \( L \int [y(t, \alpha)] = H_2(s, \alpha), L \int [y(t, \alpha)] = K_2(s, \alpha)
\)

Now, we solve the fuzzy differential equation by the fuzzy Laplace transform method.

**Example 2.4.** Consider the initial value problem

\[
\begin{cases}
    y'(t) = y(t) & 0 \leq t \leq T \\
    y(0) = (y(0, \alpha), \dot{y}(0, \alpha)).
\end{cases}
\]

by using the fuzzy Laplace transform method, we have

\[
L[y'(t)] = L[y(t)], \text{ and } L[y'(t)] = \int_0^s y'(t) e^{-st} dt \text{ in (i)-differentiable then by using Case (i), we have } L[y'(t)] = (s \circ L[y(t)]) \otimes y(0)
\]

Therefore, \( L[y(t)] = s \circ L[y(t)] \otimes y(0) \)

\[
l[y(t, \alpha)] = s \int [y(t, \alpha)] - y(0, \alpha)
\]

\[
l \int [y(t, \alpha)] = s L [y(t, \alpha)] - y(0, \alpha)
\]

Thus, the solution of system (3) is:

\[
l[y(t, \alpha)] = -y(0, \alpha) \left( \frac{s}{s^2 - 1} \right) + y(0, \alpha) \left( -\frac{1}{s^2 - 1} \right)
\]

\[
l \int [y(t, \alpha)] = -y(0, \alpha) \left( \frac{s}{s^2 - 1} \right) + y(0, \alpha) \left( -\frac{1}{s^2 - 1} \right)
\]
Thus
\[
\dot{y}(t, \alpha) = -y(0, \alpha) I^{-1} \left( \frac{s}{s^2 - 1} \right) + \dot{y}(0, \alpha) I^{-1} \left( -\frac{1}{s^2 - 1} \right)
\]
\[
\ddot{y}(t, \alpha) = -\ddot{y}(0, \alpha) I^{-1} \left( \frac{s}{s^2 - 1} \right) + \ddot{y}(0, \alpha) I^{-1} \left( -\frac{1}{s^2 - 1} \right)
\]
Finally, we have:
\[
\dot{y}(t, \alpha) = e^{-t} \left( \frac{y(0, \alpha) - \dot{y}(0, \alpha)}{2} \right) - e^{t} \left( \frac{y(0, \alpha) + \ddot{y}(0, \alpha)}{2} \right)
\]
\[
\ddot{y}(t, \alpha) = e^{-t} \left( -\frac{y(0, \alpha) + \dot{y}(0, \alpha)}{2} \right) - e^{t} \left( \frac{y(0, \alpha) + \ddot{y}(0, \alpha)}{2} \right)
\]
If \(y(t)\) in (ii)-differentiable, then by using Case II, we have
\[
L[y(t)] = -(y(0)) \Theta(-s L [y(t)])
\]
\[
l[y(t, \alpha)] = s l[y(t, \alpha)] - \dot{y}(0, \alpha)
\]
\[
l\left[\ddot{y}(t, \alpha)\right] = s l\left[\dot{y}(t, \alpha)\right] - y(0, \alpha)
\]
Hence, the solution of system (4) is:
\[
l[y(t, \alpha)] = -\dot{y}(0, \alpha) \left( \frac{1}{1 + s} \right)
\]
\[
l\left[\ddot{y}(t, \alpha)\right] = -\ddot{y}(0, \alpha) \left( \frac{1}{1 + s} \right)
\]
Thus
\[
\dot{y}(t, \alpha) = -\dot{y}(0, \alpha) I^{-1} \left( \frac{1}{1 + s} \right)
\]
\[
\ddot{y}(t, \alpha) = -\ddot{y}(0, \alpha) I^{-1} \left( \frac{1}{1 + s} \right)
\]
Finally, we have:
\[
\dot{y}(t, \alpha) = -\dot{y}(0, \alpha) e^{-t}
\]
\[
\ddot{y}(t, \alpha) = -\ddot{y}(0, \alpha) e^{-t}
\]

**Remark 2.5.** By following the above procedure, the solution of fuzzy IVP with initial conditions can be obtained. The solution of simultaneous fuzzy linear differential equations and fuzzy BVPs can also be obtained.

Now, we discuss the application of fixed points in existence and the uniqueness of the solution of the ordinary differential equation.

IVP’s existence and uniqueness can be easily established using the fixed point technique. Banach fixed point theorem can be applied to derive the existence and
uniqueness of the solution of an initial value problem. The function used in IVP satisfies the Lipschitz condition.

**Definition 2.6.** Contraction Mapping:
Let $X$ be a complete normed linear space (Banach Space). A mapping $F : X \to X$ is called a contraction if $\|Fx - Fy\| \leq \alpha \|x - y\|, \forall x, y \in X$, for some $\alpha < 1$

Example-. If $F(x) = \frac{x}{2}$, then $F$ is a contraction for $\alpha = \frac{1}{2}$

**Definition 2.7.** Banach Fixed-Point Theorem (or Banach Contraction Principle):
If $F : X \to X$ is a contraction, then $F$ has a unique fixed point; say it is $x_1$. $Fx_1 = x_1$

Further, the sequence $\{x_n\}$ defined by $x_n = Fx_n (\forall n = 1, 2, 3, \ldots )$, converges to the unique fixed point $x_1$ of $F$.

**Definition 2.8.** Generalized Banach contraction principle:
If $F^n$ is contraction, $F : X \to X$ is a Banach space for $n \geq 1$, then $F$ has a unique fixed point

**Theorem 2.9.** Consider an Initial Value Problem (IVP)

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Let $f(x, y)$ be a continuous function defined on a domain $D \subseteq \mathbb{R}^2$. Let $f$ be Lipschitz continuous concerning $y$ on $D$. Then, there exists a unique solution to the IVP on an interval $|x - x_0| \leq h$, where $h = \min (a, \frac{b}{M})$, $M = \text{Max} |f(x, y)|$

$(x, y) \in R$, and $R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b \} \subseteq D$

Further, the unique solution can be computed from the successive approximation scheme $y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt, y_0(t) = y_0 (\forall n = 0, 1, 2, \ldots )$

**Proof:** The solvability of IVP follows
If the integral equation $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$ is solvable, let $X = C[x_0, x_1]$ set of all continuous functions defined on $[x_0, x_1]$. Define $\|x\| = \sup_{t \in [x(t)]} |x(t)| (X, ||.|| )$ is complete normed linear space.

Define

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \quad (5)$$

Define an operator $F : C[x_0, x_1] \to C[x_0, x_1]$ by $(Fy)(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$.

If $F$ has a fixed point, that is, there exists a $y$ such that $y = Fy$, then the fixed point $y$ is a solution to the integral equation (5).

Now, we will show $F^n$ is contraction for some large $n$.

$$(Fy)(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$$

Let $y_1, y_2 \in C[x_0, x_1]$, then

$$| (Fy_1)(x) - (Fy_2)(x) | = \left| \int_{x_0}^{x} f(t, y_1(t) - f(t, y_2(t))dt \right|$$

$$\leq \alpha \int_{x_0}^{x} |y_1(t) - y_2(t)|dt \quad (6)$$
\[ \alpha \int_{x_0}^{x} \sup_{t \in [x_0, x]} |y_1(t) - y_2(t)| \, dt \leq \alpha \int_{x_0}^{x} \|y_1(t) - y_2(t)\| \, dt \leq \alpha (x - x_0) \|y_1 - y_2\| \]  

(7)

On taking sup, we have

\[ \|(F^2 y_1) - (F^2 y_2)\| \leq \frac{\alpha^2}{2!} (x_1 - x_0)^2 \|y_1 - y_2\| \]

Similarly,

\[ \|(F^3 y_1) - (F^3 y_2)\| \leq \frac{\alpha^3}{3!} (x_1 - x_0)^3 \|y_1 - y_2\| \]

\[ \|(F^n y_1) - (F^n y_2)\| \leq \frac{\alpha^n}{n!} (x_1 - x_0)^n \|y_1 - y_2\| \]  

(8)

In (8), \( x_1 - x_0 < 1 \) if \( n \) is large enough. 
This implies that \( F^n \) is a contraction for large \( n \), and \( F \) has a unique fixed point. 
This implies that there is a unique solution to the integral equation. 
This shows that there is a unique solution to IVP. 
Further, \( x_{n+1} = F x_n \), \( \{x_n\} \) converges to the unique solution of the IVP. 
This implies that \( y_{n+1} = F y_n \). 
Which implies that \( y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t)) \, dt, y_0(x) = y_0 \) 

**Example 2.10.** Consider the IVP \( \frac{dy}{dx} = \frac{2y}{x}, x_0 = 1, y_0 = 1 \) 
Here \( f(x, y) = \frac{2y}{x} \), \( x_0 = 1, y_0 = 1 \)

\[ y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t)) \, dt \]

\[ y_1(x) = 1 + \int_{1}^{x} f(t, y_0(t)) \, dt \]

\[ y_1(x) = \int_{1}^{x} \frac{2}{t} \, dt \]

\[ y_1(x) = 2 \log(x) \]
$$y_2(x) = 1 + \int_1^x f(t, y_1(t)) dt$$
$$y_2(x) = \int_1^x \frac{\log t}{t} dt$$
$$y_2(x) = \int_1^x \frac{4\log t}{t} dt$$
$$y_2(x) = 2(x^2 - 1)$$
$$y_3(x) = \int_1^x f(t, y_2(t)) dt$$
$$y_3(x) = 1 + \int_1^x f(t, y_2(t)) dt$$
$$y_3(x) = 1 + \int_1^x \frac{4(t^2 - 1)}{t} dt$$
$$y_3(x) = 1 + 4 \int_1^x \left( t - \frac{1}{t} \right) dt$$
$$y_3(x) = 1 + 2 \left( x^2 - 1 \right) - 4 \left( \log x \right)$$
$$y_3(x) = (2x^2 - 1) - 4 \left( \log x \right)$$

On keep continuing this process up to the nth time and then taking $n \rightarrow \infty$ we have

$$y(x) = x^2$$

**Example 2.11.** Consider the IVP $\frac{dy}{dx} = y, y(0) = 1$
Here $f(x, y) = y, x_0 = 0, y_0 = 1$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t)) dt$$
$$y_1(x) = 1 + \int_{0}^{x} y_0(t) dt$$
$$y_1(x) = 1 + \int_{0}^{x} 1 dt = 1 + x$$
$$y_2(x) = 1 + \int_{0}^{x} y_1(t) dt = 1 + x + \frac{x^2}{2}$$
$$y_3(x) = 1 + \int_{0}^{x} y_2(t) dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Taking $n \rightarrow \infty$, then we have $y_n(x) \rightarrow e^x$
Thus $y(x) = e^x$ is a solution for a given IVP.
Stability of the solution: In these questions/examples, the functions \( f(x, y) = \frac{2y}{x} \) and \( f(x, y) = y \) satisfies the Lipschitz condition concerning the initial condition \( y_0 = 1 \), so the answer is stable concerning initial data.

3. Conclusion

Here in this chapter, we discussed the application of a fuzzy set and a fixed point in dynamic systems. We also tried to discuss applications of fuzzy sets and fixed points in other directions. We gave a solution of fuzzy ordinary differential equations with initial conditions by the fuzzy Laplace transform method and provided a solution to the existence and uniqueness problem of ODE of the first order by the fixed point technique. We solved examples of existence and uniqueness problems and checked the stability of the solution also.

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