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Abstract

In 1961, Bargmann introduced the classical Fock space \( F(\mathbb{C}) \) and in 1984, Cholewinsky introduced the generalized Fock space \( F_{2,\nu}(\mathbb{C}) \). These two spaces are the aim of many works, and have many applications in mathematics, in physics and in quantum mechanics. In this work, we introduce and study the Fock space \( F_{3,\nu}(\mathbb{C}) \) associated to the generalized Bessel operator \( L_{3,\nu} \). The space \( F_{3,\nu}(\mathbb{C}) \) is a reproducing kernel Hilbert space (RKHS). This is the reason for defining the orthogonal projection operator, the Toeplitz operators and the Hankel operators associated to this space. Furthermore, we give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator \( T : F_{3,\nu}(\mathbb{C}) \to H \), where \( H \) be a Hilbert space. Finally, we come up with some results regarding the extremal functions, when \( T \) is a difference operator and an integral operator, respectively. Finally, we remark that it is now natural to raise the problem of studying the Bessel-type Segal-Bargmann transform associated to the space \( F_{3,\nu}(\mathbb{C}) \). This problem is difficult and will be an open topic. This topic requires more details for the harmonic analysis associated to the operator \( L_{3,\nu} \). We have the idea to continue this research in a future paper.

Keywords: Bessel-type Fock space, Heisenberg-type uncertainty principle, Toeplitz operators, Hankel operators, Tikhonov regularization problem, extremal function

1. Introduction

In [1], Bargmann has studied the Fock space \( F(\mathbb{C}) \), is a Hilbert space consisting of entire functions on \( \mathbb{C} \), square integrable with respect to the measure

\[
\text{dm}(z) := \frac{1}{\pi} e^{-|z|^2} \, dx \, dy, \quad z = x + iy.
\]

This space is equipped with the inner product

\[
(f, g)_{F(\mathbb{C})} := \int_{\mathbb{C}} f(z) \overline{g(z)} \, dm(z),
\]

and has the reproducing kernel \( k(z, w) = e^{zw} \). In [2], Cholewinsky has constructed a generalized Fock space \( F_{2,\nu}(\mathbb{C}) \) consisting of even entire functions on \( \mathbb{C} \), square integrable with respect to the measure
where $K_\nu, \nu > 0$, is the modified Bessel function of the second kind and index $\nu$, called also the Macdonald function [3]. The generalized Fock space $F_{2,\nu}(C)$ is associated to the Bessel operator

$$L_{2,\nu} := \frac{d^2}{dz^2} + \frac{2\nu}{z} \frac{d}{dz},$$

and has the reproducing kernel

$$k_{2,\nu}(z, w) = I_{2,\nu}(zw) = \sum_{n=0}^{\infty} \alpha_n(z, \nu),$$

where

$$\alpha_n(z, \nu) = 2^{2n} n! \Gamma(n + \nu + 1/2) \Gamma(\nu + 1/2).$$

The study of several generalizations of the Fock spaces has a long and rich history in many different settings [4–8]. In this work, we will try to generalize Bessel-type Fock space, to give some properties concerning Toeplitz operators and Hankel operators of this space; and to establish Heisenberg-type uncertainty principle for this generalized Fock space. The generalized Bessel operator (or hyper-Bessel operator [9]) is the third-order singular differential operator given by

$$L_{3,\nu} := \frac{d^3}{dz^3} + \frac{3\nu}{z^2} \frac{d^2}{dz^2} - \frac{3\nu}{z^2} \frac{d}{dz},$$

where $\nu$ is a nonnegative real number. When $\nu = 0$ this operator becomes the third derivative operator for which some analysis was studied by Widder [10] and for some special value of $\nu$ the operator $L_{3,\nu}$ appeared as a radial part of the generalized Airy equation of a nonlinear diffusion type partial differential equation in $\mathbb{R}^d$. Recently, in a nice and long paper, Cholewinski and Reneke [11] studied and extended, for the operator $L_{3,\nu}$, the well known theory related to some singular differential operator of second order for which the literature is extensive. Next, Fitouhi et al. [12, 13] established a harmonic analysis related to this operator (for examples the eigenfunctions, the generalized translation, the Fourier-Airy transform, the heat equation, the heat polynomials, the transmutation operators, ...). Recently the Airy operator has gained considerable interest in various field of mathematics [9, 14] and in certain parts of quantum mechanics [15]. The results of this work will be useful when discussing the Fock space associated to this operator. This space is the background of some applications in this contribution. Especially, we give an application of the theory of extremal functions and reproducing kernels of Hilbert spaces, to examine the extremal function for the Tikhonov regularization problem associated to a bounded linear operator $T : F_{2,\nu}(C) \rightarrow H$, where $H$ be a Hilbert space. We come up with some results regarding the extremal functions associated to a difference operator $D$ and to an integral operator $P$. 

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The examination of the extremal functions is studied in several directions, in the Fourier analysis [16, 17], in the Sturm-Liouville hypergroups [18], and in the Fourier-Dunkl analysis [19, 20]...

The contents of the paper are as follows. In Section 2, we study the Toeplitz operators and the Hankel operators on the Bessel-type Fock space $F_{3,\nu}(\mathbb{C})$, and we establish Heisenberg-type uncertainty principle for this space. In Section 3, we give an application of the theory of reproducing kernels to the Tikhonov regularization problem for a difference operator and for an integral operator, respectively. In the last section, we summarize the obtained results and describe the future work.

2. Bessel-type Fock space

In this section we introduce the Toeplitz and the Hankel operators on the Bessel-type Fock space $F_{3,\nu}(\mathbb{C})$. And we establish an uncertainty inequality of Heisenberg-type on the space $F_{3,\nu}(\mathbb{C})$.

2.1 Toeplitz and Hankel operators on $F_{3,\nu}(\mathbb{C})$

Let $z \in \mathbb{C}$ and $\omega_k = e^{2\pi i k / 3}$, $k = 1, 2, 3$. A function $u(z)$ is called 3-even if $u(\omega_k z) = u(z)$.

For $\lambda \in \mathbb{C}$, the initial problem

$$L_{3,\nu} u(z) = \lambda^3 u(z), \quad u(0) = 1, u^{(k)}(0) = 0, \quad k = 1, 2$$

admits a unique analytic solution on $\mathbb{C}$ (see [11]), which will be denoted by $I_{3,\nu}(\lambda z)$ and expanded in a power series as

$$I_{3,\nu}(\lambda z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^{3n}}{\alpha_n(3, \nu)}, \quad (1)$$

where

$$\alpha_n(3, \nu) = \frac{3^{3n} n! \Gamma(n + 1/3) \Gamma(n + \nu + 2/3)}{\Gamma(1/3) \Gamma(\nu + 2/3)}.$$ 

The function $I_{3,\nu}(\lambda z)$ is 3-even and defined as the hypergeometric function [11],

$$I_{3,\nu}(\lambda z) = {}_0F_2 \left[ \begin{array}{c} 1 \ \rac{1}{3}, \nu + \frac{2}{3}, \frac{(\lambda z)^3}{3} \end{array} \frac{3}{2} \right].$$

In particular, $|I_{3,\nu}(\lambda z)| \leq e^{|\lambda z|}$ and $I_{3,0}(\lambda z) = \cos 3(-\lambda z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^{3n}}{3^{3n} n!}.$

In the following we denote by

- $O_\nu$ the function (see [11], p. 12) defined for $t \geq 0$ by

$$O_\nu(t) = \frac{2^{1/3}}{3^{(3\nu+5)/2} \Gamma(1/3) \Gamma(\nu + 2/3)} \int_0^\infty x^{(\nu-1)/2} e^{-x/2} K_{\nu-1/3} \left( \frac{4x}{27} \right) dx,$$
where \( K_\nu \) is the Macdonald function.

- \( dm_{3,\nu}(z) \), \( \nu > 0 \), the measure defined on \( \mathbb{C} \) by

\[
dm_{3,\nu}(z) = \frac{3}{\pi} |z|^{-2} O_\nu \left( |z|^6 \right) dxdy, \quad z = x + iy.
\]

This weighted measure is used by Cholewinski-Reneke in their interesting paper [11] for computing the \( \nu \)-Airy heat function.

- \( L^2(\mathbb{C}) \), the space of measurable functions \( f \) on \( \mathbb{C} \) satisfying

\[
\|f\|_{L^2(\mathbb{C})}^2 := \int_{\mathbb{C}} |f(z)|^2 dm_{3,\nu}(z) < \infty.
\]

- \( H_{3,*}(\mathbb{C}) \), the space of 3-even entire functions on \( \mathbb{C} \).

Let \( \nu > 0 \). We define the Bessel-type Fock space \( F_{3,\nu}(\mathbb{C}) \), to be the pre-Hilbert space of functions in \( H_{3,*}(\mathbb{C}) \cap L^2(\mathbb{C}) \), equipped with the inner product

\[
(f,g)_{F_{3,\nu}(\mathbb{C})} := \int_{\mathbb{C}} f(z) \overline{g(z)} dm_{3,\nu}(z),
\]

and the norm \( \|f\|_{F_{3,\nu}(\mathbb{C})} := \|f\|_{L^2_{\nu}(\mathbb{C})} \).

If \( f, g \in F_{3,\nu}(\mathbb{C}) \) with \( f(z) = \sum_{n=0}^{\infty} a_n z^{3n} \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^{3n} \), then

\[
(f,g)_{F_{3,\nu}(\mathbb{C})} = \sum_{n=0}^{\infty} a_n \overline{b_n} \alpha_{3,\nu}(n),
\]

where \( \alpha_{3,\nu}(n) \) are the constants given by (1).

If \( f \in F_{3,\nu}(\mathbb{C}) \), then \( |f(z)| \leq e^{z^3/2} \|f\|_{F_{3,\nu}(\mathbb{C})}, z \in \mathbb{C} \). The map \( f \to f(z), z \in \mathbb{C} \), is a continuous linear functional on \( F_{3,\nu}(\mathbb{C}) \). Thus from Riesz theorem [21], the space \( F_{3,\nu}(\mathbb{C}) \) has a reproducing kernel. The function \( k_{3,\nu}(z,w) \) for, by

\[
k_{3,\nu}(z,w) = I_{3,\nu}(3w),
\]

is a reproducing kernel for the Bessel-type Fock space \( F_{3,\nu}(\mathbb{C}) \), that is \( k_{3,\nu}(z, \cdot) \in F_{3,\nu}(\mathbb{C}) \), and for all \( f \in F_{3,\nu}(\mathbb{C}) \), we have \( \{f, k_{3,\nu}(z, \cdot)\} = f(z) \).

The space \( F_{3,\nu}(\mathbb{C}) \) equipped with the inner product \( (\cdot, \cdot)_{F_{3,\nu}(\mathbb{C})} \) is a reproducing kernel Hilbert space (RKHS); and the set \( \left\{ \frac{z^n}{\sqrt{\alpha_{3,\nu}(n)}} \right\}_{n \in \mathbb{N}} \) forms a Hilbert’s basis for the space \( F_{3,\nu}(\mathbb{C}) \).

In the next part of this section we study the Toeplitz operators and the Hankel operators on the Bessel-type Fock space \( F_{3,\nu}(\mathbb{C}) \). These operators generalize the classical operators [5]. We consider the orthogonal projection operator \( P_{\nu} : L^2(\mathbb{C}) \to F_{3,\nu}(\mathbb{C}) \) defined for \( z \in \mathbb{C} \), by

\[
P_{\nu} f(z) := \{f, k_{3,\nu}(z, \cdot)\}_{L^2(\mathbb{C})},
\]
where $k_{3,\nu}$ is the reproducing kernel given by (3). Then we have
\[ P_{\nu}^* P_{\nu} = P_{\nu}, \quad P_{\nu}^* = P_{\nu}, \quad \|P_{\nu}\| = 1, \quad \|I - P_{\nu}\| \leq 1. \]

Let $\phi \in L^\infty(\mathbb{C})$. The multiplication operators $M_\phi : L^2_{3}(\mathbb{C}) \to L^2_{3}(\mathbb{C})$ are the operators defined for $z \in \mathbb{C}$, by
\[ M_\phi f(z) := \phi(z) f(z). \]

The Bessel-type Toeplitz operators $T_\phi : F_{3,\nu}(\mathbb{C}) \to F_{3,\nu}(\mathbb{C})$ are the operators defined for $z \in \mathbb{C}$, by
\[ T_\phi f(z) := P_{\nu} M_\phi f(z). \]

Let $\phi \in L^\infty(\mathbb{C})$. Then we have
\[ \|T_\phi\| \leq \|\phi\|_\infty, \quad T_\phi^* = T_\phi. \]

However, if $\phi \in L^\infty(\mathbb{C})$ has compact support, then $T_\phi$ is a compact operator.

Let $\phi, \varphi \in L^\infty(\mathbb{C})$. Then we have
\[ \|H_\phi\| \leq \|\phi\|_\infty, \quad H_\phi^* = P_{\nu} M_\phi (I - P_{\nu}), \quad T_{\phi \varphi} - T_\phi T_\varphi = H_\phi^* H_\varphi. \]

### 2.2 Heisenberg-type uncertainty principle for $F_{3,\nu}(\mathbb{C})$

Let $U_{3,\nu}(\mathbb{C})$ be the prehilbertian space of 3-even entire functions, equipped with the inner product
\[ \langle f, g \rangle_{U_{3,\nu}(\mathbb{C})} := \int_{\mathbb{C}} f(z) \overline{g(z)} |z|^{3\nu} dm_{3,\nu}(z). \]

If $f, g \in U_{3,\nu}(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{3n}$ and $g(z) = \sum_{n=0}^{\infty} b_n z^{3n}$, then
\[ \langle f, g \rangle_{U_{3,\nu}(\mathbb{C})} = \sum_{n=0}^{\infty} a_n \overline{b_n} \alpha_{n+1}(3, \nu), \quad \|f\|_{U_{3,\nu}(\mathbb{C})}^2 = \sum_{n=0}^{\infty} |a_n|^2 \alpha_{n+1}(3, \nu). \]

The space $U_{3,\nu}(\mathbb{C})$ is a Hilbert space with Hilbert’s basis $\left\{ \frac{z^{3n}}{\sqrt{\alpha_{n+1}(3, \nu)}} \right\}_{n \in \mathbb{N}}$ and reproducing kernel
\[ f_{3,\nu}(z, w) = \sum_{n=0}^{\infty} \frac{\overline{(z w)}^{3n}}{\sqrt{\alpha_{n+1}(3, \nu)}} = \frac{1}{(2\pi)^3} (I_{3,\nu}(z w) - 1). \]

Using the fact that
\[ a_{n+1}(3, \nu) = 3(n + 1)(3n + 1)(3n + 3\nu + 2)a_n(3, \nu) \geq a_n(3, \nu), \quad n \in \mathbb{N}, \quad (4) \]

then the space \( U_{3, \nu}(\mathbb{C}) \) is a subspace of the Bessel-type Fock space \( F_{3, \nu}(\mathbb{C}) \).

Let \( M \) be the multiplication operator defined by

\[ Mf(x) = x^3 f(x). \]

**Lemma 1.** For \( f \in U_{3, \nu}(\mathbb{C}) \) then \( L_{3, \nu}f \) and \( Mf \) belong to \( F_{3, \nu}(\mathbb{C}) \). And for \( f, g \in U_{3, \nu}(\mathbb{C}) \) one has

\[ \langle L_{3, \nu}f, g \rangle_{F_{3, \nu}(\mathbb{C})} = \langle f, Mg \rangle_{F_{3, \nu}(\mathbb{C})}. \]

**Proof.** Let \( f \in U_{3, \nu}(\mathbb{C}) \) with \( f(x) = \sum_{n=0}^{\infty} a_n x^{3n} \). By straightforward calculation we obtain

\[ L_{3, \nu}(x^{3n}) = 3n(3n - 2)(3n + 3\nu - 1)x^{3n-3}, \quad n \in \mathbb{N}^*. \]

Thus

\[ L_{3, \nu}(x^{3n}) = \sum_{n=0}^{\infty} 3(n + 1)(3n + 1)(3n + 3\nu + 2)a_{n+1} x^{3n}, \quad (5) \]

and by (4) we deduce that

\[ \|L_{3, \nu}f\|_{L^2(\mathbb{C})}^2 = \sum_{n=1}^{\infty} 3n(3n - 2)(3n + 3\nu - 1)|a_n|^2 \alpha_{3n}(\nu) \leq \|f\|_{U_{3, \nu}(\mathbb{C})}^2. \]

On the other hand for \( Mf \) one has

\[ Mf(x) = \sum_{n=1}^{\infty} a_{n-1} x^{3n}, \]

and

\[ \|Mf\|_{L^2(\mathbb{C})}^2 = \sum_{n=1}^{\infty} |a_{n-1}|^2 a_n(3, \nu) = \|f\|_{U_{3, \nu}(\mathbb{C})}^2. \]

Therefore \( L_{3, \nu}f \) and \( Mf \) belong to \( F_{3, \nu}(\mathbb{C}) \).

Let \( f, g \in U_{3, \nu}(\mathbb{C}) \) with \( f(x) = \sum_{n=0}^{\infty} a_n x^{3n} \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^{3n} \). From relations (4) and (5) we have

\[ L_{3, \nu}f(x) = \sum_{n=0}^{\infty} a_{n+1} x^{3n}. \]

Therefore and according to (2) we obtain

\[ \langle L_{3, \nu}f, g \rangle_{F_{3, \nu}(\mathbb{C})} = \sum_{n=0}^{\infty} a_{n+1} b_n \alpha_{n+1}(3, \nu) = \sum_{n=1}^{\infty} a_n b_{n-1} \alpha_{3n}(\nu) = \langle f, Mg \rangle_{F_{3, \nu}(\mathbb{C})}. \]

This completes the proof of the lemma. \( \square \)
Let $V_{3,\nu}(\mathbb{C})$ be the prehilbertian space of 3-even entire functions, equipped with the inner product

$$
(f, g)_{V_{3,\nu}(\mathbb{C})} := \int_{\mathbb{C}} f(z) \overline{g(z)} |z|^2 \, dm_{3,\nu}(z).
$$

If $f, g \in V_{3,\nu}(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{3n}$ and $g(z) = \sum_{n=0}^{\infty} b_n z^{3n}$, then

$$
(f, g)_{V_{3,\nu}(\mathbb{C})} = \sum_{n=0}^{\infty} a_n \overline{b_n} a_{n+2}(3, \nu), \quad ||f||_{V_{3,\nu}(\mathbb{C})}^2 = \sum_{n=0}^{\infty} |a_n|^2 a_{n+2}(3, \nu).
$$

The space $V_{3,\nu}(\mathbb{C})$ is a Hilbert space with Hilbert’s basis \( \left\{ \frac{z^n}{\sqrt{a_{n+2}(3, \nu)}} \right\}_{n \in \mathbb{N}} \) and reproducing kernel

$$
K_{3,\nu}(z, w) = \sum_{n=0}^{\infty} \left( \frac{\overline{z} w}{\nu} \right)^n a_{n+2}(3, \nu) = \frac{1}{(\overline{z} w)^2} \left( I_{3,\nu}(\overline{z} w) - \frac{(z w)^{3}}{9 \nu + 6} \right).
$$

The space $V_{3,\nu}(\mathbb{C})$ is a subspace of the space $U_{3,\nu}(\mathbb{C})$. Let $[L_{3,\nu}, M]$ the commutator operator defined by

$$
[L_{3,\nu}, M] := L_{3,\nu} M - M L_{3,\nu}.
$$

We easily have

**Lemma 2.** For $f \in V_{3,\nu}(\mathbb{C})$ we have $[L_{3,\nu}, M] f \in V_{3,\nu}(\mathbb{C})$ and

$$
||M f||_{F_{3,\nu}(\mathbb{C})}^2 = ||L_{3,\nu} f||_{F_{3,\nu}(\mathbb{C})}^2 + \langle [L_{3,\nu}, M] f, f \rangle_{F_{3,\nu}(\mathbb{C})}.
$$

We will use the following result of functional analysis.

**Lemma 3.** (See [22, 23]). Let $A$ and $B$ be self-adjoint operators on a Hilbert space $H$ ($A^* = A, B^* = B$). Then we have

$$
||(A - a f)(B - b f)||_H \geq \frac{1}{2} ||[A, B] f||_H^2,
$$

for all $f \in \text{Dom} ([A, B])$, and all $a, b \in \mathbb{R}$.

We obtain the following Heisenberg-type uncertainty principle.

**Theorem 1.** Let $f \in V_{3,\nu}(\mathbb{C})$. For all $a, b \in \mathbb{R}$, we have

$$
||L_{3,\nu} M - a f||_{F_{3,\nu}(\mathbb{C})} ||L_{3,\nu} M + ib f||_{F_{3,\nu}(\mathbb{C})} \geq ||M f||_{F_{3,\nu}(\mathbb{C})}^2 - ||L_{3,\nu} f||_{F_{3,\nu}(\mathbb{C})}^2. \quad (6)
$$

**Proof.** Let us consider the following two operators on $V_{3,\nu}(\mathbb{C})$ by

$$
A = L_{3,\nu} M, \quad B = i(L_{3,\nu} - M).
$$

By Lemmas 1 and 2, the operators $A$ and $B$ satisfies the following properties.

i. For $f, g \in V_{3,\nu}(\mathbb{C})$, we have

$$
\langle Af, g \rangle_{F_{3,\nu}(\mathbb{C})} = \langle f, Ag \rangle_{F_{3,\nu}(\mathbb{C})}, \quad \langle Bf, g \rangle_{F_{3,\nu}(\mathbb{C})} = \langle f, Bg \rangle_{F_{3,\nu}(\mathbb{C})}.
$$
Thus the inequality (6) follows from Lemmas 2 and 3. □

3. Reproducing kernel theory

Let $T : F_{3,\nu}^\mathbb{C} \rightarrow H$ be a bounded linear operator from $F_{3,\nu}^\mathbb{C}$ into a Hilbert space $H$. By using the theory of reproducing kernels of Hilbert space and building on the ideas of Saitoh [24–26] we examine the extremal function associated to the operator $T$ on the Bessel-type Fock space $F_{3,\nu}^\mathbb{C}$.

**Theorem 2.** For any $h \in H$ and for any $\lambda > 0$, the problem

$$
\inf_{f \in F_{3,\nu}^\mathbb{C}} \left\{ \lambda \|f\|_{F_{3,\nu}^\mathbb{C}}^2 + \|Tf - h\|_H^2 \right\}
$$

has a unique minimizer given by

$$
f_{\lambda,T}^*(h) = (\lambda I + T^*T)^{-1}T^*h.
$$

**Proof.** The problem (7) is solved elementarily by finding the roots of the first derivative $d\Phi(f) = \lambda \|f\|_{F_{3,\nu}^\mathbb{C}}^2 + \|Tf - h\|_H^2 - \|h\|_H^2$. Note that for convex functions the equation $d\Phi(f) = 0$ is a necessary and sufficient condition for the minimum at $f$. The calculation provides

$$
d\Phi(f) = 2\lambda f + 2T^*(Tf - h),
$$

and the assertion of the theorem follows at once. □

In this section we examine the extremal functions associated to a difference operator $D$, and to an integral operator $P$, respectively.

3.1 The difference operator

Let $D$ be the difference operator defined by

$$
Df(z) := \frac{1}{z^3}(f(z) - f(0)).
$$

If $f(z) = \sum_{n=0}^\infty a_n z^{2n}$, then $Df(z) = \sum_{n=0}^\infty a_{n+1} z^{2n}$.

In this subsection, we determine the extremal function $f_{\lambda,D}^*$ associated to the difference operator $D$ on the space $F_{3,\nu}^\mathbb{C}$.

**Theorem 3.**

i. The operator $D$ maps continuously from $F_{3,\nu}^\mathbb{C}$ into $F_{3,\nu}^\mathbb{C}$, and

$$
\|Df\|_{F_{3,\nu}^\mathbb{C}} \leq \|f\|_{F_{3,\nu}^\mathbb{C}}.
$$
ii. For \( f \in F_{3,\nu}(C) \) with \( f(z) = \sum_{n=0}^{\infty} a_n z^{3n} \), we have

\[
D^* f(z) = \sum_{n=1}^{\infty} \frac{a_{n-1}(3, \nu)}{a_n(3, \nu)} a_n z^{3n}, \quad D^* D f(z) = \sum_{n=1}^{\infty} \frac{a_{n-1}(3, \nu)}{a_n(3, \nu)} a_n z^{3n}.
\]

iii. For any \( h \in F_{3,\nu}(C) \) and for any \( \lambda > 0 \), the problem

\[
\inf_{f \in F_{3,\nu}(C)} \left\{ \lambda \|f\|^2_{F_{3,\nu}(C)} + \|Df - h\|^2_{F_{3,\nu}(C)} \right\}
\]

has a unique minimizer given by

\[
f^{*}_{\lambda,h}(z) = (h, \Psi_z)_{F_{3,\nu}(C)},
\]

where

\[
\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(z^{3n+3} w^{3n})}{\lambda a_{n+1}(3, \nu) + a_n(3, \nu)}, \quad w \in C.
\]

Proof.

i. If \( f \in F_{3,\nu}(C) \) with \( f(z) = \sum_{n=0}^{\infty} a_n z^{3n} \), then

\[
\|Df\|^2_{F_{3,\nu}(C)} = \sum_{n=0}^{\infty} a_n(3, \nu)|a_{n+1}|^2 \leq \sum_{n=1}^{\infty} a_n(3, \nu)|a_n|^2 \leq \|f\|^2_{F_{3,\nu}(C)}.
\]

ii. If \( f, g \in F_{3,\nu}(C) \) with \( f(z) = \sum_{n=0}^{\infty} a_n z^{3n} \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^{3n} \), then

\[
(Df, g)_{F_{3,\nu}(C)} = \sum_{n=0}^{\infty} a_n(3, \nu) a_{n+1} b_n = \sum_{n=1}^{\infty} a_{n-1}(3, \nu) a_n b_{n-1} = (f, D^* g)_{F_{3,\nu}(C)},
\]

where

\[
D^* g(z) = \sum_{n=1}^{\infty} \frac{a_{n-1}(3, \nu)}{a_n(3, \nu)} b_{n-1} z^{3n}.
\]

And therefore

\[
D^* D f(z) = \sum_{n=1}^{\infty} \frac{a_{n-1}(3, \nu)}{a_n(3, \nu)} a_n z^{3n}.
\]

iii. We put \( h(z) = \sum_{n=0}^{\infty} b_n z^{3n} \) and \( f^{*}_{\lambda,h}(z) = \sum_{n=0}^{\infty} c_n z^{3n} \). From (8) we have \( (I + D^* D) f^{*}_{\lambda,h}(z) = D^* h(z) \). By (ii) we deduce that

\[
c_0 = 0, \quad c_n = \frac{a_{n-1}(3, \nu) b_{n-1}}{\lambda a_n(3, \nu) + a_{n-1}(3, \nu)}, \quad n \in \mathbb{N}^*.
\]
Thus
\[ f^{*}_{J,L}(h)(z) = \sum_{n=0}^{\infty} \frac{\alpha_n(3, \nu) h_n z^{3n+3}}{\lambda \alpha_{n+1}(3, \nu) + \alpha_n(3, \nu)} = \langle h, \Psi_z \rangle_{F_{\lambda}(C)}, \] (9)

where
\[ \Psi_z(w) = \sum_{n=0}^{\infty} \frac{(z \nu)^{3n+3} w^{3n}}{\lambda \alpha_{n+1}(3, \nu) + \alpha_n(3, \nu)} \].

This completes the proof of the theorem. \( \square \)

The extremal function \( f^{*}_{J,L}(h) \) possesses the following properties.

**Theorem 4.** If \( \lambda > 0 \) and \( h \in F_{3,\nu}(C) \), then

\[ \| f^{*}_{J,L}(h)(z) \| \leq \frac{1}{2\sqrt{\lambda}} \left( I_{3,\nu} \left( |z|^2 \right) \right)^{1/2} \| h \|_{F_{\lambda}(C)}, \]

\[ \| f^{*}_{J,L}(h) \|_{F_{\lambda}(C)} \leq \frac{1}{2\sqrt{\lambda}} \| h \|_{F_{\lambda}(C)}. \]

**Proof.** Let \( \lambda > 0 \) and \( h \in F_{3,\nu}(C) \) with \( h(z) = \sum_{n=0}^{\infty} h_n z^{3n} \).

i. From (9) we have

\[ \| f^{*}_{J,L}(h)(z) \| \leq \| \Psi_z \|_{F_{\lambda}(C)} \| h \|_{F_{\lambda}(C)}. \]

And by using the fact that \( (x+y)^2 \geq 4xy \) we obtain

\[ \| \Psi_z \|_{F_{\lambda}(C)}^2 = \sum_{n=0}^{\infty} \frac{\alpha_n(3, \nu) |z|^{6n+6}}{|\lambda \alpha_{n+1}(3, \nu) + \alpha_n(3, \nu)|^2} \leq \frac{1}{4\lambda} \sum_{n=0}^{\infty} \frac{|z|^{6n+6}}{\lambda \alpha_{n+1}(3, \nu) + \alpha_n(3, \nu)} \leq \frac{1}{4\lambda} I_{3,\nu} \left( |z|^2 \right). \]

This proves (i).

ii. From (9) we have

\[ \| f^{*}_{J,L}(h) \|_{F_{\lambda}(C)}^2 = \sum_{n=1}^{\infty} \alpha_n(3, \nu) \left[ \frac{\alpha_{n-1}(3, \nu) |h_{n-1}|^2}{\lambda \alpha_n(3, \nu) + \alpha_{n-1}(3, \nu)} \right]. \]

Using the fact that \( (x+y)^2 \geq 4xy \) we obtain

\[ \| f^{*}_{J,L}(h) \|_{F_{\lambda}(C)}^2 \leq \frac{1}{4\lambda} \sum_{n=1}^{\infty} \alpha_{n-1}(3, \nu) |h_{n-1}|^2 = \frac{1}{4\lambda} \| h \|_{F_{\lambda}(C)}^2. \]

This proves (ii) and completes the proof of the theorem. \( \square \)

As in the same way of Theorem 4 we also obtain.

**Remark 1.** If \( \lambda > 0 \) and \( h \in F_{3,\nu}(C) \), then

\[ |Df^{*}_{J,L}(h)(z)|, |Df^{*}_{J,L}(Dh)(z)| \leq \frac{1}{2\sqrt{\lambda}} \left( I_{3,\nu} \left( |z|^2 \right) \right)^{1/2} \| f \|_{F_{\lambda}(C)}. \]
The extremal function \( f^*_{\lambda,D}(h) \) possesses also the following approximation formulas.

**Theorem 5.** If \( \lambda > 0 \) and \( h \in F_{\lambda,3}(C) \), then

\[
\lim_{\lambda \to 0^+} \| D f^*_{\lambda,D}(h) - h \|_{F_{\lambda,3}(C)}^2 = 0,
\]

and

\[
\lim_{\lambda \to 0^+} D f^*_{\lambda,D}(h)(z) = h(z).
\]

**Proof.** Let \( \lambda > 0 \) and \( h \in F_{\lambda,3}(C) \) with \( h(z) = \sum_{n=0}^{\infty} h_n z^{3n} \). From (9) we have

\[
D f^*_{\lambda,D}(h)(z) - h(z) = \sum_{n=0}^{\infty} \frac{-\lambda \alpha_{n+1}(3,\nu) h_n}{\lambda \alpha_{n+1}(3,\nu) + \alpha_n(3,\nu)} z^{3n}.
\]

Therefore

\[
\| D f^*_{\lambda,D}(h) - h \|_{F_{\lambda,3}(C)}^2 = \sum_{n=0}^{\infty} \alpha_n(3,\nu) \left[ \frac{\lambda \alpha_{n+1}(3,\nu) h_n}{\lambda \alpha_{n+1}(3,\nu) + \alpha_n(3,\nu)} \right]^2.
\]

Using the fact that

\[
\alpha_n(3,\nu) \left[ \frac{\lambda \alpha_{n+1}(3,\nu) h_n}{\lambda \alpha_{n+1}(3,\nu) + \alpha_n(3,\nu)} \right]^2 \leq \alpha_n(3,\nu) |h_n|^2,
\]

and

\[
\frac{\lambda \alpha_{n+1}(3,\nu) h_n}{\lambda \alpha_{n+1}(3,\nu) + \alpha_n(3,\nu)} |z|^{3n} \leq |h_n| |z|^{3n},
\]

we obtain the results from dominated convergence theorem. □

**Remark 2.** If \( \lambda > 0 \) and \( h \in F_{3,\nu}(C) \), then

\[
\lim_{\lambda \to 0^+} \| f^*_{\lambda,D}(Dh) - (h - h_0) \|_{F_{3,\nu}(C)} = 0,
\]

and

\[
\lim_{\lambda \to 0^+} f^*_{\lambda,D}(Dh)(z) = h(z) - h(0),
\]

where \( h_0(z) = h(0) \).

### 3.2 The integral operator

Let \( P \) be the integral operator defined by

\[
P f(z) = \langle f, \Phi_z \rangle_{F_{\lambda,3}(C)} = \int_{C} f(w) \Phi_z(w) d \mu_{3,\nu}(w),
\]
Theorem 6.

i. The operator $P$ maps continuously from $F_{3,\nu}(\mathbb{C})$ into $F_{3,\nu}(\mathbb{C})$, and

$$\|Pf\|_{F_{3,\nu}(\mathbb{C})} \leq \sqrt{27(\nu+1)}\|f\|_{F_{3,\nu}(\mathbb{C})}.$$ 

ii. For $f \in F_{3,\nu}(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$P^*f(z) = \sum_{n=0}^{\infty} \frac{a_{n+1}(3,\nu) a_n z^n}{(\sqrt{n+1})^3}.$$ 

iii. For any $h \in F_{3,\nu}(\mathbb{C})$ and for any $\lambda > 0$, the problem

$$\inf_{f \in F_{3,\nu}(\mathbb{C})} \left\{ \lambda \|f\|_{F_{3,\nu}(\mathbb{C})}^2 + \|Pf - h\|_{F_{3,\nu}(\mathbb{C})}^2 \right\}$$

has a unique minimizer given by

$$f_{\lambda,P}(h)(z) = \langle h, \Psi_z \rangle_{F_{3,\nu}(\mathbb{C})},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\sqrt{n})^3 (\sqrt{n+1}) \alpha_n(3,\nu) z^n}{(\sqrt{n+1})^3} a_n(3,\nu), \quad w \in \mathbb{C}.$$ 

Proof.

i. If $f \in F_{3,\nu}(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|Pf\|_{F_{3,\nu}(\mathbb{C})}^2 = \sum_{n=1}^{\infty} \frac{a_n(3,\nu) a_{n-1}^2}{n^3} \leq 27(\nu+1) \sum_{n=0}^{\infty} \frac{a_n(3,\nu) a_n^2}{n^3} = 27(\nu+1)\|f\|_{F_{3,\nu}(\mathbb{C})}^2.$$ 

ii. If $f, g \in F_{3,\nu}(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle Pf, g \rangle_{F_{3,\nu}(\mathbb{C})} = \sum_{n=1}^{\infty} \frac{a_n(3,\nu)}{(\sqrt{n})^3} \overline{b}_n = \sum_{n=0}^{\infty} \frac{a_{n+1}(3,\nu)}{(\sqrt{n+1})^3} \overline{b}_{n+1} = \langle f, P^*g \rangle_{F_{3,\nu}(\mathbb{C})}.$$
where
\[ P^*g(z) = \sum_{n=0}^{\infty} \frac{\alpha_{n+1}(3, \nu) b_{n+1}}{(n+1)^3} z^{3n}. \]

And therefore
\[ P^* P^f(z) = \sum_{n=0}^{\infty} \frac{\alpha_{n+1}(3, \nu) a_n}{(n+1)^3} z^{3n}. \]

iii. We put \( h(z) = \sum_{n=0}^{\infty} c_n z^{3n} \) and \( f^*_j, p(h)(z) = \sum_{n=0}^{\infty} c_n z^{3n} \). From (8) we have \((\lambda I + P^* P)f^*_j, p(h)(z) = P^* h(z)\). By (ii) we deduce that
\[ c_n = \left( \frac{\sqrt{n+1}}{Z n+1} \right)^3 a_{n+1}(3, \nu) h_{n+1} \quad \text{with} \quad \lambda n^3 a_{n-1}(3, \nu) + a_n(3, \nu) = 0 \quad \text{and} \quad c_0 = 1. \]

Thus
\[ f^*_j, p(h)(z) = \sum_{n=1}^{\infty} \left( \frac{\sqrt{n}}{Z n^3} \right)^3 a_{n-1}(3, \nu) h_n + a_n(3, \nu) = \langle h, \Psi_z \rangle_{F_{\nu_1}(C)}, \]

where
\[ \Psi_z(w) = \sum_{n=1}^{\infty} \left( \frac{\sqrt{n}}{Z n^3} \right)^3 a_{n-1}(3, \nu) h_n + a_n(3, \nu). \]

This completes the proof of the theorem. \( \square \)

The extremal function \( f^*_j, p(h) \) possesses the following properties.

**Theorem 7.** If \( \lambda > 0 \) and \( h \in F_{\lambda,j}(C) \), then
\[ \|f^*_j, p(h)(z)\|_{F_{\nu_1}(C)} \leq \frac{1}{2\sqrt{\lambda}} \|h\|_{F_{\nu_1}(C)}, \]
\[ \|f^*_j, p(h)\|_{F_{\nu_1}(C)} \leq \frac{1}{2\sqrt{\lambda}} \|h\|_{F_{\nu_1}(C)}. \]

**Proof.** Let \( \lambda > 0 \) and \( h \in F_{\lambda,j}(C) \) with \( h(z) = \sum_{n=0}^{\infty} c_n z^{3n} \).

i. From (10) we have
\[ |f^*_j, p(h)(z)| \leq \|\Psi_z\|_{F_{\nu_1}(C)} \|h\|_{F_{\nu_1}(C)}. \]

And by using the fact that \((x + y)^2 \geq 4xy\) we obtain
\[ \|\Psi_z\|_{F_{\nu_1}(C)}^2 = \sum_{n=1}^{\infty} \frac{n^3 a_{n-1}(3, \nu) h_n}{[\lambda n^3 a_{n-1}(3, \nu) + a_n(3, \nu)]^3} \leq \frac{1}{4\lambda} \sum_{n=0}^{\infty} \frac{|w|^{6n}}{|w|^{6n} - 6} \leq \frac{1}{4\lambda} I_{3, \nu}(\|z\|^2). \]
This proves (i).

ii. From (10) we have

$$\|f_{\lambda,P}(h)\|_{F_{\lambda,c}(C)}^2 = \sum_{n=0}^{\infty} a_n(3,\nu) \left[ \frac{(\sqrt{n+1})^3 a_{n+1}(3,\nu) \|h_{n+1}\|}{\lambda(n+1)^3 a_n(3,\nu) + a_{n+1}(3,\nu)} \right]^2.$$  

Using the fact that $(x + y)^2 \geq 4xy$ we obtain

$$\|f_{\lambda,P}(h)\|_{F_{\lambda,c}(C)}^2 \leq \frac{1}{4\lambda} \sum_{n=0}^{\infty} a_{n+1}(3,\nu) \|h_{n+1}\|^2 \leq \frac{1}{4\lambda} \|h \|_{F_{\lambda,c}(C)}^2.$$  

This proves (ii) and completes the proof of the theorem.  

**Remark 3.** If $\lambda > 0$ and $h \in F_{3,\lambda}(C)$, then

$$|Pf_{\lambda,P}^+(h)(z)|, |Pf_{\lambda,P}^+(Ph)(z)| \leq \frac{3\sqrt{3}}{2} \sqrt{\frac{\nu + 1}{\lambda}} \left( f_{3,\lambda} \left( \| \bar{z} \|^2 \right) \right)^{1/2} \|h\|_{F_{\lambda,c}(C)}.$$  

The extremal function $f_{\lambda,P}^+(h)$ possesses also the following approximation formulas.

**Theorem 8.** If $\lambda > 0$ and $h \in F_{3,\lambda}(C)$, then

$$\lim_{\lambda \to 0^+} \|Pf_{\lambda,P}^+(h) - (h - h_0)\|_{F_{\lambda,c}(C)}^2 = 0,$$

and

$$\lim_{\lambda \to 0^+} Pf_{\lambda,P}^+(h)(x) = h(x) - h(0).$$

**Proof.** Let $\lambda > 0$ and $h \in F_{3,\lambda}(C)$ with $h(x) = \sum_{n=0}^{\infty} h_n x^{3n}$. From (10) we have

$$Pf_{\lambda,P}^+(h)(x) - (h(x) - h(0)) = \sum_{n=1}^{\infty} -\frac{\lambda n^3 a_{n-1}(3,\nu) h_n}{\lambda n^3 a_n(3,\nu) + a_{n-1}(3,\nu) x^{3n}}.$$  

Therefore

$$\|Pf_{\lambda,P}^+(h) - (h - h_0)\|_{F_{\lambda,c}(C)}^2 = \sum_{n=1}^{\infty} a_n(3,\nu) \left[ \frac{\lambda n^3 a_{n-1}(3,\nu) \|h_n\|}{\lambda n^3 a_n(3,\nu) + a_{n-1}(3,\nu) \|x\|^3} \right]^2.$$  

Using the fact that

$$a_n(3,\nu) \left[ \frac{\lambda n^3 a_{n-1}(3,\nu) \|h_n\|}{\lambda n^3 a_n(3,\nu) + a_{n-1}(3,\nu) \|x\|^3} \right]^2 \leq a_n(3,\nu) \|h_n\|^2,$$

and

$$\frac{\lambda n^3 a_{n-1}(3,\nu) \|h_n\|}{\lambda n^3 a_n(3,\nu) + a_{n-1}(3,\nu) \|x\|^3} \|x\|^3 \leq \|h_n\| \|x\|^3,$$

we obtain the results from dominated convergence theorem.  

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As in the same of Theorem 8 we also obtain.

**Remark 4.** If $\lambda > 0$ and $h \in F_{3,\lambda}(\mathbb{C})$, then

$$\lim_{\lambda \to 0^+} \| f^{*}_{\lambda} \langle Ph \rangle - h \|_{F_{3,\lambda}(\mathbb{C})}^2 = 0,$$

and

$$\lim_{\lambda \to 0^+} f^{*}_{\lambda} \langle Ph \rangle(z) = h(z).$$

4. **Conclusion and perspectives**

Bargmann [1] in 1961 introduced the classical Fock space $F(\mathbb{C})$ and Cholewinsky [2] in 1984 introduced the generalized Fock space $F_{2,\lambda}(\mathbb{C})$. The Bessel-type Fock space $F_{3,\lambda}(\mathbb{C})$ introduced in this work generalizes these analytic spaces. We are studied the Tikhonov regularization problem associated to this Hilbert space, and we are established the extremal function for this problem. Finally, in a future paper we have the idea to study the Bessel-type Segal-Bargmann transform, in which we will prove inversion formula, Plancherel formula and some uncertainty inequalities for this transform.

**Conflicts of interest**

The author declares that there is no conflict of interests regarding the publication of this paper.

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