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Relaxation Dynamics of Point Vortices

Ken Sawada and Takashi Suzuki

Abstract

We study a model describing relaxation dynamics of point vortices, from quasi-stationary state to the stationary state. It takes the form of a mean field equation of Brownian point vortices derived from Chavanis, and is formulated by our previous work as a limit equation of the patch model studied by Robert-Someria. This model is subject to the micro-canonical statistic laws; conservation of energy, that of mass, and increasing of the entropy. We study the existence and nonexistence of the global-in-time solution. It is known that this profile is controlled by a bound of the negative inverse temperature. Here we prove a rigorous result for radially symmetric case. Hence $E/M^2$ large and small imply the global-in-time and blowup in finite time of the solution, respectively. Where $E$ and $M$ denote the total energy and the total mass, respectively.

Keywords: point vortex, quasi-equilibrium, relaxation dynamics

1. Introduction

Our purpose is to study the system

$$
\begin{align*}
\omega_t + \nabla \cdot \omega \nabla \psi &= \nabla \cdot (\nabla \omega + \beta \omega \nabla \psi) \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \omega}{\partial \nu} + \beta \frac{\partial \psi}{\partial \nu} |_{\partial \Omega} &= 0, \quad \omega|_{t=0} = \omega_0(x)
\end{align*}
$$

(1)

with

$$
-\Delta \psi = \omega \quad \text{in } \Omega, \quad \psi|_{\partial \Omega} = 0, \quad \beta = -\frac{\int_{\Omega} \nabla \omega \cdot \nabla \psi}{\int_{\Omega} |\nabla \psi|^2},
$$

(2)

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$, $\nu$ is the outer unit normal vector on $\partial \Omega$, and

$$
\nabla = \left( \begin{array}{c} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{array} \right), \quad \nabla^\perp = \left( \begin{array}{c} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{array} \right), \quad x = (x_1, x_2).
$$

(3)

The unknown $\omega = \omega(x, t) \in \mathbb{R}$ stands for a mean field limit of many point vortices,
\[ \omega(x,t)dx = \sum_{i=1}^{N} \alpha_i \delta_{\omega_i(t)}(dx). \quad (4) \]

It was derived, first, for Brownian point vortices by [1, 2], with \( \beta = \beta(t) \) standing for the inverse temperature. Then, [3, 4] reached it by the Lynden-Bell theory [5] of relaxation dynamics, that is, as a model describing the movement of the mean field of many point vortices, from quasi-stationary state to the stationary state. This model is consistent to the Onsager theory [6–12] on stationary states and also the patch model proposed by [13, 14], that is,

\[ \omega(x,t) = \sum_{i=1}^{N_p} \sigma_i \Omega_i(t)(x), \quad (5) \]

where \( N_p, \sigma_i, \) and \( \Omega_i(t) \) denote the number of patches, the vorticity of the \( i \)-th patch, and the domain of the \( i \)-th patch, respectively [15–17].

This chapter is concerned on the one-sided case of

\[ \omega_0 = \omega_0(x) > 0. \quad (6) \]

If this initial value is smooth, there is a unique classical solution to (1)–(4) local in time, denoted by \( \omega = \omega(x,t) \), with the maximal existence time \( T = T_{\text{max}} \in [0, +\infty] \). More precisely, the strong maximum principle to (1) guaranties

\[ \omega = \omega(x,t) > 0 \quad \text{on } \Omega \times [0,T). \quad (7) \]

Then, the Hopf lemma to the Poisson equation in (2) ensures

\[ \frac{\partial \psi}{\partial n}|_{\partial \Omega} < 0, \quad (8) \]

and hence the well-definedness of

\[ -\beta = \left[ \frac{\nabla \omega \cdot \nabla \psi}{|\nabla \psi|^2} \right]. \quad (9) \]

We confirm that system (1)–(3) satisfies the requirements of isolated system of thermodynamics. First, the mass conservation is derived from (1) as

\[ \frac{d}{dt} \int_{\Omega} \omega = 0, \quad (10) \]

because

\[ \nu \cdot \nabla \psi|_{\partial \Omega} = 0 \quad (11) \]

holds by (2). Second, the energy conservation follows as

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \psi\|^2 = (\nabla \psi, \nabla \psi) = (\omega, \psi) \\
= (\omega \nabla^2 \psi, \nabla \psi) - (\nabla \omega + \beta \omega \nabla \psi, \nabla \psi) \\
= -(\nabla \omega, \nabla \psi) - \beta \int_{\Omega} \omega |\nabla \psi|^2 = 0
\]

(12)
by (1) and (2), because
\[ \nabla \cdot \nabla \psi = 0, \quad (13) \]
where \((, )\) denotes the \(L^2\) inner product. Third, the entropy increasing is achieved, writing (1) as
\[ \omega_t = \nabla \cdot \omega (\nabla \psi) + \frac{\partial}{\partial \nu} (\log \omega + \beta \psi) \bigg|_{\partial \Omega} = 0. \quad (14) \]
In fact, it then follows that
\[ \int_{\Omega} \omega_t (\log \omega + \beta \psi) = \int_{\Omega} \omega \nabla \cdot (\log \psi + \beta \psi) - \omega |\nabla (\log \omega + \beta \psi)|^2 \, dx \quad (15) \]
with
\[ \int_{\Omega} \omega \nabla \cdot (\log \omega + \beta \psi) = \int_{\Omega} \nabla \cdot \nabla \psi \]
from (11) and
\[ \nabla \cdot \nabla = \nabla \cdot \nabla^\perp = 0. \quad (17) \]
Since
\[ \int_{\Omega} \omega \log \omega = \frac{d}{dt} \int_{\Omega} \omega (\log \omega - 1), \quad \int_{\Omega} \omega \psi = \frac{1}{2} \frac{d}{dt} \|\nabla \psi\|_2^2 = 0, \quad (18) \]
We thus end up with the mass conservation
\[ M = \int_{\Omega} \omega, \quad (19) \]
the energy conservation
\[ E = \| \nabla \psi \|_2^2 = (\psi, \omega), \quad (20) \]
and the entropy increasing
\[ \frac{d}{dt} \int_{\Omega} \omega (\log \omega - 1) = - \int_{\Omega} \omega |\nabla (\log \omega + \beta \psi)|^2 \leq 0. \quad (21) \]
Henceforth, \(C > 0\) stands for a generic constant. In the previous work [4] we studied radially symmetric solutions and obtained a criterion for the existence of the solution global in time. Here, we refine the result as follows, where \(B(0, 1)\) denotes the unit ball.

**Theorem 1** Let
\[ \Omega = B(0, 1), \quad \omega_0 = \omega_0(r), \quad \omega_0 < 0, \quad 0 < r = |x| \leq 1. \quad (22) \]
Then there is $C_0 > 0$ such that

$$C_0 \| \omega_0 \|_2^2 \leq E_0 \Rightarrow T = +\infty, \quad \| \omega(\cdot, t) \|_\infty \leq C, \quad t \geq 0,$$

where

$$\omega = \min_{\Omega} \omega_0 > 0.$$ (24)

**Theorem 2** Under the assumption of (22) there is $\delta_0 > 0$ such that

$$\frac{E}{M^2} < \delta_0 \Rightarrow T < +\infty.$$ (25)

**Remark 1** Since

$$\| \omega_0 \|_2^2 = \left( \int_{\Omega} \omega_0^2 \right)^{3/2} \geq \left( \int_{\Omega} \omega_0^{4/3} \right)^{3/2},$$

$$= \omega \left( \int_{\Omega} \omega_0^{4/3} \right)^{3/2} \geq \omega |\Omega|^{-1/2} \left( \int_{\Omega} \omega_0 \right)^2 = \omega |\Omega|^{-1/2} M^2$$ (26)

the assumption (23) implies

$$\frac{E}{M^2} \geq C_0 |\Omega|^{-1/2}.$$ (27)

Therefore, roughly, the conditions $E/M^2 \gg 1$ and $E/M^2 \ll 1$ imply $T = +\infty$ and $T < +\infty$, respectively.

**Remark 2** The assumption (22) implies

$$\beta = \beta(t) < 0, \quad 0 \leq t < T,$$ (28)

and then we obtain Theorem 1. In other words, the conclusion of this theorem arises from (28), without (22).

**Remark 3** Since

$$\frac{E}{M^2} = \frac{\int_{\Omega} |\nabla \psi|^2}{\left( \int_{\Omega} \omega \right)^2},$$ (29)

it holds that

$$\frac{E}{M^2} = \| \nabla c \|_2^2, \quad c = \frac{(-\Delta)^{-1} \omega_0}{\int_{\Omega} \omega_0} = \frac{\psi_0}{\int_{\Omega} \omega_0}$$ (30)

where $\psi_0 = (-\Delta)^{-1} \omega_0$.

The system (1)–(4) thus obeys a profile of the micro-canonical ensemble. In a system associated with the canonical ensemble, the inverse temperature $\beta$ is a constant in (1) independent of $t$, with the third equality in (2) eliminated:

$$\omega_t + V \cdot \omega V^4 \psi = \nabla \cdot \left( \nabla \omega + \beta \omega \nabla \psi \right), \quad \frac{\partial \omega}{\partial t} + \beta \omega \frac{\partial \psi}{\partial t} = 0, \quad \omega|_{t=0} = \omega_0(x) > 0$$

$$-\Delta \psi = \omega, \quad \psi|_{\partial \Omega} = 0.$$ (31)
Then there arise the mass conservation
\[ \frac{d}{dt} \int_{\Omega} \omega = 0, \tag{32} \]
and the free energy decreasing
\[ \frac{d}{dt} \int_{\Omega} \omega (\log \omega - 1) + \frac{\beta}{2} |\nabla \psi|^2 \, dx = - \int_{\Omega} \omega |\nabla (\log \omega + \beta \psi)|^2 \leq 0. \tag{33} \]
The system (31) without vortex term,
\[ \omega_t = \nabla \cdot (\nabla \omega + \beta \omega \nabla \psi), \quad \frac{\partial \omega}{\partial \nu} + \beta \omega \frac{\partial \psi}{\partial \nu} \bigg|_{\partial \Omega} = 0, \quad \omega|_{t=0} = \omega_0(x) > 0 \tag{34} \]
is called the Smoluchowski-Poisson equation. This model is concerned on the thermodynamics of self-gravitating Brownian particles [18] and has been studied in the context of chemotaxis [19–23]. We have a blowup threshold to (34) as a consequence of the quantized blowup mechanism [19, 23]. The results on the existence of the bounded global-in-time solution [24–26] and blowup of the solution in finite time [27] are valid even to the case that \( \beta \) is a function of \( t \) as in \( \beta = \beta(t) \), provided with the vortex term \( \nabla \cdot \omega \nabla^2 \psi \) on the right-hand side. We thus obtain the following theorems.

**Theorem 3** It holds that
\[ -\beta(t) \leq \delta, \quad \|\omega_0\|_1 < 8\pi \delta^{-1} \Rightarrow T = +\infty, \quad \|\omega(\cdot, t)\|_\infty \leq C \tag{35} \]
in (31), where \( \delta > 0 \) is arbitrary.

**Theorem 4** It holds that
\[ -\beta(t) \geq \delta, \quad \|\omega_0\|_1 > 8\pi \delta^{-1} \Rightarrow \exists \omega_0 > 0, \quad \|\omega_0\|_1 > 8\pi \delta^{-1} \text{ such that } T < +\infty \tag{36} \]
in (31), where \( \delta > 0 \) is arbitrary.

**Remark 4** In the context of chemotaxis in biology, the boundary condition of \( \psi \) is required to be the form of Neumann zero. The Poisson equation in (34) is thus replaced by
\[ -\Delta \psi = \omega - \frac{1}{|\Omega|} \int_{\Omega} \omega, \quad \frac{\partial \psi}{\partial \nu} \bigg|_{\partial \Omega} = 0 \tag{37} \]
or
\[ -\Delta \psi + \psi = \omega, \quad \frac{\partial \psi}{\partial \nu} \bigg|_{\partial \Omega} = 0 \tag{38} \]
by [28] and [29], respectively. In this case there arises the boundary blowup, which reduces the value \( 8\pi \) in Theorems 3–4 to \( 4\pi \). The value \( 8\pi \) in Theorems 3–4, therefore, is a consequence of the exclusion of the boundary blowup [30]. This property is valid even for (37) or (38) of the Poisson part, if (22) is assumed.

**Remark 5** The requirement to \( \omega_0 \) in Theorem 4 is the concentration at an interior point, which is not necessary in the case of (22). Hence Theorems 3 and 4 are refined as
\[ -\beta(t) \leq \delta, \quad \|\omega_0\|_1 < 8\pi \delta^{-1} \Rightarrow T = +\infty, \quad \|\omega(\cdot, t)\|_\infty \leq C \tag{39} \]
and

$$-\beta(t) \geq \delta, \quad \|\omega_0\|_1 > 8\pi\delta^{-1} \Rightarrow T < +\infty,$$

(40)

if (22) holds in (35). The main task for the proof of Theorems 1 and 2, therefore, is a control of $\beta = \beta(t)$ in (1).

This paper is composed of four sections and an appendix. Section 2 is devoted to the study on the stationary solutions, and Theorems 1 and 2 are proven in Sections 3 and 4, respectively. Then Theorem 4 is confirmed in Appendix.

2. Stationary states

First, we take the canonical system (31) with $\beta$ independent of $t$. By (32) and (33), its stationary state is defined by

$$\log \omega + \beta \psi = \text{constant}, \quad \omega = \omega(x) > 0, \quad \int_{\Omega} \omega = M. \tag{41}$$

Then it holds that

$$\omega = \frac{Me^{-\beta \psi}}{\int_{\Omega} e^{-\beta \psi}}, \tag{42}$$

and hence

$$-\Delta \psi = \frac{Me^{-\beta \psi}}{\int_{\Omega} e^{-\beta \psi}}, \quad \psi|_{\partial \Omega} = 0. \tag{43}$$

There arises an ordered structure arises in $\beta < 0$, as observed by [11], as a consequence of a quantized blowup mechanism [19, 20, 31]. In the micro-canonical system (1) and (2), the value $\beta$ in (43) has to be determined by $E$ besides $M$.

Equality (21), however, still ensures (41) and hence (42) in the stationary state even for (1)–(3). Writing

$$v = -\beta \psi, \quad \mu = -\frac{\beta M}{\int_{\Omega} e^{-\beta \psi}}, \tag{44}$$

we obtain

$$-\Delta v = \mu e^v \quad \text{in} \quad \Omega, \quad v|_{\partial \Omega} = 0, \quad \frac{E}{M^2} = \frac{\|\nabla v\|_2^2}{\left(\int_{\Omega} e^{-\beta \psi}\right)^2}, \tag{45}$$

by (30) and (43).

This system is the stationary state of (1) and (2) introduced by [4]. The first two equalities

$$-\Delta v = \mu e^v, \quad v|_{\partial \Omega} = 0 \tag{46}$$

comprise a nonlinear elliptic eigenvalue problem and the unknown eigenvalue $\mu$ is determined by the third equality,
The elliptic theory ensures rather detailed features of the set of solutions to (46). Here we note the following facts [31].

1. There is \( \overline{\mu} = \mu(\Omega) > 0 \) such that the problem (46) does not admit a solution for \( \mu > \overline{\mu} \).

2. Each \( \mu \leq 0 \) admits a unique solution.

3. Each \( 0 < \delta < \overline{\mu} \) admits a constant \( C = C(\delta) > 0 \) such that \( \|v\|_\infty \leq C \) for any solution \( v = v(x) \).

4. There is a family of solutions \( \{ (\mu, v) \} \) such that \( \mu \downarrow 0 \) and \( \|v\|_\infty \rightarrow +\infty \).

We show the following theorem, consistent to Theorem 2.

**Theorem 5** If \( \Omega = B(0, 1) \subset \mathbb{R}^2 \), there is \( \delta > 0 \) such that any solution \( v, \mu(\Omega) \) to (45) admits

\[
\frac{E}{M^2} \geq \delta. \tag{48}
\]

**Proof:** If \( \mu = 0 \), it holds that \( v = 0 \). We have \( \pm v > 0 \) exclusively in \( \Omega \), provided that \( \pm \mu > 0 \), respectively. By the elliptic theory [32], therefore, any solution \( v \) to (46) is radially symmetric as in \( v = v(r) \), \( r = |x| \). We have, furthermore, \( \pm v_r < 0 \) in \( 0 < r \leq 1 \), if \( \pm \mu > 0 \), respectively.

Then it holds that \( \psi = \psi(r) \), and hence

\[- \frac{1}{r} (r\psi)'_r = \omega \quad \text{in} \quad 0 < r \leq 1, \quad \psi|_{r=1} = 0 \quad \tag{49}
\]

by (42) and (43), which implies

\[-r\psi_r(r) = \int_0^1 s\omega(s)ds > 0, \quad 0 < r \leq 1. \quad \tag{50}
\]

We thus obtain \( \mu \neq 0 \), in particular.

If \( \mu < 0 \) we have \( \beta > 0 \) by (44), and therefore, \( \psi_r > 0 \) in \( 0 < r \leq 1 \) by \( v_r > 0 \) there. It is a contradiction, and hence \( \mu > 0 \). In this case, the solution \( v = v(r) \) to (46) is explicit [31]. The numbers of the solution is 0, 1, and 2, according to \( \mu > 2 \), \( \mu = 2 \), and \( 0 < \mu < 2 \), respectively, and if \( 0 < \mu \leq 2 \) the solutions \( v = v_\pm \) are given as

\[v_\pm(r) = \log \frac{8\gamma_r r}{(1 + \gamma_r r^2)^2}, \quad \gamma_\pm = \frac{4}{\mu} \left\{ 1 - \frac{\mu}{4} \pm \left( 1 - \frac{\mu}{2} \right)^{1/2} \right\}. \quad \tag{51}
\]

In fact, we have \( \gamma_+ = \gamma_- \) for \( \mu = 2 \).

This solution is parametrized by

\[\sigma = \int_\Omega \mu \nu^\mu \in (0, 8\pi). \quad \tag{52}
\]

Hence each \( 0 < \sigma < 8\pi \) admits a unique solution \( (\nu, \mu) \) to (46), and \( v = v_+ \) and \( v = v_- \) according as \( \sigma \geq 4\pi \) and \( \sigma \leq 4\pi \), respectively. It holds also that \( \mu \downarrow 0 \) if either
\[ \sigma \uparrow 8\pi \text{ or } \sigma \downarrow 0. \text{ Thus we have only to confirm that } E/M^2 \text{ is bounded, both as } \sigma \uparrow 8\pi \text{ and } \sigma \downarrow 0. \]

As \( \sigma \uparrow 8\pi \), we have
\[ v = v_+(x) \rightarrow 4 \log \frac{1}{|x|} \text{ locally uniformly on } \Omega \setminus \{0\} \quad (53) \]
and hence
\[ \|\nabla v\|_2^2 \rightarrow +\infty, \quad \int_{\partial \Omega} -\frac{\partial v}{\partial v} \rightarrow 8\pi, \quad (54) \]
which implies
\[ \lim_{\sigma \uparrow 8\pi} \frac{E}{M^2} = +\infty. \quad (55) \]

As \( \sigma \downarrow 0 \), on the other hand, we have
\[ v = v_-(x) \rightarrow 0 \text{ uniformly in } \Omega. \quad (56) \]

Since \( \mu \downarrow 0 \), furthermore, there arises that
\[ \gamma = \gamma_- = -\frac{4}{\mu} \left( 1 - \frac{\mu}{4} - \left( 1 - \frac{\mu}{2} \right)^{1/2} \right) = \mu(1 + o(1)). \quad (57) \]

It holds also that
\[ v(r) = \log \frac{8\gamma}{\mu} - 2 \log (1 + \mu^2) \quad (58) \]
and hence
\[ v_\nu(r) = -\frac{4\mu r}{(1 + \mu^2)^2} = -4 \mu r(1 + o(1)) \text{ uniformly on } \overline{\Omega}. \quad (59) \]

Then, (59) implies
\[ \|\nabla v\|_2^2 = 2\pi \int_0^1 v_\nu^2 r \, dr = 2\pi \cdot 16\mu^2 \cdot \int_0^1 r^3 \, dr \cdot (1 + o(1)) \]
\[ = 8\pi \mu^2 (1 + o(1)) \quad (60) \]
as well as
\[ \left( \int_{\partial \Omega} -\frac{\partial v}{\partial v} \right)^2 = 16\mu^2 \cdot 2\pi(1 + o(1)). \quad (61) \]

It thus follows that
\[ \lim_{\sigma \downarrow 0} \frac{E}{M^2} = \frac{1}{4} \quad (62) \]
and hence the conclusion. \( \square \)
3. Proof of Theorem 1

The first observation is the following lemma.

**Lemma 1** Under the assumption of (22), it holds that
\[ \beta = \beta(t) < 0, \quad \omega_r(r,t) < 0, \quad 0 < r \leq 1, \quad 0 \leq t < T. \] (63)

**Proof:** We have (7) and hence
\[ \psi_r(r,t) < 0, \quad 0 < r \leq 1, \quad 0 \leq t < T \] (64)
by (49), which implies, in particular,
\[ \beta = -\frac{(\nabla \omega, \nabla \psi)}{\int \omega |\nabla \psi|^2} < 0 \] (65)
at \( t = 0 \) by (22).
Since \( \omega = \omega(r,t) \) and \( \psi = \psi(r,t) \), we obtain \( \nabla^2 \psi = 0 \), and hence
\[ \omega_r = \omega_{rr} + r \omega_r + \beta \psi_r \omega_r - \beta \omega^2 \] (66)
by (1). Then \( z = \omega_r \) satisfies
\[ z_t = z_{rr} - \frac{1}{r^2} z + \frac{1}{r} z_r + \beta \psi_r z + \beta \psi_z z_r - 2 \beta \omega z, \quad 0 < r \leq 1, \quad 0 \leq t < T \] (67)
\[ z|_{r=0} = 0, \quad z|_{t=0} = \omega_r(r) < 0, \quad 0 < r \leq 1 \]
and
\[ z = -\beta \omega_z, \quad r = 1, \quad 0 \leq t < T. \] (68)

Putting
\[ m(t) = \min_{\omega_r} z(\cdot, t) = \omega_r(\cdot, t)|_{r=1}, \] (69)
we obtain \( m(0) < 0 \) from the assumption. If there is \( 0 < t_0 \) such that
\[ m(t) < 0, \quad 0 \leq t < t_0 < T, \quad m(t_0) = 0, \] (70)
we obtain \( z(r,t) > 0 \) for \( 0 \leq t < t_0, \quad 0 < r \leq 1 \), and \( t = t_0, \quad 0 < r < 1 \) by the strong maximum principle. By (64), we have (65) for \( 0 \leq t \leq t_0 \), that is,
\[ \beta = -\frac{\int_0^1 \omega_r \psi_z dr}{\int_0^1 \omega \psi_r^2 dr} < 0, \quad 0 \leq t \leq t_0, \] (71)
and hence
\[ z = -\beta \omega z < 0, \quad r = 1, \quad t = t_0, \] (72)
a contradiction. It holds that \( z = \omega_r < 0 \) for \( 0 \leq t < T, \quad r = 1 \), and hence
\[ \beta = -\frac{\int_0^1 \omega_r \psi_r r dr}{\int_0^1 \omega \psi_r^2 r dr} < 0, \quad 0 \leq t < T. \] (73)
The proof of Theorem 3 relies on the fact
\[
\beta \geq - C, \quad \int_{\Omega} \omega (\log \omega - 1) \leq C \Rightarrow T = +\infty, \quad \|\omega(\cdot, t)\|_{\infty} \leq C. \quad (74)
\]

This property is known for the Smoluchowski-Poisson equation (34), but the proof is valid even to (31) with vortex term. Having (21), therefore, we have to provide the inequality \(\beta \geq - C\).

The inequality \(\beta < 0\), on the other hand, is sufficient for the following arguments.

**Lemma 2** If \(\beta \leq 0\), \(0 \leq t < T\), it holds that
\[
\omega \geq \omega_0 \quad \text{min} \quad \Omega_0 > 0 \quad \text{on} \quad \Omega \times (0, T).
\]

**Proof:** Since (17) we obtain
\[
\omega_t + \nabla^+ \psi \cdot \nabla \omega = \Delta \omega + \beta \nabla \psi \cdot \nabla \omega + \beta \Delta \psi
\]
\[
= \Delta \omega + \beta \nabla \psi \cdot \nabla \omega - \beta \omega^2
\]
\[
\geq \Delta \omega + \beta \nabla \psi \cdot \nabla \omega \quad \text{in} \quad \Omega \times (0, T)
\]
with
\[
- \frac{\partial \omega}{\partial \nu} = \beta \omega \frac{\partial \psi}{\partial \nu} > 0 \quad \text{on} \quad \partial \Omega \times [0, T)
\]
by (8). Then the result follows from the comparison theorem. \(\square\)

**Lemma 3** Under the assumption of the previous lemma, there is \(C_0 = C_0(\Omega) > 0\) such that
\[
C_0 \|\omega_0\|^2 \leq E \omega \Rightarrow \|\omega(\cdot, t)\|_2 \leq \|\omega_0\|_2, \quad - \beta(t) \leq \alpha \equiv \frac{\|\omega_0\|^2}{E \omega}, \quad 0 \leq t < T. \quad (78)
\]

**Proof:** Using (11) and (17), we obtain
\[
\int_{\Omega} \nabla \cdot (\omega \nabla^+ \psi) \omega = \int_{\Omega} \omega \nabla \omega \cdot \nabla^+ \psi = \frac{1}{2} \int_{\Omega} \omega \nabla \omega \cdot \nabla \psi
\]
\[
= -\frac{1}{2} \int_{\Omega} \omega^2 \nabla \cdot \nabla^+ \psi = 0. \quad (79)
\]

Hence (1) with (2) implies
\[
\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \|\nabla \omega\|_2^2 = -\beta \int_{\Omega} \omega \nabla \psi \cdot \nabla \omega = -\frac{\beta}{2} (\nabla \psi, \nabla \omega^2)
\]
\[
= -\frac{\beta}{2} \int_{\partial \Omega} \omega \frac{\partial \psi}{\partial \nu} + \frac{\beta}{2} (\Delta \psi, \omega^2) \leq \frac{\beta}{2} \|\omega\|^3
\]
by \(\beta \leq 0\) and (88). Since
\[
\int_{\Omega} \nabla \omega \cdot \nabla \psi = \int_{\Omega} \omega \frac{\partial \psi}{\partial \nu} + \int_{\Omega} \omega (-\Delta \psi) \leq \int_{\Omega} \omega^2
\]
follows from (8), furthermore, it holds that

\[-\beta = \frac{\int \omega \nabla \cdot \nabla \psi}{\int \omega \nabla \psi^2} \leq \omega^{-1} \cdot \frac{\|\omega\|^2_2}{\|\nabla \psi\|^2_2} = \frac{1}{E_\omega} \|\omega\|^2_2. \quad (82)\]

Then inequality (80) induces

\[\frac{1}{2} \frac{d}{dt} \|\omega\|^2_2 + \|\nabla \omega\|^2_2 \leq \frac{1}{2E_\omega} \|\omega\|^2_2 \cdot \|\omega\|^3_3. \quad (83)\]

Here we use the Gagliardo-Nirenberg inequality (see (4.16) of [19]) in the form of

\[\|\omega\|^2_3 \leq C \|\omega\|^2_H \|\omega\|^3_2 = C \|\omega\|^2_2 \|\nabla \omega\|^2_2 + \|\omega\|^2_3, \quad (84)\]

to obtain

\[\frac{1}{2} \frac{d}{dt} \|\omega\|^2_2 + \|\nabla \omega\|^2_2 \leq \frac{C}{E_\omega} \|\omega\|^2_2 \|\nabla \omega\|^2_2 + \|\omega\|^2_3 \leq \frac{1}{2} \|\nabla \omega\|^2_2 + \frac{C^2}{8E_\omega^2} \|\omega\|^4_2 + \frac{C}{2E_\omega} \|\omega\|^5_2 \quad (85)\]

and hence

\[\frac{d}{dt} \|\omega\|^2_2 + \|\nabla \omega\|^2_2 \leq \frac{C}{E_\omega} \|\omega\|^3_2 \left( \frac{C}{E_\omega} \|\omega\|^2_2 + 1 \right) \quad (86)\]

Then, Poincaré-Wirtinger’s inequality ensures

\[\frac{d}{dt} \|\omega\|^2_2 + \mu \|\omega\|^2_2 \leq \frac{C}{E_\omega} \left( \frac{C}{E_\omega} \|\omega\|^5_2 + \|\omega\|^3_2 \right) \|\omega\|^2_2 \quad (87)\]

where \(\mu = \mu(\Omega) > 0\) is a constant.

Writing

\[y(t) = \frac{C}{E_\omega} \|\omega\|^3_2 \quad (88)\]

we obtain

\[\frac{d}{dt} \|\omega\|^2_2 + \mu \|\omega\|^2_2 \leq (y^2 + y) \|\omega\|^2_2 \quad (89)\]

and therefore, if

\[y^2 + y < \mu/2 \quad (90)\]

holds at \(t = 0\), it keeps to hold that

\[\frac{d}{dt} \|\omega\|^2_2 \leq 0 \quad (91)\]

and (90) for \(0 \leq t < T\). Then, we obtain
\[ \| \omega(\cdot, t) \|_2 \leq \| \omega_0 \|_2, \quad 0 \leq t < T, \quad (92) \]

and hence
\[ -\beta(t) \leq \frac{\| \omega_0 \|_2^2}{E_0} = \alpha, \quad 0 \leq t < T \quad (93) \]

by (82).

The condition \( y(0) < \frac{\delta}{C_0} \) means
\[ C_0 \| \omega_0 \|_2 \leq E_0 \quad (94) \]

for \( C_0 > 0 \) sufficiently large, and hence we obtain the conclusion. \( \square \)

**Proof of Theorem 1:** By the parabolic regularity, it suffices to show that
\[ \| \omega(\cdot, t) \|_{\infty} \leq C, \quad 0 \leq t < T \quad (95) \]

under the assumption. We have readily shown
\[ \| \omega(\cdot, t) \|_2 \leq \beta(t), \quad 0 \leq t < T \quad (96) \]

by Lemma 3. Then, the conclusion (95) is obtained similarly to (34). See [26] for more details.

In fact, we have
\[
\int_{\Omega} \nabla \cdot (\omega \nabla \psi) \, d\Omega = -\int_{\Omega} \omega \nabla \cdot \nabla \omega \cdot \nabla \omega = -p \int_{\Omega} \omega \nabla \cdot \nabla \omega \cdot \nabla \omega
\]
\[ = -\frac{p}{p+1} \int_{\Omega} \nabla \cdot \nabla \omega \cdot \nabla \omega^{p+1} = \frac{p}{p+1} \int_{\Omega} \omega^{p+1} \cdot \nabla \cdot \nabla \omega = 0 \quad (97) \]

for \( p > 0 \) by (11) and (34). Then it follows that
\[
\frac{1}{p+1} \frac{d}{dt} \int_{\Omega} \omega^{p+1} + \frac{4p}{(p+1)^2} \| \nabla \omega^{p+1} \|_2^2 = -\beta \int_{\Omega} \omega \nabla \psi \cdot \nabla \omega
\]
\[ = -\beta \frac{p}{p+1} \int_{\Omega} \nabla \cdot \nabla \omega \cdot \nabla \omega^{p+1} \leq -\beta \frac{p}{p+1} \int_{\Omega} \omega^{p+1} (\Delta \psi)
\]
\[ = -\beta \frac{p}{p+1} \int_{\Omega} \omega^{p+1} \leq C \int_{\Omega} \omega^{p+2} \quad (98) \]

by \( \beta < 0 \) and (8). Then, Moser’s iteration scheme ensures (95) as in [33].

**4. Proof of Theorem 2**

We begin with the following lemma.

**Lemma 4** Under the assumption of (22), it holds that
\[ -\beta(t) \geq \delta, \quad 0 \leq t < T, \quad M = \| \omega_0 \|_1 > \frac{8\pi}{\delta} \Rightarrow T < +\infty \quad (99) \]

in (31), where \( \delta > 0 \) is a constant.
Proof: We have \( \omega = \omega(r, t) \) and \( \psi = \psi(r, t) \) for \( r = |x| \) under the assumption, which implies \( \nabla \psi = 0 \). Then we obtain

\[
\nabla \cdot \omega \nabla \psi = \nabla \omega \cdot \nabla \psi = 0 \tag{100}
\]

by (17). It holds also that

\[
\nabla \cdot (\omega \nabla \psi) = \nabla \cdot \left( \omega \frac{x}{r^2} \right) = \left( \nabla \cdot \frac{x}{r^2} \right) \omega \nabla \psi + \frac{x}{r^2} \cdot \nabla (\omega \nabla \psi) = \frac{1}{r} \omega \nabla \psi + \frac{\omega (\nabla \psi) r}{r^2} = \frac{1}{r} (r \omega \nabla \psi),
\]

and therefore, there arises that

\[
\omega_t = \frac{1}{r} (r \omega + \beta \omega \nabla \psi), \quad \omega_r + \beta \omega \nabla \psi \mid_{r=1} = 0. \tag{102}
\]

from (31). Then (102) implies

\[
\begin{align*}
\frac{d}{dt} \int_0^1 \omega r^3 \, dr &= \int_0^1 \omega_t r^3 \, dr + \int_0^1 (r \omega + \beta \omega \nabla \psi) r^2 \, dr \\
&= -\int_0^1 2r^2 (\omega_r + \beta \omega \nabla \psi) \, dr \\
&= -2r^2 \omega \bigg|_{r=0}^{r=1} + \int_0^1 4r \omega - 2 \beta \omega \nabla \psi r^2 \, dr.
\end{align*}
\]

Here we use (50) derived from the Poisson part of (31), that is,

\[
-r \omega \nabla \psi = A(r, t) = \int_0^s \omega (s, t) \, ds. \tag{104}
\]

Putting

\[
\lambda = \int_0^1 \omega_r \, dr = \frac{M}{2\pi}, \tag{105}
\]

we obtain

\[
\begin{align*}
\frac{d}{dt} \int_0^1 \omega r^3 \, dr &= -2 \omega \bigg|_{r=1}^{r=1} + 4\lambda + 2\beta \int_0^1 AA \, dr \\
&= -2 \omega \bigg|_{r=1}^{r=1} + 4\lambda + \beta A^2 \bigg|_{r=0}^{r=1} \\
&= -2 \omega \bigg|_{r=1}^{r=1} + 4\lambda + \beta \lambda^2 \\
&< 4\lambda \left( \beta + \frac{M}{8\pi} \right) \leq 4\lambda \left( -\delta + \frac{M}{8\pi} \right).
\end{align*}
\]

Since \(-\delta + \frac{M}{8\pi} < 0\), therefore, \(T = +\infty\) is impossible, and we obtain \(T < +\infty\). \(\square\)

**Lemma 5** Under the assumption (22), there is \(\delta > 0\) such that

\[
\frac{E}{M^2} < \delta, \quad \beta(t) \leq 0, \quad 0 \leq t < T \Rightarrow \beta(t) \leq -\frac{1}{CE^{3/2}} 0 \leq t < T. \tag{107}
\]
Proof: First, Lemma 1 implies
\[ \omega \geq \omega_* \equiv \omega|_{\partial \Omega}. \] (108)

Second, we have
\[
\begin{align*}
\int_{\Omega} \nabla \psi \cdot \nabla \omega &= \int_{\partial \Omega} \frac{\partial \psi}{\partial n} \omega - \int_{\Omega} (-\Delta \psi) \omega = \omega_* \int_{\partial \Omega} \frac{\partial \psi}{\partial n} + \|\omega\|_2^2 \\
&= \omega_* \int_{\Omega} \Delta \psi + \|\omega\|_2^2 = \|\omega\|_2^2 - \omega_* M,
\end{align*}
\] (109)
and hence
\[
-\beta = \frac{\int_{\Omega} \nabla \psi \cdot \nabla \omega}{\int_{\Omega} |\nabla \psi|^2} = \frac{\|\omega\|_2^2 - \omega_* M}{\int_{\Omega} |\nabla \psi|^2}. \] (110)

Here, we use the Gagliardo-Nirenberg inequality in the form of
\[ \|w\|_4^4 \leq C \|w\|_2 \|w\|_{H^1}, \] (111)
which implies
\[ \int_{\Omega} \omega |\nabla \psi|^2 \leq \|\omega\|_2 \|\nabla \psi\|_4^4 \leq C \|\omega\|_2 \|\nabla \psi\|_2 \|\nabla \psi\|_{H^1} \leq CE^{1/2} \|\omega\|_2^2 \] (112)
by the elliptic estimate of the Poisson equation in (2),
\[ \|\psi\|_{H^2} \leq C \|\omega\|_2. \] (113)

We have, on the other hand,
\[ \omega_* M \leq \frac{M}{E} \int_{\Omega} \omega |\nabla \psi|^2 \] (114)
by (110), and therefore,
\[ -\beta \geq \frac{1}{C E^{1/2}} - \frac{E}{M} \geq \frac{1}{2CE^{1/2}}, \] (115)
provided that
\[ \frac{E}{M^2} < C. \] (116)

Then the conclusion follows. \[ \square \]

Proof of Theorem 2: By Lemma 5, there is \( \delta_0 > 0 \) such that
\[ \frac{E}{M^2} < \delta \Rightarrow -\beta \geq -\frac{1}{C E^{1/2}} \equiv \delta_1, \] (117)
and then, Lemma 4 ensures

\[ M > \frac{8\pi}{\delta_1} \Rightarrow T < +\infty. \quad (118) \]

The assumption in (118) means

\[ \frac{E}{M^2} < \left( \frac{1}{8\pi} \right)^2, \quad (119) \]

and hence we obtain the conclusion. □

### Appendix Proof of Theorem 4

This theorem is valid to the general case of \( \Omega \) and \( \omega_0 \) without (22). We assume \( \delta = 1 \) without loss of generation, so that

\[ \beta \leq -1. \quad (120) \]

We follow the argument [27] concerning (34) with the Poisson part replaced by (42) or (43). Thus we have to take case of the vortex term \( \nabla \cdot \omega \nabla \varphi \), time varying \( \beta = \beta(t) \), and the Dirichlet boundary condition in (31).

We recall the cut-off function used in [34] (see also Chapter 5 of [19]). Hence each \( x_0 \in \Omega \) and \( 0 < R \leq 1 \) admit \( \varphi = \varphi_{x_0, R} \in C^2(\Omega) \) with

\[ \frac{\partial \varphi}{\partial \nu} \bigg|_{\partial \Omega} = 0, \quad 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ in } \Omega \cap B(x_0, R/2), \quad \varphi = 0 \text{ in } \Omega \setminus B(x_0, R), \quad (121) \]

and

\[ |\nabla \varphi| \leq C R^{-1} \varphi^{1/2}, \quad |\nabla^2 \varphi| \leq C R^{-2} \varphi^{1/2}. \quad (122) \]

In more details, we take a cut-off function, denoted by \( \psi \), satisfying (121), using a local conformal mapping, and then put \( \varphi = \psi^4 \).

Let \( \varphi \in C^2(\Omega) \), \( \frac{\partial \varphi}{\partial \nu} \bigg|_{\partial \Omega} = 0. \quad (123) \)

be given. First, we have

\[ \frac{d}{dt} \int_{\Omega} \omega \varphi = \int_{\Omega} \omega \nabla^2 \varphi \cdot \nabla \varphi - (\nabla \omega + \beta \omega \nabla \varphi) \cdot \nabla \varphi \ dx \]

\[ = \int_{\Omega} \omega \nabla^2 \varphi \cdot \nabla \varphi + \omega \Delta \varphi - \beta \omega \nabla \varphi \cdot \nabla \varphi \ dx \quad (124) \]

by (11). It holds that
Let, furthermore, \( x_0 \in \Omega \) and \( 0 < R \ll 1 \) in the above equality. Then,

\[
\varphi = |x - x_0|^2 \varphi_{x_0, R}
\]

(126)
satisfies the requirement (123).

It holds that

\[
\nabla \varphi = 2(x - x_0)\varphi_{x_0, R} + |x - x_0|^2 \nabla \varphi_{x_0, R}
\]

(127)
and hence

\[
|\nabla \varphi| \leq C|x - x_0| \left( \varphi_{x_0, R} + |x - x_0| R^{-1/2} \varphi_{x_0, R}^{1/2} \right) \leq C|x - x_0|^{1/2}. \]

(128)

We obtain, furthermore,

\[
|x' - x_0| \geq 2R, \quad |x - x_0| \leq R \Rightarrow |x - x'| \geq R,
\]

(129)
and hence

\[
|\nabla x G(x, x')| \leq CR^{-1}
\]

(130)
in this case. Then it follows that

\[
|II| \leq CR^{-1} M \int_{\Omega} |x - x_0|^{1/2} \omega(x, t) \ dx \leq CR^{-1} M^{1/2} A^{1/2},
\]

(131)
where

\[
A = \int_{\Omega} |x - x_0|^2 \varphi_{x_0, R} \omega.
\]

(132)

We have, on the other hand,

\[
1 = \iint_{\Omega^2} \omega(x, t) \varphi_{x_0, 2R}^{1/2}(x') \nabla \varphi(x) \omega(x', t) \ dxdx'
\]

\[
= \frac{1}{2} \iint_{\Omega^2} \left[ \varphi_{x_0, 2R}(x') \nabla \varphi(x) \cdot \nabla x G(x, x') + \varphi_{x_0, 2R}(x) \nabla \varphi(x') \cdot \nabla x G(x, x') \right] \omega \otimes \omega,
\]

(133)
where \( G = G(x, x') \) is the Green's function to

\[
-\Delta \varphi = \omega, \quad \omega|_{\partial \Omega} = 0
\]

(134)
and
Here we use the local property of the Green’s function
\[ G(x, x') = \Gamma(x - x') + K(x, x'), \quad K \in C^2(\Omega \times \Omega) \cap C^2(\Omega \times \Omega), \]
where
\[ \Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|} \]
stands for the fundamental solution to \(-\Delta\).
Let
\[ \rho^2_{x_0, R}(x, x') = \varphi_{x_0, 2R}(x') \nabla \varphi(x) \cdot \nabla K(x, x') + \varphi_{x_0, 2R} \nabla \varphi(x') \cdot \nabla K(x, x'). \]
Since (128) implies
\[ |\varphi_{x_0, 2R}(x') \nabla \varphi(x)| \leq C |x - x_0|^{1/2} \rho^2_{x_0, R}(x), \]
it holds that
\[ |\rho^2_{x_0, R}(x, x')| \leq C \left(|x - x_0|^{1/2} \rho^2_{x_0, R}(x) + |x' - x_0|^{1/2} \rho^2_{x_0, R}(x')\right). \]
Then, we obtain
\[ I = \frac{1}{2} \iint_{\Omega \times \Omega} \rho^0_{x_0, R}(x, x') \omega \otimes \omega + III \]
with
\[ |III| \leq CM^{3/2}A^{1/2} \leq CR^{-1}M^{3/2}A^{1/2}, \]
where
\[ \rho^0_{x_0, R}(x, x') = \nabla \Gamma(x - x') \cdot (\varphi_{x_0, 2R}(x') \nabla \varphi(x) - \varphi_{x_0, 2R}(x) \nabla \varphi(x')). \]
Here, we have
\[ \nabla \Gamma(x) = -\frac{x}{2\pi|x|^2}, \]
and therefore,
\[ \rho^0_{x_0, R}(x, x') = \rho^2_{x_0, R}(x, x') + \rho^3_{x_0, R}(x, x') \]
of
\[ \rho^2_{x_0, R}(x, x') = -\frac{1}{2\pi} \frac{x - x'}{|x - x'|^2} \varphi_{x_0, 2R}(x') \cdot (\nabla \varphi(x) - \nabla \varphi(x')), \]
\[ \rho^3_{x_0, R}(x, x') = -\frac{1}{2\pi} \frac{x - x'}{|x - x'|^2} \left(\varphi_{x_0, 2R}(x') - \varphi_{x_0, 2R}(x)\right) \cdot \nabla \varphi(x). \]
Since (128) implies
\[
|\rho^3_{x_0,R}(x,x')| \leq CR^{-1} |\nabla \varphi(x)| \leq CR^{-1} |x - x_0|^{1/2} \varphi_{x_0,R}(x),
\]
there arises that
\[
I = \frac{1}{2} \iint_{\Omega \times \Omega} \rho^2_{x_0,R}(x,x') \omega \otimes \omega + IV,
\]
with
\[
|IV| \leq CR^{-1} M^{3/2} A^{1/2},
\]
similarly.

We have, furthermore,
\[
\begin{align*}
\nabla \varphi(x) - \nabla \varphi(x') &= 2(x - x') \varphi_{x_0,R}(x) + 2(x' - x_0)(\varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')) \\
&= |x' - x_0|^2 (\nabla \varphi_{x_0,R}(x) - \nabla \varphi_{x_0,R}(x')) + \left( |x - x_0|^2 - |x' - x_0|^2 \right) \nabla \varphi_{x_0,R}(x),
\end{align*}
\]
and hence
\[
\rho^2_{x_0,R}(x,x') = -\frac{1}{\pi} \varphi_{x_0,2R}(x') \varphi_{x_0,R}(x) + \rho^4_{x_0,R}(x,x') + \rho^5_{x_0,R}(x,x') + \rho^6_{x_0,R}(x,x')
\]
with
\[
|\rho^2_{x_0,R}(x,x')| \leq C|x - x'|^{-1} \varphi_{x_0,2R}(x') |x' - x_0| |\varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')| \leq CR^{-1} |x' - x_0| \varphi_{x_0,2R}(x'),
\]
\[
|\rho^4_{x_0,R}(x,x')| \leq C|x - x'|^{-1} \varphi_{x_0,2R}(x') |x' - x_0|^2 |\nabla \varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')| \leq CR^{-2} |x' - x_0|^2 \varphi_{x_0,2R}(x') \leq CR^{-2} |x - x_0| \varphi_{x_0,2R}(x'),
\]
and
\[
|\rho^6_{x_0,R}(x,x')| \leq C|x - x'| \varphi_{x_0,2R}(x') |x - x_0|^2 - |x' - x_0|^2 \cdot |\nabla \varphi_{x_0,R}(x)| \leq CR^{-1} |x - x_0| + |x' - x_0| \varphi_{x_0,R}(x') \varphi_{x_0,2R}(x') \leq C(R^{-1} |x - x_0| \varphi_{x_0,R}(x) + R^{-1} |x' - x_0| \varphi_{x_0,2R}(x'))
\]
by
\[
|x - x_0|^2 - |x' - x_0|^2 = |(x - x', x + x' - 2x_0)| \leq |x - x'|(|x - x_0| + |x' - x_0|).
\]

The residual terms are thus treated similarly, and it follows that
\[
\left| I + \frac{1}{2\pi} \int_{\Omega} \omega \varphi_{x_0,R} \cdot \int_{\Omega} \omega \varphi_{x_0,2R} \right| \leq CR^{-1} M^{3/2} A^{1/2},
\]
which results in
\[
\left| \int_{\Omega} \omega \nabla \psi \cdot \nabla \varphi + \frac{1}{2\pi} \int_{\Omega} \omega \varphi_{x_R} \cdot \int_{\Omega} \omega \varphi_{x_2} \right| \leq C R^{-1} M^{3/2} A^{1/2}. \tag{158}
\]

We can argue similarly to the vortex term in (124). This time, from
\[
\nabla^\perp \Gamma(x) : x = 0
\tag{159}
\]

it follows that
\[
\left| \int_{\Omega} \omega \nabla^\perp \psi \cdot \nabla \varphi \right| \leq C R^{-1} M^{3/2} A^{1/2}. \tag{160}
\]

Concerning the principal term of (124), we use
\[
\Delta \varphi = 4 \varphi_{x_R} + 4(x - x_0) \cdot \nabla \varphi_{x_R} + |x - x_0|^2 \Delta \varphi_{x_R}. \tag{161}
\]

From
\[
|\nabla \varphi_{x_R}| \leq C R^{-1} |x - x_0|^3 \tag{162}
\]

and
\[
|\nabla \varphi_{x_R}| \leq C R^{-2} |x - x_0|^2 \tag{163}
\]

it follows that
\[
\left| \int_{\Omega} \omega \Delta \varphi - 4 \int_{\Omega} \omega \varphi_{x_R} \right| \leq C \int_{\Omega} R^{-1} |x - x_0|^3 \omega \tag{164}
\]
\[
\leq C R^{-1} M^{1/2} A^{1/2}. \tag{165}
\]

Let \( M_1 = M_{x_R} \) and \( M_2 = M_{x_2} \) for
\[
M_{x_R} = \int_{\Omega} \omega \varphi_{x_R}. \tag{166}
\]

Then, using (120), we end up with
\[
\frac{dA}{dt} \leq 4 M_1 - \frac{M_1^2}{2\pi} + C R^{-1} \left( M^{1/2} + M^{1/2} \right) A^{1/2} + C(M_2 - M_1). \tag{167}
\]

Inequality (166) implies \( T < + \infty \) if \( A(0) \ll 1 \), as is observed by [27] (see also Chapter 5 of [19]). Here we describe the proof for completeness.

The first observation is the monotonity formula
\[
\left| \frac{d}{dt} \int_{\Omega} \omega \varphi \right| \leq C(M + M^2) \| \nabla \varphi \|_{L^2}, \tag{167}
\]

derived from (124) and the symmetry of the Green's function: \( G(x, x') = G(x', x) \). The proof is the same as in (34) and is omitted.
Second, we put \( I_1 = I_{x_0,R} \) and \( I_2 = I_{x_0,2R} \) for
\[
I_{x,R} = \int_{\Omega} |x - x_0|^2 \omega \varphi_{x,R}.
\]
(168)

Then it holds that
\[
M_2 - M_1 \leq 2R^{-1} \int_{R < |x - x_0| < 2R} |x - x_0| \varphi_{x,2R} \omega \leq 2M^{3/2} R^{-1} \frac{1}{2}
\]
and
\[
A_2 = A_1 + \int_{\Omega} |x - x_0|^2 \left( \varphi_{x,2R} - \varphi_{x,R} \right) \omega \leq A_1 + 4R^2 \int_{\Omega} \left( \varphi_{x,2R} - \varphi_{x,R} \right) \omega,
\]
(169)

which implies
\[
\frac{dA_1}{dt} \leq 4M_1 - M^2 \frac{1}{2} + CR^{-1} \left( M^{3/2} + M^{1/2} \right) A_1^{1/2}
+ C \left( M^{3/2} + M^{1/2} \right) \left( \int_{\Omega} \left( \varphi_{x,2R} - \varphi_{x,R} \right) \omega \right)^{1/2}.
\]
(170)

Here, we use (167) to ensure
\[
\left| \frac{d}{dt} \left( 4M_1 - M^2 \frac{1}{2\pi} \right) \right| \leq C(M + M^2) R^{-2}
\]
(171)

and
\[
\left| \frac{d}{dt} \int_{\Omega} \left( \varphi_{x,2R} - \varphi_{x,R} \right) \omega \right| \leq C(M + M^2) R^{-2}.
\]
(172)

Then, it follows that
\[
4M_1 - M^2 \frac{1}{2\pi} \leq 4M_1(0) - M^2 \frac{1}{2\pi} + CBa \left( R^{-1} \frac{1}{2} \right)
\]
and
\[
\int_{\Omega} \left( \varphi_{x,2R} - \varphi_{x,R} \right) \omega \leq \int_{\Omega} \left( \varphi_{x,2R} - \varphi_{x,R} \right) \omega_0 + CBa \left( R^{-1} \frac{1}{2} \right)
\]
\[
\leq 2R^{-1} A_2(0) + CBa \left( R^{-1} \frac{1}{2} \right)
\]
(173)

for
\[
B = M^{3/2} + M^{1/2}, \quad a(s) = s^2 + s.
\]
(174)
Thus we obtain

\[
\frac{dA}{dt} \leq 4M_1(0) - \frac{M_1(0)^2}{2\pi} + CR^{-1}BA_1^{1/2} + CBA_2(0)^{1/2} + CBa\left(R^{-1}t^{1/2}\right)
\]

\[= J(0) + CBa\left(R^{-1}t^{1/2}\right) + CB^{-1}A_1^{1/2}\]

for

\[J = 4M_1 - \frac{M_1^2}{4\pi} + CBR^{-1}A_2^{1/2}.\]  

(177)

Assume \(M_1(0) > 8\pi\), and put

\[-4\delta = 4M_1(0) - \frac{M_1(0)^2}{2\pi} < 0.\]  

(179)

Let, furthermore,

\[1 \int_{\Omega} |x - x_0|^2 |\varphi_{x_0}(y)|^2 \leq \eta.\]  

(180)

Now we define \(s_0\) by

\[CBa(s_0) = \delta\]  

(181)

in (177), and take \(0 < \eta \ll 1\) such that

\[\eta \leq \delta^2\]  

(182)

Then, if \(R\) and \(T_0\) satisfy \(R^{-2}T_0 = \eta^{-1}\), it holds that

\[A_1(0) \leq R^2\eta < 2\delta T_0.\]  

(183)

Making \(0 < \eta \ll 1\), furthermore, we may assume

\[J(0) + CBR^{-1}A_1(0)^{1/2} \leq -4\delta + CBR^{-1}A_2(0)^{1/2} \leq -4\delta + CB\eta^{1/2} \leq -3\delta,\]  

(184)

which results in

\[\frac{dA_1}{dt} \leq J(0) + CBa\left(R^{-1}T_0^{1/2}\right) + BR^{-1}A_1(t)^{1/2}
\]

\[= J(0) + \delta + CBR^{-1}A_1^{1/2}, \quad 0 \leq t < T_0,
\]

(185)

provided that \(T \geq T_0\).

A continuation argument to (184)–(185) guarantees

\[\frac{dA_1}{dt} \leq -2\delta, \quad 0 \leq t < T_0,
\]

(186)

and then we obtain
by (183), a contradiction.

\[ A_1(T_0) \leq A_1(0) - 2\delta T_0 < 0 \]  

(187)

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References


