Chapter

Abel and Euler Summation Formulas for $SBV(\mathbb{R})$ Functions

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Abstract

The purpose of this paper is to show that the natural setting for various Abel and Euler-Maclaurin summation formulas is the class of special function of bounded variation. A function of one real variable is of bounded variation if its distributional derivative is a Radom measure. Such a function decomposes uniquely as sum of three components: the first one is a convergent series of piece-wise constant function, the second one is an absolutely continuous function and the last one is the so-called singular part, that is a continuous function whose derivative vanishes almost everywhere. A function of bounded variation is special if its singular part vanishes identically. We generalize such space of special function of bounded variation to include higher order derivatives and prove that the functions of such spaces admit a Euler-Maclaurin summation formula. Such a result is obtained by deriving in this setting various integration by part formulas which generalizes various classical Abel summation formulas.

Keywords: Euler summation, Abel summation, bounded variation functions, special bounded variation functions, Radon measure

1. Introduction

Abel and the Euler-Maclaurin summation formulas are standard tool in number theory (see e.g. [1, 2]).

The space of *special functions of bounded variation* (SBV) is a particular subclass of the classical space of bounded variation functions which is the natural setting for a wide class of problems in the calculus of variations studied by Ennio De Giorgi and his school: see e.g. [3, 4].

The purpose of this paper is to show that this class of functions (and some subclasses introduced here of function of a single real variable) is the natural settings for (an extended version of) the Euler-Maclaurin formula.

Let us describe now what we prove in this paper.

In Section 2 we obtain some “integration by parts”-like formulas for functions of bounded variations which imply the various “Abel summation” techniques (Propositions (0.6), (0.7), and the relative examples) and in Section 3 we give some criterion for the absolute summability of some series obtained by sampling the values of a bounded variations function.

The last section contains the proofs of the main result of this paper (Theorem (0.1)) that we will now describe.

We denote by $C^1(\mathbb{R})$ (resp. $C^k([a,b])$), $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ respectively the space of continuously differentiable functions (resp. k-times differentiables functions on the closed interval $[a, b]$), the space of Lebesgue (absolutely) integrable functions and the space of essentially bounded Borel functions on $\mathbb{R}$.

Given $f : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$ we set

$$f(x^+) = \lim_{h \to 0^+} f(x+h), \tag{1}$$

$$f(x^-) = \lim_{h \to 0^-} f(x+h), \tag{2}$$

$$\delta f(x) = f(x^+) - f(x^-). \tag{3}$$

We denote by $BV(\mathbb{R})$ the space of bounded variation complex functions on $\mathbb{R}$; we refer to [5, 6] for the main properties of functions in $BV(\mathbb{R})$.

Any real function of bounded variation can be written as a difference of two non decreasing functions. It follows that if $f \in BV(\mathbb{R})$ then $f(x^+), f(x^-)$ and $\delta f(x)$ exist for each $x \in \mathbb{R}$ and the set $\{x \in \mathbb{R} | \delta f(x) \neq 0\}$ is an arbitrary at most countable subset of $\mathbb{R}$. Moreover, the derivative $f'(x)$ exists for almost all $x \in \mathbb{R}$ and $f'(x) \in L^1(\mathbb{R})$.

Let $f \in BV(\mathbb{R})$. We denote by df the unique Radon measure on $\mathbb{R}$ such that for each open interval $]a, b[\subset \mathbb{R}$

$$df(]a, b[) = f(a^-) - f(b^+). \tag{4}$$

We recall that f is *special* if for any bounded Borel function u

$$\int_{\mathbb{R}} u(x) df(x) = \int_{\mathbb{R}} u(x) f'(x) dx + \sum_{x \in \mathbb{R}} u(x) \delta f(x). \tag{5}$$

We denote by $SBV(\mathbb{R})$ the space of all special functions of bounded variation. We also say that $f \in BV_{loc}(\mathbb{R})$ (resp. $f \in SBV_{loc}(\mathbb{R})$) if for each $a, b \in \mathbb{R}$, with $a < b$ the function

$$f(x) = \begin{cases} 0 & \text{if } x < a \text{ or } x > b, \\ f(x) & \text{if } a \leq x \leq b, \end{cases} \tag{6}$$

is in $BV(\mathbb{R})$ (resp. $SBV(\mathbb{R})$).

We define $SBV^n(\mathbb{R})$ inductively setting

$$SBV^1(\mathbb{R}) = SBV(\mathbb{R}), \tag{7}$$

and for each integer $n > 1$

$$SBV^n(\mathbb{R}) = \{ f \in SBV(\mathbb{R}) | f' \in SBV^{n-1}(\mathbb{R}) \} \tag{8}$$

We denote by B_n and $B_n(x)$, $n = 1, 2, \ldots$ respectively the Bernoulli numbers and the Bernoulli functions. Let us recall that

$$B_1(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ x - [x] - \dfrac{1}{2} & \text{if } x \in \mathbb{R} \backslash \mathbb{Z}, \end{cases} \tag{9}$$

where $[x]$ stands for the greatest integer less than or equal to x and $B_n(x)$, $n = 2, 3, \ldots$ are the unique continuous functions such that

$$B_n(x+1) = B_n(x), \tag{10}$$

$$B_n'(x) = nB_{n-1}(x), \tag{11}$$

$$\int_0^1 B_n(x)dx = 0. \tag{12}$$

Moreover $B_{2n+1} = 0$ for $n > 0$ and $B_n = B_n(0)$ for $n > 1$.

The main results of this paper is the following theorem.

Theorem 0.1 Let $f \in SBV^m(\mathbb{R})$, $m \geq 1$ and suppose $f, \dots, f^{(m)} \in L^1(\mathbb{R})$. Then

$$\sum_{n \in \mathbb{Z}} \frac{f(n^+) + f(n^-)}{2} = \int_{\mathbb{R}} f(x)dx + \sum_{x \in \mathbb{R}} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k!} B_k(x)\delta f^{(k-1)}(x) + \frac{(-1)^{m-1}}{m!} \int_{\mathbb{R}} B_m(x) f^{(m)}(x)dx. \tag{13}$$

Remark. The sum "$\sum_{x \in \mathbb{R}}$" in the right hand side of the above "Euler-Maclaurin formula" (13) is actually a sum over the subset of the $x \in \mathbb{R}$ such that some of the terms $B_k(x)\delta f^{(k-1)}(x)$ do not vanish. We point out that such a set can be an arbitrary at most countable subset of $\mathbb{R}$.

Remark. Let p and q, $p < q$ be two integers and let f be a function of class C^m on the interval $[p, q]$. Set $f(x) = 0$ when x is outside of the interval $[p, q]$. Then the classical Euler-Maclaurin formula (see, e.g. Section 9.5 of [7])

$$\sum_{k=p}^{q-1} f(k) = \int_p^q f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} \left(f^{(k-1)}(q) - f^{(k-1)}(p) \right) + \frac{(-1)^{m-1} B_m}{m!} \int_p^q B_m(x) f^{(m)}(x)dx, \tag{14}$$

follows easily from Theorem 0.1.

Remark. Any $f \in BV(\mathbb{R})$ decomposes uniquely as $f = f_1 + f_2 + f_3$, where $f_1(x)$ can be written in the form

$$f_1(x) = \sum_{n=1}^{+\infty} \varphi_n(x) \tag{15}$$

where each $\varphi_n(x)$ is a piece-wise constant function, $f_2(x)$ is an absolutely continuous function and $f_3(x)$ is a singular function, that is $f_3(x)$ is continuous and $f'_{3(x)} = 0$ for almost all $x \in \mathbb{R}$. Then $f = f_1 + f_2 + f_3$ is special if, and only if, $f_3 = 0$ and in this case, for each bounded Borel function $u(x)$,

$$\int_{\mathbb{R}} u(x)df_1(x) = \sum_{x \in \mathbb{R}} u(x)\delta f(x), \tag{16}$$

$$\int_{\mathbb{R}} u(x)df_2(x) = \int_{\mathbb{R}} u(x)f'(x)dx. \tag{17}$$

In this paper we do not need of the existence of such a decomposition.

2. Integration by parts formulas

Our starting point is the following theorem:

Theorem 0.2 Let $f, g : \mathbb{R} \to \mathbb{C}$ two complex function. Assume that $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \in BV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f(x^+)dg(x) + \int_{\mathbb{R}} g(x^-)df(x) = 0, \tag{18}$$

$$\int_{\mathbb{R}} f(x^-)dg(x) + \int_{\mathbb{R}} g(x^+)df(x) = 0, \tag{19}$$

$$\int_{\mathbb{R}} \frac{f(x^+) + f(x^-)}{2} dg(x) + \int_{\mathbb{R}} \frac{g(x^+) + g(x^-)}{2} df(x) = 0. \tag{20}$$

Proof: Let $a, b \in \mathbb{R}$ with $a < b$. Theorem 7.5.9 of [5] yields

$$\int_{]a,b[} f(x^+)dg(x) + \int_{]a,b[} g(x^-)df(x) = f(b^-)g(b^-) - f(a^+)g(a^+), \tag{21}$$

$$\int_{]a,b[} f(x^-)dg(x) + \int_{]a,b[} g(x^+)df(x) = f(b^-)g(b^-) - f(a^+)g(a^+). \tag{22}$$

Since $f \in L^1(\mathbb{R})$ then necessarily

$$\lim_{b \to +\infty} f(b^-) = \lim_{a \to -\infty} f(a^+) = 0. \tag{23}$$

Since $g \in L^\infty(\mathbb{R})$ then $g(x^+)$ and $g(x^-)$ are bounded and we also have

$$\lim_{b \to +\infty} f(b^-)g(b^-) = \lim_{a \to -\infty} f(a^+)g(a^+) = 0. \tag{24}$$

and hence one obtains the formulas (18) and (19) taking the limits as $a \to -\infty$ and $b \to +\infty$ respectively in (21) and (22).

Formula (20) is obtained summing memberwise (18) and (19) and dividing by two. □

Next we prove:

Theorem 0.3 Let $f, g : \mathbb{R} \to \mathbb{C}$ two complex function. Assume that $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \in SBV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and suppose that $g' \in L^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f(x)g'(x)dx + {\sum_{x \in \mathbb{R}}}' f(x^+)\delta g(x) + \int_{\mathbb{R}} g(x^-)df(x) = 0, \tag{25}$$

$$\int_{\mathbb{R}} f(x)g'(x)dx + {\sum_{x \in \mathbb{R}}}' f(x^-)\delta g(x) + \int_{\mathbb{R}} g(x^+)df(x) = 0, \tag{26}$$

$$\int_{\mathbb{R}} f(x)g'(x)dx + {\sum_{x \in \mathbb{R}}}' \frac{f(x^+) + f(x^-)}{2} \delta g(x) + \int_{\mathbb{R}} \frac{g(x^+) + g(x^-)}{2} df(x) = 0. \tag{27}$$

where

$$ {\sum_{x \in \mathbb{R}}}' := \lim_{\substack{a \to -\infty \\ b \to +\infty}} \sum_{a < x < b}. \tag{28}$$

Moreover, if the function f also is continuous then

$$\int_{\mathbb{R}} f(x)g'(x)dx + \int_{\mathbb{R}} g(x)df(x) = 0, \tag{29}$$

Proof: Given $a, b \in \mathbb{R}$, $a < b$ set

$$g(a,b,x) = \begin{cases} 0 & x \leq a, \\ g(x) & a < x < b, \\ 0 & x \geq b. \end{cases} \tag{30}$$

The function $h(x) = g(a,b,x)$ is in $SBV(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Hence, formula (18) yields

$$\int_{\mathbb{R}} f(x^+)dh(x) + \int_{\mathbb{R}} h(x^-)df(x) = 0. \tag{31}$$

Since $h \in SBV(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x^+)dh(x) = \int_a^b f(x^+)g'(x)dx + \sum_{x \in \mathbb{R}} f(x^+)\delta g(a,b,x). \tag{32}$$

But $f(x^+) = f(x)$ for almost all $x \in \mathbb{R}$ and hence

$$\int_{\mathbb{R}} f(x^+)dh(x) = \int_a^b f(x)g'(x)dx + \sum_{x \in \mathbb{R}} f(x^+)\delta g(a,b,x), \tag{33}$$

which combined with (31) yields

$$\int_a^b f(x)g'(x)dx + \sum_{x \in \mathbb{R}} f(x^+)\delta g(a,b,x) + \int_{\mathbb{R}} g(a,b,x^-)df(x) = 0. \tag{34}$$

Using the definition of $g(a,b,x)$ we have

$$\sum_{x \in \mathbb{R}} f(x^+)\delta g(a,b,x) = f(a^+)g(a^+) + \sum_{a<x<b} f(x^+)\delta g(x), \tag{35}$$

and hence

$$\sum_{a<x<b} f(x^+)\delta g(x) = -f(a^+)g(a^+) - \int_a^b f(x)g'(x)dx - \int_{\mathbb{R}} g(a,b,x^-)df(x). \tag{36}$$

As in the proof of the previous theorem we have

$$\lim_{a \to -\infty} f(a^+)g(a^+) = 0. \tag{37}$$

Since $f \in L^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$ then $fg' \in L^1(\mathbb{R})$ and hence

$$\lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f(x)g'(x)dx = \int_{\mathbb{R}} f(x)g'(x)dx. \tag{38}$$

The Radon measure $df(x)$ is bounded and the functions $x \mapsto g(a,b,x^-)$ are equibounded with respect to a and b; by the Lebesgue dominated convergence we have

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}} \int_{\mathbb{R}} g(a,b,x^-)df(x) = \int_{\mathbb{R}} g(x^-)df(x). \tag{39}$$

From (36) it follows that

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}} \sum_{a<x<b} f(x^+)\delta g(x) = \sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) = -\int_{\mathbb{R}} f(x)g'(x)dx - \int_{\mathbb{R}} g(x^-)df(x) \tag{40}$$

which is equivalent to (25).

The proof of (26) is obtained in a similar manner using (19) instead of (18), and (27) is obtained summing memberwise (25) and (26) and dividing by two.

If the function g is continuous then $g(x^+) = g(x^-) = g(x)$ for each $x \in \mathbb{R}$,

$$\sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) = 0, \tag{41}$$

and (29) follows from, e.g., (25).

□

Example. This example shows that in the hypoteses of Theorem (0.3) the series

$$\sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) \tag{42}$$

is not, in general, absolutely convergent. Indeed, set

$$f(x) = \begin{cases} 0 & \text{if } x \le 1/2, \\ 1/x^2 & \text{if } x > 1/2, \end{cases} \tag{43}$$

and

$$g(x) = \begin{cases} 1 & \text{if } \sqrt{2n-1} < x \le \sqrt{2n}, \quad n \in \mathbb{Z}, \\ 0 & \text{if } \sqrt{2n} < x \le \sqrt{2n+1}, \quad n \in \mathbb{Z}. \end{cases} \tag{44}$$

Then the integral

$$\int_{\mathbb{R}} f'(x)g(x)dx = \sum_{n=1}^{+\infty} \int_{\sqrt{2n-1}}^{\sqrt{2n}} df(x) = -\sum_{n=1}^{+\infty} \frac{1}{2n(2n-1)} \tag{45}$$

is absolutely convergent, but the series

$$\sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \tag{46}$$

is convergent but not absolutely convergent.

We also have the following theorem.

Theorem 0.4 Let $f, g : \mathbb{R} \to \mathbb{C}$ two complex function. Assume that $f \in SBV(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \in SBV_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and suppose that $g' \in L^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f'(x)g(x)dx + \int_{\mathbb{R}} f(x)g'(x)dx + \sum_{x\in\mathbb{R}} \delta f(x)g(x^+) + \sum_{x\in\mathbb{R}}{}' f(x^-)\delta g(x) = 0, \tag{47}$$

$$\int_{\mathbb{R}} f'(x)g(x)dx + \int_{\mathbb{R}} f(x)g'(x)dx + \sum_{x\in\mathbb{R}} \delta f(x)g(x^-) + {\sum_{x\in\mathbb{R}}}' f(x^+)\delta g(x) = 0, \tag{48}$$

$$\begin{aligned}&\int_{\mathbb{R}} f'(x)g(x)dx + \int_{\mathbb{R}} f(x)g'(x)dx + \sum_{x\in\mathbb{R}} \frac{g(x^+)+g(x^-)}{2}\delta f(x)\\ &+ {\sum_{x\in\mathbb{R}}}' \frac{f(x^+)+f(x^-)}{2}\delta g(x) = 0,\end{aligned} \tag{49}$$

where

$${\sum_{x\in\mathbb{R}}}' := \lim_{\substack{a\to-\infty\\ b\to+\infty}} \sum_{a<x<b} \tag{50}$$

If the function g also is continuous then

$$\int_{\mathbb{R}} f'(x)g(x)dx + \int_{\mathbb{R}} f(x)g'(x)dx + \sum_{x\in\mathbb{R}} g(x)\delta f(x) = 0, \tag{51}$$

Proof: Let f and g be as in the theorem. By formula (26) we have

$$\int_{\mathbb{R}} f(x)g'(x)dx + {\sum_{x\in\mathbb{R}}}' f(x^-)\delta g(x) + \int_{\mathbb{R}} g(x^+)df(x) = 0. \tag{52}$$

Since $f \in SBV(\mathbb{R})$, using the fact that $g(x^+) = g(x)$ for almost all $x \in \mathbb{R}$, we obtain

$$\int_{\mathbb{R}} g(x^+)df(x) = \int_{\mathbb{R}} g(x)f'(x)dx + \sum_{x\in\mathbb{R}} g(x^+)\delta f(x). \tag{53}$$

Then (52) and (53) yield (47). Formulas (48) and (49) are obtained in a similar manner using respectively Formulas (25) and (27) instead of (26).

If the function g is continuous then $g(x^+) = g(x^-) = g(x)$ for each $x \in \mathbb{R}$,

$${\sum_{x\in\mathbb{R}}}' f(x^+)\delta g(x) = 0, \tag{54}$$

and (51) follows from, e.g., (47). □

Theorem 0.4 generalizes to high order derivatives.

Theorem 0.5 Let $f, g : \mathbb{R} \to \mathbb{C}$ two complex function. Let $m > 0$ be a positive integer. Assume that $f \in SBV^m(\mathbb{R})$ with $f, \dots, f^{(m)} \in L^1(\mathbb{R})$ and $g \in SBV^m_{loc}(\mathbb{R})$ with $g, \dots, g^{(m)} \in L^\infty(\mathbb{R})$. Then

$$\begin{aligned}&(-1)^{(m-1)}\int_{\mathbb{R}} f^{(m)}(x)g(x)dx + \int_{\mathbb{R}} f(x)g^{(m)}(x)dx\\ &+\sum_{x\in\mathbb{R}}\sum_{k=1}^{m}(-1)^{k-1}\delta f^{(k-1)}(x)g^{(m-k)}(x^+)\\ &+{\sum_{x\in\mathbb{R}}}'\sum_{k=1}^{m}(-1)^{k-1}f^{(k-1)}(x^-)\delta g^{(m-k)}(x) = 0,\end{aligned} \tag{55}$$

$$(-1)^{(m-1)}\int_{\mathbb{R}} f^{(m)}(x)g(x)dx + \int_{\mathbb{R}} f(x)g^{(m)}(x)dx + \sum_{x\in\mathbb{R}}\sum_{k=1}^{m}(-1)^{k-1}\delta f^{(k-1)}(x)g^{(m-k)}(x^-) + \sum_{x\in\mathbb{R}}{}'\sum_{k=1}^{m}(-1)^{k-1}f^{(k-1)}(x^+)\delta g^{(m-k)}(x) = 0, \tag{56}$$

$$(-1)^{(m-1)}\int_{\mathbb{R}} f^{(m)}(x)g(x)dx + \int_{\mathbb{R}} f(x)g^{(m)}(x)dx + \sum_{x\in\mathbb{R}}\sum_{k=1}^{m}(-1)^{k-1}\delta f^{(k-1)}(x)\frac{g^{(m-k)}(x^-)+g^{(m-k)}(x^+)}{2} + \sum_{x\in\mathbb{R}}{}'\sum_{k=1}^{m}(-1)^{k-1}\frac{f^{(k-1)}(x^-)+f^{(k-1)}(x^+)}{2}\delta g^{(m-k)}(x) = 0, \tag{57}$$

Proof: We prove first the formula (55). The proof is by induction on m. When $m = 1$ (55) reduces to (47). Assume that (55) holds for $m-1$, that is

$$(-1)^{(m-2)}\int_{\mathbb{R}} f^{(m-1)}(x)g(x)dx + \int_{\mathbb{R}} f(x)g^{(m-1)}(x)dx + \sum_{x\in\mathbb{R}}\sum_{k=1}^{m-1}(-1)^{k-1}\delta f^{(k-1)}(x)g^{(m-k-1)}(x^+) + \sum_{x\in\mathbb{R}}{}'\sum_{k=1}^{m-1}(-1)^{k-1}f^{(k-1)}(x^-)\delta g^{(m-k-1)}(x) = 0. \tag{58}$$

Replacing f with f', k with $k+1$ and changing the sign we obtain

$$(-1)^{(m-1)}\int_{\mathbb{R}} f^{(m)}(x)g(x)dx - \int_{\mathbb{R}} f'(x)g^{(m-1)}(x)dx + \sum_{x\in\mathbb{R}}\sum_{k=2}^{m}(-1)^{k-1}\delta f^{(k-1)}(x)g^{(m-k)}(x^+) + \sum_{x\in\mathbb{R}}{}'\sum_{k=2}^{m}(-1)^{k-1}f^{(k-1)}(x^-)\delta g^{(m-k)}(x) = 0. \tag{59}$$

Replacing g with $g^{(m-1)}$ in (47) we obtain

$$\int_{\mathbb{R}} f'(x)g^{(m-1))}(x)dx + \int_{\mathbb{R}} f(x)g^{m}(x)dx + \sum_{x\in\mathbb{R}}\delta f(x)g^{(m-1)}(x^+) + \sum_{x\in\mathbb{R}}{}' f(x^+)\delta g^{(m-1)}(x) = 0. \tag{60}$$

Summing (59) and (60) we obtain (55).
The proofs of (56) and (57) are similar.

□

We say that a function $f \in SBV_{loc}(\mathbb{R})$ is a *step function* if $f'(x) = 0$ for almost every $x \in \mathbb{R}$.

The following propositions are easy consequences of Theorem (0.4).

Proposition 0.6 Let $[u, v] \subset \mathbb{R}$ be a bounded closed interval and let f be an absolutely continuous function on the closed interval $[u, v]$. Let $g \in SBV_{loc}(\mathbb{R})$ be a step function. Then

$$\int_u^v f'(x)g(x)dx = f(v)g(v^-) - f(u)g(u^+) - \sum_{u<x<v} f(x)\delta g(x). \tag{61}$$

Proof: First we extend the functions f as zero outside of the interval $[u, v]$. We may also assume that the function g is zero outside of a bounded open interval containing the closed interval $[u, v]$. Observe that then $f(u^+) = f(u), f(v^-) = f(v)$ and $f(u^-) = f(v^+) = 0$ and therefore $\delta f(u) = f(u)$, $\delta f(v) = -f(v)$ and $\delta f(x) = 0$ for $x \neq u, v$. By (47), we have

$$\int_{\mathbb{R}} f'(x)g(x)dx + \int_{\mathbb{R}} f(x)g'(x)dx + \sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) + \sum_{x\in\mathbb{R}} g(x^-)\delta f(x) = 0. \tag{62}$$

Since g is a step function then $g'(x) = 0$ for almost all $x \in \mathbb{R}$ and hence it follows that

$$\int_{\mathbb{R}} f'(x)g(x)dx = -\sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) - \sum_{x\in\mathbb{R}} g(x^-)\delta f(x). \tag{63}$$

The function f by construction has compact support, and hence, as $f(v^+) = 0$, we have

$$\begin{aligned}\sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) &= f(u^+)(g(u^+) - g(u^-)) + \sum_{u<x<v} f(x^+)\delta g(x) \\ &= f(u)g(u^+) - f(u)g(u^-) + \sum_{u<x<v} f(x^+)\delta g(x),\end{aligned} \tag{64}$$

and

$$\sum_{x\in\mathbb{R}} g(x^-)\delta f(x) = g(u^-)\delta f(u) + g(v^-)\delta f(v) = f(u)g(u^-) - f(v)g(v^-). \tag{65}$$

Summing memberwise the last two formulas we obtain

$$\begin{aligned}\sum_{x\in\mathbb{R}}{}' f(x^+)\delta g(x) + \sum_{x\in\mathbb{R}} g(x^-)\delta f(x) = &-f(v)g(v^-) + f(u)g(u^+) \\ &+ \sum_{u<x<v} f(x^+)\delta g(x),\end{aligned} \tag{66}$$

as desired. □

Proposition 0.7 Let $f, g \in SBV_{loc}(\mathbb{R})$ be two step function. Let $[u, v] \subset \mathbb{R}$ be a bounded closed interval. Then

$$\sum_{u<x<v} g(x^+)\delta f(x) = f(v^-)g(v^-) - f(u^+)g(u^+) - \sum_{u<x<v} f(x^-)\delta g(x). \tag{67}$$

Proof: Set both the functions f and g to zero outside the closed interval $[u, v]$. Then formula (47) yields

$$\sum_{x \in \mathbb{R}} f(x^+)\delta g(x) + \sum_{x \in \mathbb{R}} g(x^-)\delta f(x) = 0. \tag{68}$$

But then

$$\sum_{x \in \mathbb{R}} f(x^+)\delta g(x) = f(u^+)g(u^+) + \sum_{u<x<v} f(x^+)\delta g(x), \tag{69}$$

and

$$\sum_{x \in \mathbb{R}} g(x^-)\delta f(x) = -f(v^-)g(v^-) + \sum_{u<x<v} g(x^-)\delta f(x); \tag{70}$$

hence

$$f(u^+)g(u^+) + \sum_{u<x<v} f(x^+)\delta g(x) - f(v^-)g(v^-) + \sum_{u<x<v} g(x^-)\delta f(x) = 0, \tag{71}$$

which is equivalent to (67). □

Example 1. (Abel summation I) Let (a_n), $n \in \mathbb{Z}$ be a sequence of complex numbers such that $a_n = 0$ for $n < < 0$. Then the function

$$A(x) = \sum_{n<x} a_n \tag{72}$$

is a step function in $SBV_{loc}(\mathbb{R})$. If $f \in C^1[u, v]$ then Proposition (0.6) yields

$$\int_u^v f'(x)A(x)dx = f(v)A(v^-) - f(u)A(u^+) - \sum_{u<n<v} f(n)a_n. \tag{73}$$

Example 2. (Abel summation II) Let (a_n), (b_n), $n \in \mathbb{Z}$ be two sequence of complex numbers. Let $f, g \to \mathbb{C}$ be defined respectively setting $f(x) = a_n$ and $g(x) = b_n$ when $n \leq x < n+1$, $n \in \mathbb{Z}$. Clearly $f, g \in SBV_{loc}(\mathbb{R})$ and they are two step functions. Let be given two integers p and q, $p < q$. Set $u = p$ and $v = q + 1$. Then it is easy to show that

$$\sum_{u<x<v} g(x^+)\delta f(x) = \sum_{n=p+1}^{q} b_n(a_n - a_{n-1}) \tag{74}$$

and

$$\sum_{u<x<v} f(x^-)\delta g(x) = \sum_{n=p+1}^{q} a_{n-1}(b_n - b_{n-1}); \tag{75}$$

hence, Proposition (0.7) yields

$$\sum_{n=p+1}^{q} b_n(a_n - a_{n-1}) = a_q b_q - a_p b_p - \sum_{n=p+1}^{q} a_{n-1}(b_n - b_{n-1}). \tag{76}$$

3. Sampling estimates

In this section we give some conditions which ensures the absolute convergence of series of the form $\sum_{x \in E}(f(x^-) + f(x^+))/2$ where f is a function absolutely integrable of bounded variation and E is a countable subset of $\mathbb{R}$.

The basic estimate is given in the following lemma.

Lemma 0.8 Let $A \subset \mathbb{R}$ be an open subset and let $F \subset A$ be a finite subset of A. Assume that there exist $a > 0$ such that

$$\begin{aligned} &x_1, x_2 \in F, \ \ x_1 \neq x_2 \Rightarrow |x_1 - x_2| \geq a, \\ &x \in F, \ \ y \in \mathbb{R} \backslash A \Rightarrow |x - y| \geq a/2. \end{aligned} \tag{77}$$

Then, for any complex function$f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ we have

$$\left| \sum_{x \in F} \frac{f(x^-) + f(x^+)}{2} \right| \leq \frac{1}{a} \int_A |f(x)| dx + \frac{1}{2} \int_A |df|(x) \tag{78}$$

Proof: Let define

$$g(x) = \begin{cases} 0, & \text{if } \ x < -1/2 \ \text{ or } \ x = 0 \ \text{ or } \ x \geq 1/2, \\ x + 1/2, & \text{if } \ -1/2 \leq x < 0, \\ x - 1/2, & \text{if } \ 0 \leq x < 1/2, \end{cases} \tag{79}$$

and set

$$G(x) = \sum_{y \in F} g\left(\frac{x - y}{a}\right). \tag{80}$$

For each $x \in \mathbb{R}$ we have

$$\frac{G(x^-) + G(x^+)}{2} = G(x) \tag{81}$$

By Eq. (27)

$$-\sum_{x \in \mathbb{R}}{}' \frac{f(x^+) + f(x^-)}{2} \delta G(x) = \int_{\mathbb{R}} f(x) G'(x) dx + \int_{\mathbb{R}} G(x) df(x). \tag{82}$$

We also have

$$\delta G(x) = \begin{cases} -1 & \text{if } \ x \in F, \\ 0 & \text{if } \ x \in \mathbb{R} \backslash F, \end{cases} \tag{83}$$

which implies

$$-\sum_{x \in \mathbb{R}}{}' \frac{f(x^+) + f(x^-)}{2} \delta G(x) = \sum_{x \in F} \frac{f(x^-) + f(x^+)}{2}. \tag{84}$$

Set

$$E = \bigcup_{x \in F}]x - a, x + a[. \tag{85}$$

Then $F \subset E \subset A$ and

$$G'(x) = \begin{cases} 1/a & \text{if } \ x \in E, \\ 0 & \text{if } \ x \in \mathbb{R} \backslash E, \end{cases} \tag{86}$$

and hence

$$\int_{\mathbb{R}} f(x)G'(x)dx = \frac{1}{a}\int_E f(x)dx. \tag{87}$$

Moreover we have $G(x) = 0$ if $x \in \mathbb{R} \backslash E$ and hence

$$\begin{aligned}\sum_{x \in F} \frac{f(x^-) + f(x^+)}{2} &= \int_{\mathbb{R}} f(x)G'(x)dx + \int_{\mathbb{R}} G(x)df(x) \\ &= \frac{1}{a}\int_E f(x)dx + \int_E G(x)df(x).\end{aligned} \tag{88}$$

Taking modules, and observing that $|G(x)| \leq 1/2$ for each $x \in E$, we obtain

$$\begin{aligned}\left|\sum_{x \in F} \frac{f(x^-) + f(x^+)}{2}\right| &\leq \frac{1}{a}\int_E |f(x)|dx + \int_E |G(x)||df|(x). \\ &\leq \frac{1}{a}\int_A |f(x)|dx + \frac{1}{2}\int_A |df|(x),\end{aligned} \tag{89}$$

as required.

Corollary 0.9 Let $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ and let $E \subset R$ be a countable subset. If there exists a real constant $a > 0$ such that for each pair of distinct $x_1, x_2 \in E$ we have $|x_1 - x_2| \geq a$ then

$$\sum_{x \in E} \left|\frac{f(x^-) + f(x^+)}{2}\right| < +\infty. \tag{90}$$

Proof: It suffices to choose $A = \mathbb{R}$; lemma (0.8) yields easily the assertion.

□

4. Proof of Theorem 0.1

Inserting $B_m(x)$ instead of $g_m(x)$ in formula (57) of Theorem 0.5 we easily obtain

$$\begin{aligned}\sum_{n \in \mathbb{Z}}{}' \frac{f(n^+) + f(n^-)}{2} &= \int_{\mathbb{R}} f(x)dx + \sum_{x \in \mathbb{R}} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k!} B_k(x) \delta f^{(k-1)}(x) \\ &+ \frac{(-1)^{m-1}}{m!}\int_{\mathbb{R}} B_m(x) f^{(m)}(x)dx.\end{aligned} \tag{91}$$

By Corollary 0.9 it follows that

$$\sum_{n \in \mathbb{Z}}{}' \frac{f(n^+) + f(n^-)}{2} = \sum_{n \in \mathbb{Z}} \frac{f(n^+) + f(n^-)}{2} \tag{92}$$

is an absolutely convergent series, and hence Theorem 0.1 follows.

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