Chapter

The Periodic Restricted EXPAR(1) Model

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Abstract

In this chapter, we discuss the nonlinear periodic restricted EXPAR(1) model. The parameters are estimated by the quasi maximum likelihood (QML) method and we give their asymptotic properties which lead to the construction of confidence intervals of the parameters. Then we consider the problem of testing the nullity of coefficients by using the standard Likelihood Ratio (LR) test, simulation studies are given to assess the performance of this QML and LR test.

Keywords: nonlinear time series, periodic restricted exponential autoregressive model, quasi maximum likelihood estimation, confidence interval, LR test

1. Introduction

Since the 1920*s*, linear models with Gaussian noise have occupied a prominent place, they have played an important role in the specification, prevision and general analysis of time series and many specific problems were solved by them. Nevertheless, many physical and natural processes exhibit nonlinear characteristics that are not taken into account with linear representation and are better explicated and fitted by nonlinear models. For example, ecological and environmental fields present phenomena close to the theory of nonlinear oscillations, such as limit cycle behavior remarked in the famous lynx or sunspot series, leading to the consideration of more complex models from the 1980*s* onwards. A first nonlinear model possible is the Volterra series which plays the same role as the Wold representation, for linear series. The interest of this representation is rather theoretical than practical, for this reason, specific parametric nonlinear models were presented as the *ARCH* and Bilinear models suitable for financial and economic data, threshold AutoRegressif (*TAR*) and exponential *AR* (*EXPAR*) models suitable for ecological and meteorological data. These nonlinear models have been applied with great success in many important real-life problems. Basics of nonlinear time series analysis can be found in [1–3] and references therein.

Amplitude dependent frequency, jump phenomena and limit cycle behavior are familiar features of nonlinear vibration theory and to reproduce them [4, 5] introduced the exponential autoregressive (*EXPAR*) models. The start was by taking an autoregressive (*AR*) model Y_t, say, and then make the coefficients dependent in an exponential way of Y_{t-1}^2.

Several papers treated the probabilistic and statistic aspects of *EXPAR* models. A direct method of estimation is proposed by [5], it consists to fix the nonlinear coefficient in the exponential term at one of a grid of values and then estimate the other parameters by linear least squares and use the AIC criterion to select the final parameters, necessary and sufficient conditions of stationarity and geometric ergodicity for the *EXPAR*(1) model are given by [6], the problem of estimation of nonlinear time series in a general framework by conditional least squares CLS and maximum likelihood ML methods is treated by [7] with application in *EXPAR* models, a forecasting method is proposed by [8], the *LAN* property was shown in [9] and asymptotically efficient estimates was constructed there for the restricted *EXPAR*(1), a genetic algorithm for estimation is used in [10], Bayesian analysis of these models is introduced in [11], a parametric and nonparamtric test for the detection of exponential component in *AR*(1) is constructed by [12], sup-tests are constructed by [13] with the trilogy Likelihood Ratio (LR), Wald and Lagrange Multiplier (LM) for linearity in a general nonlinear *AR*(1) model with *EXPAR*(1) as special cases, the extended Kalman filter (*EKF*) is used in [14]. Given that nonlinear estimation is time consuming [15] proposed to estimate heuristically the nonlinear parameter from the data and this is a very interesting remark because when the nonlinear parameter is known we get the Restricted *EXPAR* model. The applications of the *EXPAR* model are multiple: ecology, hydrology, speech signal, macroeconomic and others see, for example, [16–21].

On the other hand, fitted seasonal time series exhibiting nonlinear behavior such cited before and having a periodic autocovariance structure by *SARIMA* models will be inadequate. These models are linear and the seasonally adjusted data may still show seasonal variations because the structure of the correlations depends on the season. The solution is the use of a periodic version of *EXPAR* models. The notion of periodicity, introduced by [22], was used to fit hydrological and financial series and allowed the emergence of new classes of time series models such as Periodic *GARCH*, Periodic Bilinear, *MPAR* model. Motivated by all this, we introduced recently the Periodic restricted *EXPAR*(1) model see [23], which consists of having different restricted *EXPAR*(1) for each cycle and we established a most stringent test of periodicity since a periodic model is more complicated than a nonperiodic one and its consideration must be justified. We studied the problem of estimation by the least squares (*LS*) method in [24] and the test of Student was used for testing the nullity of the coefficients in the application. Traditionally, the step of estimation must be followed by tests of nullity of coefficients and the major tests used are Wald, LR and LM tests. We used a Wald test for testing the nullity of one coefficient and consequently testing linearity in [25].

In this chapter, we will present the quasi maximum likelihood (*QML*) estimation of the parameters, which are the *LS* estimators in [24] under the assumption that the density is Gaussian, these estimators are asymptotically normal under quite general conditions. This will play a role in the construction of the confidence interval for the parameters and then we treat the problem of testing the nullity of parameters which lead us to a linearity test using the standard and well known LR test. This test is based on the comparison between the maximum of the constrained and unconstrained quasi log likelihood, see for example [26] or [27], the null hypothesis is accepted, if the difference is small enough or equivalently H_0 ought to be rejected for large values of the difference. The problem is standard because the periodic model is restricted, i.e. the nonlinear parameter is known and for the other parameters 0 is an interior point of the parameter space, then the LR statistic asymptotically follows the χ^2 distribution under H_0 just like the Wald test, but we chose the former because it does not require estimation of the information matrix. It is known that the two tests are asymptotically equivalent and may be identical see [26] for more details.

The chapter is organized as follows. In Section 2, we introduce the Restricted *PEXPAR* model and we present the asymptotic normality of the QML estimators and we construct confidence intervals of the parameters. Section 3 provides the LR test for nullity of one coefficient and a test for linearity, a small simulation shows the efficiency of these tests.

2. The Periodic Restricted $EXPAR(1)$ model and QML estimation

2.1 Restricted $PEXPAR(1)$ model

Let $\{Y_t\}_{t\geq 1}$ be a seasonal stochastic process with period S $(S\geq 2)$.

Definition 1

The process $\{Y_t\}_{t\geq 1}$ is a Periodic Restricted EXPonential AutoRegressive model (restricted $PEXPAR(1)$) of order 1 if it is a solution of the nonlinear difference equation given by

$$Y_t = \left(\varphi_{t,1} + \varphi_{t,2}\exp\left(-\gamma Y_{t-1}^2\right)\right)Y_{t-1} + \varepsilon_t, \quad t\in\mathbb{N}, \tag{1}$$

where $\{\varepsilon_t\}_{t\geq 1}$ is $iid\left(0,\sigma_t^2\right)$, $\varphi_{t,1}$ and $\varphi_{t,2}$ are the autoregressive parameters and $\gamma > 0$ is the known nonlinear parameter. A heuristic determination of γ from data is

$$\hat{\gamma} = -\frac{\log\varepsilon}{\max\limits_{1\leq t\leq n} Y_t^2}, \tag{2}$$

where ε is a small number and n is the number of observations. (cf. [15]).

The autoregressive parameters and the innovation variance are periodic of period S, that is,

$$\varphi_{t+kS,1} = \varphi_{t,1}, \varphi_{t+kS,2} = \varphi_{t,2} \text{ and } \sigma_{t+kS}^2 = \sigma_t^2 \;, \forall k, \; t\in\mathbb{N}. \tag{3}$$

To point out the periodicity, let $t = i + S\tau, i = 1, \dots, S$ and $\tau\in\mathbb{N}$, then Eq. (1) becomes

$$Y_{i+S\tau} = \left(\varphi_{i,1} + \varphi_{i,2}\exp\left(-\gamma Y_{i+S\tau-1}^2\right)\right)Y_{i+S\tau-1} + \varepsilon_{i+S\tau}, i = 1, \dots, S, \quad \tau\in\mathbb{N} \tag{4}$$

In Eq. (4), $Y_{i+S\tau}$ is the value of Y_t during the i-th season of the cycle τ and $\varphi_{i,1}$, $\varphi_{i,2}$ are the model parameters at the season i. It is clear that the parameters depend on $Y_{i+S\tau-1}$ in the sense that for large $|Y_{i+S\tau-1}|$ we have $\varphi_{i,1}+\varphi_{i,2}\exp\left(-\gamma Y_{i+S\tau-1}^2\right) \sim \varphi_{i,1}$ while for small $|Y_{i+S\tau-1}|$: $\varphi_{i,1}+\varphi_{i,2}\exp\left(-\gamma Y_{i+S\tau-1}^2\right) \sim \varphi_{i,1}+\varphi_{i,2}$ of course the change is done smoothly between these regimes. In application, the restricted $PEXPAR(1)$ model is fitted to seasonal time series displaying nonlinearity features like amplitude dependent frequency.

These forms of models are new in the literature of the time series it is interesting to make several simulations to see their characteristics. An important fact is their property of non normality as is shown by histogram in **Figure 1** and confirmed by the test of Shapiro Wilk where the $p - value = 0.008226$ is less than 0.05. The realization of the process (A) is given in **Figure 1** from it and from the correlogram we can see that the process is stationary in each season due to the fast decay to 0 as h increases. Another interesting fact, that these types of models can exhibit, is the limit cycle behavior which is a well known feature in nonlinear vibrations and is one of possible mode of oscillations. Such phenomena is shown in **Figure 2** from model (B).

$$\text{Model (A)}: \begin{cases} Y_{1+2\tau} = \left(-0.3 + 2\exp\left(-Y_{2\tau}^2\right)\right)Y_{2\tau} + \varepsilon_{1+2\tau} \\ Y_{2+2\tau} = \left(-0.8 + \exp\left(-Y_{1+2\tau}^2\right)\right)Y_{1+2\tau} + \varepsilon_{2+2\tau} \end{cases}. \tag{5}$$

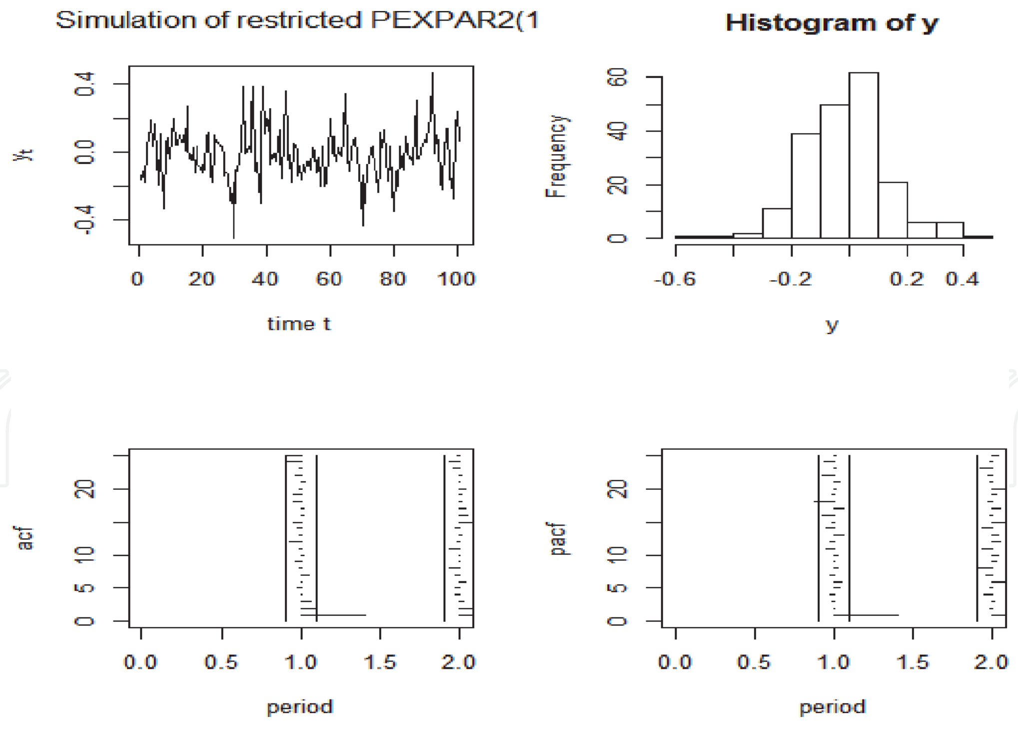

Figure 1.
Realization of (A) with corresponding histogram and correlogram.

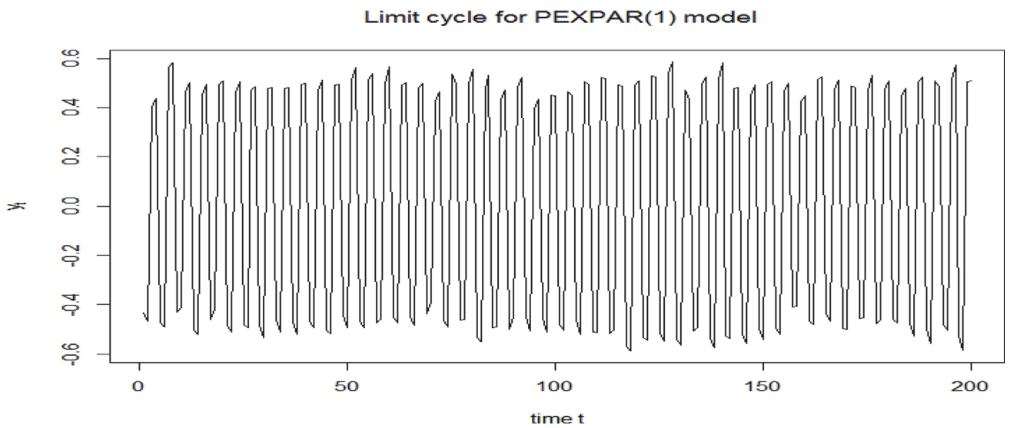

Figure 2.
Limit cycle from $PEXPAR_2(1)$ model.

$$\text{Model (B)}: \begin{cases} Y_{1+2\tau} = \left(0.2 - 1.5\exp\left(-Y_{2\tau}^2\right)\right)Y_{2\tau} + \varepsilon_{1+2\tau} \\ Y_{2+2\tau} = \left(0.8 + 0.3\exp\left(-Y_{1+2\tau}^2\right)\right)Y_{1+2\tau} + \varepsilon_{2+2\tau} \end{cases}. \tag{6}$$

2.2 QML Estimation

Let $\underline{\varphi} = \left(\underline{\varphi}'_1, \dots, \underline{\varphi}'_S\right)' \in \mathbb{R}^{2S}$ the parameter vector where $\underline{\varphi}_i = (\varphi_{i,1}, \varphi_{i,2})'$, $i = 1, \dots, S$. We want to estimate the true parameter $\underline{\varphi}_0$ from observations $Y_1, \dots, Y_n$ where $n = mS$ which means that we have m full period of data. The problem is resolved by the *QML* method and under the conditions:

$A1$: The Periodical restricted exponential autoregressive parameters $\underline{\varphi}$ satisfy the stationary periodically condition of (1). A sufficient condition is given by $|\varphi_{i,1}| < 1, \varphi_{i,2} \in \mathbb{R},\ i = 1, \dots, S$.

$A2$: The periodically ergodic process $\{Y_t; t \in \mathbb{N}\}$ is such that $E\left(Y_t^4\right) < \infty$, for any $t \in \mathbb{N}$.

Periodic stationarity has not been treated for this model so stationarity is required for each season hence $A1$. We can replace the assumption $A2$ by $E\left(\varepsilon_t^4\right) < \infty$, for any $t \in \mathbb{N}$, since $E(\varepsilon_t^4) < \infty \Rightarrow E(Y_t^4) < \infty$. Under this condition significant outliers are improbable and the existence of the information matrix is guaranteed.

Given initial value Y_0, the conditional log likelihood of the observations evaluated at $\underline{\varphi}$ depends on f. The *QML* estimator is obtained by replacing f by the $N(0, \sigma_t^2)$:

$$L_n\left(\underline{\varphi}, Y_1, \dots, Y_n\right) = -\frac{mS}{2}\log(2\pi) - \frac{m}{2}\sum_{i=1}^{S}\log\left(\sigma_i^2\right) - \sum_{i=1}^{S}\sum_{\tau=0}^{m-1}\frac{\left(Y_{i+S\tau} - \left(\varphi_{i,1} + \varphi_{i,2}\exp\left(-\gamma Y_{i+S\tau-1}^2\right)\right)Y_{i+S\tau-1}\right)^2}{2\sigma_i^2}, \tag{7}$$

(assuming) $\sigma_i \neq 0$.

Let $\hat{\underline{\varphi}}$ the *QML* estimator, one can see that maximizing L_n is equivalent to minimization of the quantity:

$$Q_n\left(\underline{\varphi}\right) = \frac{1}{n}\sum_{t=1}^{n}\left(Y_t - \left(\varphi_{t,1} + \varphi_{t,2}\exp\left(-\gamma Y_{t-1}^2\right)\right)Y_{t-1}\right)^2. \tag{8}$$

The initial value is unknown but its choice is not important for the asymptotic behavior of the *QML* estimator so we put $Y_0 = 0$, which defines the operational criterion

$$\tilde{Q}_n\left(\underline{\varphi}\right) = \frac{1}{S}\sum_{i=1}^{S}\tilde{Q}_{i,m}\left(\underline{\varphi}_i\right) \tag{9}$$

and

$$\tilde{Q}_{i,m}\left(\underline{\varphi}_i\right) = \frac{1}{m}\sum_{\tau=0}^{m-1}\left(Y_{i+S\tau} - \left(\varphi_{i,1} + \varphi_{i,2}\exp\left(-\gamma Y_{i+S\tau-1}^2\right)\right)Y_{i+S\tau-1}\right)^2. \tag{10}$$

The first order condition of the *QML* minimization problem is a system of $2S$ linear equations with $2S$ unknowns. The solution is

$$\begin{bmatrix}\hat{\varphi}_{i,1}\\ \hat{\varphi}_{i,2}\end{bmatrix} = \begin{bmatrix}\sum_{\tau=0}^{m-1} Y_{S\tau+i-1}^2 & \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}^2\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}^2\exp\left(-\gamma Y_{S\tau+i-1}^2\right) & \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}^2\exp\left(-2\gamma Y_{S\tau+i-1}^2\right)\end{bmatrix}^{-1} \times \begin{bmatrix}\sum_{\tau=0}^{m-1} Y_{S\tau+i-1}Y_{S\tau+i}\\ \sum_{\tau=0}^{m-1} Y_{S\tau+i-1}Y_{S\tau+i}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\end{bmatrix} \tag{11}$$

$$\hat{\sigma}_i^2 = \frac{1}{m}\sum_{\tau=0}^{m-1}\left(Y_{S\tau+i} - \left(\hat{\varphi}_{i,1} + \hat{\varphi}_{i,2}\exp\left(-\gamma Y_{S\tau+i-1}^2\right)\right)Y_{S\tau+i-1}\right)^2.$$

We remark that the *QML* estimator is the *LS* estimator and we can proof the next theorem in the same way.

Theorem

The *QML* estimator is strongly consistent and we have for $i = 1, \dots, S$

$$\sqrt{m}\begin{bmatrix} \hat{\varphi}_{i,1}-\varphi_{i,1} \\ \hat{\varphi}_{i,2}-\varphi_{i,2} \end{bmatrix} \underset{m\to\infty}{\overset{\mathcal{L}}{\to}} N\left(\underline{0}_2, \sigma_i^2 \begin{pmatrix} E\left(Y_{i-1}^2\right) & E\left(X_{i-1}^2 \exp\left(-\gamma Y_{i-1}^2\right)\right) \\ E\left(Y_{i-1}^2 \exp\left(-\gamma Y_{i-1}^2\right)\right) & E\left(Y_{i-1}^2 \exp\left(-2\gamma Y_{i-1}^2\right)\right) \end{pmatrix}^{-1}\right). \tag{12}$$

Furthermore, $\underline{\hat{\varphi}}_{i,m}$ and $\underline{\hat{\varphi}}_{j,m}$ are asymptotically independant, $i \neq j,\ i,j = 1, \dots, S$.

Proof

The proof is very standard in the literature of time series. The consistency is based on an ergodicity argument and for the normality a central limit version for martingale differences is used. The detail is similar to the *LSE* (see [24]) hence it is omitted. The independence of the $\varepsilon_{i+S\tau}$ implies that all the terms for $i \neq j$ are zero, this implies that $\sqrt{m}\left(\underline{\hat{\varphi}}_{i,m} - \underline{\varphi}_i\right)$ and $\sqrt{m}\left(\underline{\hat{\varphi}}_{j,m} - \underline{\varphi}_j\right),\ i \neq j,$ are asymptotically uncorrelated.

The *QML* estimators (Eq. (4)) yields a point estimator, a confidence interval (*CI*) gives a region where the parameters fall in with a given probability (usually 95% or 90%). Based on the asymptotic normality of the *QML* estimators, with asymptotic probability $1-\alpha,\ \varphi_{i,j}$ is in the interval

$$\left(\hat{\varphi}_{i,j} \pm \Phi_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{m_i}}\sqrt{(\Gamma_i)_{jj}}\right),\ j = 1, 2, i = 1, \dots, S, \tag{13}$$

where

$$\Gamma_i = \begin{pmatrix} E\left(Y_{i-1}^2\right) & E\left(Y_{i-1}^2 \exp\left(-\gamma Y_{i-1}^2\right)\right) \\ E\left(Y_{i-1}^2 \exp\left(-\gamma Y_{i-1}^2\right)\right) & E\left(Y_{i-1}^2 \exp\left(-2\gamma Y_{i-1}^2\right)\right) \end{pmatrix}^{-1}, \tag{14}$$

and $\Phi_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution. That is, the *CI* contains the true parameters in $100(1-\alpha)\%$ of all repeated samples.

To examine the performance of the *QML* estimators, we construct *CI* of the parameters from the simulation of restricted $PEXPAR_2(1)$ model with parameters: $\underline{\varphi}_1 = (-0.8, 1.2)'$ and $\underline{\varphi}_2 = (0.4, -0.9)'$ with sizes $n = 200, 500$ and 1000 and for the significance levels: $\alpha = 10\%$ and 5% and 1000 replications. From the **Tables 1–3** we deduce that the parameters are well estimated and when n increases the length of *CI* decreases showing that the estimates are consistent. Obviously, a higher confidence level produces wider *CI*.

$n = 200$	$CI(\varphi_{1,1})$	$CI(\varphi_{1,2})$	$CI(\varphi_{2,1})$	$CI(\varphi_{2,2})$
$\alpha = 10\%$	$[-0.8206, -0.7796]$	$[1.1136, 1.2576]$	$[0.3801, 0.4119]$	$[-0.9559, -0.8156]$
$\alpha = 5\%$	$[-0.8126, -0.7661]$	$[1.0893, 1.2618]$	$[0.3808, 0.4194]$	$[-0.9967, -0.8260]$

Table 1.
CI of parameters for n = 200.

$n = 500$	$CI(\varphi_{1,1})$	$CI(\varphi_{1,2})$	$CI(\varphi_{2,1})$	$CI(\varphi_{2,2})$
$\alpha = 10\%$	$[-0.8038, -0.7879]$	$[1.1551, 1.2113]$	$[0.3874, 0.4001]$	$[-0.9015, -0.8470]$
$\alpha = 5\%$	$[-0.8097, -0.7912]$	$[1.1783, 1.2453]$	$[0.3873, 0.4020]$	$[-0.9104, -0.8448]$

Table 2.
CI of parameters for n = 500.

$n = 1000$	$CI(\varphi_{1,1})$	$CI(\varphi_{1,2})$	$CI(\varphi_{2,1})$	$CI(\varphi_{2,2})$
$\alpha = 10\%$	$[-0.8027, -0.7952]$	$[1.1909, 1.2191]$	$[0.3978, 0.4040]$	$[-0.9072, -0.8793]$
$\alpha = 5\%$	$[-0.8030, -0.7938]$	$[1.1797, 1.2139]$	$[0.3958, 0.4034]$	$[-0.9119, -0.8791]$

Table 3.
CI of parameters for n = 1000.

3. Likelihood Ratio tests

3.1 Test for the Nullity of One Coefficient

The asymptotic normality of the *QML* in Eq. (12) can be exploited to perform tests on the parameters. This problem is very standard, especially when 0 is an interior point of the parameter space and can be done with the trilogy: Wald, LR and LM tests. We treated the former in [25] and in this chapter, we will use the LR test which is based upon the difference between the maximum of the likelihood under the null and under the alternative hypotheses and has the advantage of not estimating information matrix. In this section, we are interested in testing assumptions of the form

$$H_0 : \varphi_{i,2} = 0 \;\left(\text{or } H_0 : \varphi_{i,1} = 0\right) \text{ vs } H_1 : \varphi_{i,2} \neq 0 \;\left(\text{or } H_1 : \varphi_{i,1} \neq 0\right), \tag{15}$$

for some given i. Under H_1, we have the *QML* estimator $\underline{\hat{\varphi}}_i$ given by Eq. (11) and mean square error $\tilde{Q}_{i,m}\left(\underline{\hat{\varphi}}_i\right)$ given by Eq. (10) and $\underline{\tilde{\varphi}}_i = \begin{pmatrix} \tilde{\varphi}_{i,1} \\ 0 \end{pmatrix}$, is the *QML* estimator given under H_0 where

$$\tilde{\varphi}_{i,1} = \frac{\sum_{\tau=0}^{m-1} Y_{S\tau+i-1} Y_{S\tau+i}}{\sum_{\tau=0}^{m-1} Y_{S\tau+i-1}^2} \tag{16}$$

and the corresponding mean square error under the null

$$\tilde{Q}_{i,m}\left(\underline{\tilde{\varphi}}_i\right) = \frac{1}{m}\sum_{\tau=0}^{m-1}\left(Y_{i+S\tau} - \tilde{\varphi}_{i,1} Y_{i+S\tau-1}\right)^2. \tag{17}$$

The usual LR statistic is

$$\lambda_{i,m} = \frac{L\left(\underline{\tilde{\varphi}}_i, \tilde{\sigma}_i^2\right)}{L\left(\underline{\hat{\varphi}}_i, \hat{\sigma}_i^2\right)} = \left(\frac{\tilde{Q}_{i,m}\left(\underline{\hat{\varphi}}_i\right)}{\tilde{Q}_{i,m}\left(\underline{\tilde{\varphi}}_i\right)}\right)^{\frac{m}{2}} \tag{18}$$

then the test rejects H_0 at the asymptotic level α when

$$\begin{aligned} LR_{i,m} &= -2\log\lambda_{i,m} \\ &= m \log \frac{\tilde{Q}_{i,m}\left(\underline{\tilde{\varphi}}_i\right)}{\tilde{Q}_{i,m}\left(\underline{\hat{\varphi}}_i\right)} > \chi_1^2(1-\alpha), \end{aligned} \tag{19}$$

where $\chi_1^2(1-\alpha)$ is the $(1-\alpha)-$quantile of the χ^2 distribution with 1 degree of freedom.

In the same manner we can test the nullity of $\varphi_{i,1}$ by taken $\underline{\tilde{\varphi}}_i = \begin{pmatrix} 0 \\ \tilde{\varphi}_{i,2} \end{pmatrix}$ and

$$\tilde{Q}_{i,m}\left(\underline{\tilde{\varphi}}_i\right) = \frac{1}{m}\sum_{\tau=0}^{m-1}\left(Y_{i+S\tau} - \tilde{\varphi}_{i,2}\exp\left(-\gamma Y^2_{i+S\tau-1}\right)Y_{i+S\tau-1}\right)^2. \tag{20}$$

Example 1

In the simulation we focused on testing the nullity of $\varphi_{i,2}$ only. We simulated 1000 independent samples of length $n = 200$ and 500 of 3 models.

Model I: Periodic autoregressive ($PAR_2(1)$) with the parameters $\underline{\varphi} = (-0.7, 0.4)'$.

Model II: Restricted $PEXPAR_2(1)$ with the parameters $\underline{\varphi} = (-0.7, 0, 0.4, -2)'$ and $\gamma = 1$

Model III: Restricted $PEXPAR_2(1)$ with the parameters $\underline{\varphi} = (-0.7, 1.5, 0.4, -2)'$ and $\gamma = 1$.

The model I is chosen to calculate the level, the model III is chosen to calculate the power, the choice of model II is to show that the test is efficient since in the first cycle we have an $AR(1)$ and in the second cycle a restricted $EXPAR(1)$. On each realisation we fitted a restricted $PEXPAR_2(1)$ model by QML and carried out tests of $H_0 : \varphi_{i,2} = 0$ against $H_1 : \varphi_{i,2} \neq 0$. The rejection frequencies at significance level 5% and 10% are reported in **Tables 4** and **5**. **Figure 3** shows the asymptotic distribution of $LR_{i,m}$ under the null hypothesis. From the tables we see that the levels of the LR test are pretty well controlled since for $n = 500$, we note a relative rejection frequency of 5.5% for $\varphi_{1,2}$ and 5.1% for $\varphi_{2,2}$, which are not meaningfully different from the nominal 5%, the same remark is made for $\alpha = 10\%$ where the relative rejection frequency is of 9.5% and 10.3%. From model III, the rejection frequencies which represent the empirical power increase with the length n indicating the good performance and the consistency of the test. To illustrate that the asymptotic distribution of $LR_{i,m}$ under the null hypothesis is the standard χ^2_1 we have the

Model	α	$\varphi_{1,2}$	$\varphi_{2,2}$
I	5%	0.052	0.066
	10%	0.105	0.103
II	5%	0.054	0.998
	10%	0.117	1
III	5%	0.967	0.930
	10%	1	0.997

Table 4.
The rejection frequency computed on 1000 replications of simulations of length $n = 200$.

Model	α	$\varphi_{1,2}$	$\varphi_{2,2}$
I	5%	0.055	0.051
	10%	0.095	0.103
II	5%	0.053	1
	10%	0.098	1
III	5%	0.991	1
	10%	1	1

Table 5.
The rejection frequency computed on 1000 replications of simulations of length $n = 500$.

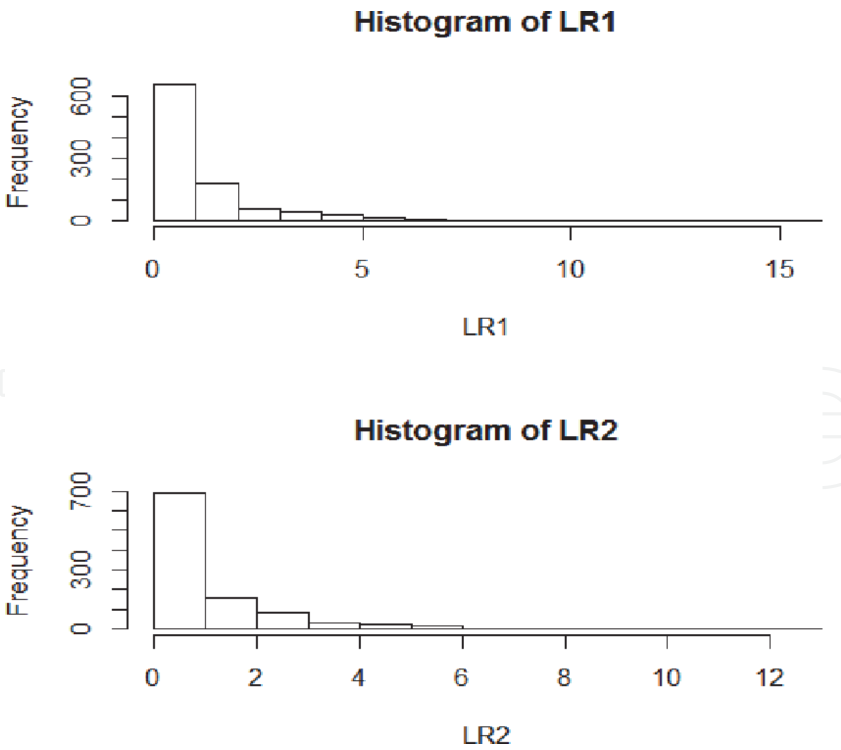

Figure 3.
Asymptotic distribution of LR.

histograms in **Figure 3** where we see that the distribution of $LR_{i,m}$ has the well known shape of χ_1^2.

3.2 Test for linearity in Restricted PEXPAR(1) model

The most important case to test is when $\varphi_{i,2} = 0, \forall i,$ which correspond to the linear periodic autoregressive model $(PAR_S(1))$ of period S. The null hypothesis is then

$$H_0 : \varphi_{i,2} = 0, \forall i \;\; \text{vs} \;\; H_1 : \exists i / \varphi_{i,2} \neq 0. \tag{21}$$

H_1 correspond to the restricted $PEXPAR_S(1)$ model, that is, the linear $PAR_S(1)$ model is nested within the nonlinear restricted model and it can be obtained by limiting the parameters $\varphi_{i,2}$ to be zero $\forall i$, hence we have a problem of testing the linearity hypothesis.

The standard LR test statistic is

$$\lambda_m = \left(\sum_{i=1}^{S} \frac{\tilde{Q}_{i,m}\left(\hat{\underline{\varphi}}_i\right)}{\tilde{Q}_{i,m}\left(\tilde{\underline{\varphi}}_i\right)} \right)^{\frac{m}{2}}. \tag{22}$$

The test rejects H_0 at the asymptotic level α when

$$\begin{aligned} LR_m &= -2 \log \lambda_m \\ &= m \sum_{i=1}^{S} \log \frac{\tilde{Q}_{i,m}\left(\tilde{\underline{\varphi}}_i\right)}{\tilde{Q}_{i,m}\left(\hat{\underline{\varphi}}_i\right)} > \chi_S^2(1-\alpha), \end{aligned} \tag{23}$$

where $\chi_S^2(1-\alpha)$ is the $(1-\alpha)-$quantile of the χ^2 distribution with S degrees of freedom which is simply the number of supplementary parameters in H_1.

Model	α	LR **test** ($\boldsymbol{n} = \mathbf{200}$)	LR **test** ($\boldsymbol{n} = \mathbf{500}$)
I	5%	0.0615	0.0581
	10%	0.1225	0.1075
II	5%	0.9999	1
	10%	0.9999	1

Table 6.
The rejection frequency computed on 10000 replications.

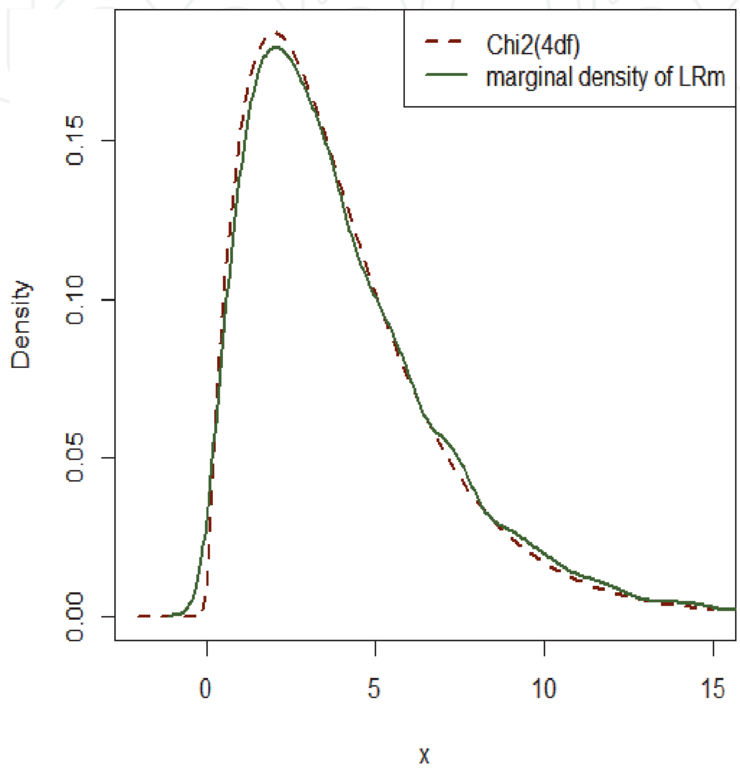

Figure 4.
Asymptotic distribution of LR_m.

Example 2

Table 6 shows the rejection frequency computed on 10000 replications of simulations of length $n = 200$ and 500 from the 2 models.

Model I: $PAR_4(1)$ with the parameters $\underline{\varphi} = (-0.8, 0.5, 0.9, -0.4)'$.

Model II: Restricted $PEXPAR_4(1)$ with $\underline{\varphi} = (-0.8, 2, 0.5, -1.5, 0.9, 1.1, -0.4, 0.6)'$ and $\gamma = 1.$, **Figure 4** shows the asymptotic distribution of LR_m under the null hypothesis. The results show that the empirical levels are acceptable, for $n = 500$, we have a relative rejection frequency of 5.81% (resp. 10.75%) which is very close to 5% (resp. 10%), the empirical power increases with the size n which means that the test is consistent. The rejection region is $\{LR_m > \chi_4^2(1-\alpha)\}$, where $\chi_4^2(1-\alpha)$ is the $(1-\alpha)-$ quantile of the χ^2 distribution with 4 degrees of freedom. From **Figure 4**, we see that the asymptotic distribution of LR_m (in full line) is close to the χ_4^2 (in dashed lines), this confirm the above theoretical result.

4. Conclusion

The periodic restricted *EXPAR* model is added to the family of nonlinear and periodic models. Interest is focused on estimation and testing problems. The periodic stationarity allows to calculate the QML estimators and derived tests of coefficients, cycle by cycle, and therefore use standard techniques. From this point of view, we can extend several results concerning the classical *EXPAR* to the periodic case.

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