Chapter

Equations of Relativistic and Quantum Mechanics (without Spin)

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Abstract

A relativistically invariant representation of the generalized momentum of a particle in an external field is proposed. In this representation, the dependence of the potentials of the interaction of the particle with the field on the particle velocity is taken into account. The exact correspondence of the expressions of energy and potential energy for the classical Hamiltonian is established, which makes identical the solutions to the problems of mechanics with relativistic and nonrelativistic approaches. The invariance of the proposed representation of the generalized momentum makes it possible to equivalently describe a physical system in geometrically conjugate spaces of kinematic and dynamic variables. Relativistic invariant equations are proposed for the action function and the wave function based on the invariance of the representation of the generalized momentum. The equations have solutions for any values of the constant interaction of the particle with the field, for example, in the problem of a hydrogen-like atom, when the atomic number of the nucleus is $Z > 137$. Based on the parametric representation of the action, the expression for the canonical Lagrangian, the equations of motion, and the expression for the force acting on the charge are derived when moving in an external electromagnetic field. The Dirac equation with the correct inclusion of the interaction for a particle in an external field is presented. In this form, the solutions of the equations are not limited by the value of the interaction constant. The solutions of the problem of charge motion in a constant electric field, the problems for a particle in a potential well and the passage of a particle through a potential barrier, the problems of motion in an exponential field (Morse), and also the problems of a hydrogen atom are given.

Keywords: quantum mechanics, relativistic invariant equations

1. Introduction

–To doubt everything or to believe everything are two equally convenient solutions; both dispense with the necessity of reflection.

Henri Poincaré (1854-1912)

–I know, I know, but suppose – just suppose! – the purity of the circle has blinded us from seeing anything beyond it!
I must begin all over with new eyes, I must rethink everything!

Hypathia (~360-415 AD)

In 1913, Bohr, based on the Balmer empirical formulas, constructed a model of atom based on the quantization of the orbital momentum [1], which was subsequently supplemented by the more general Sommerfeld quantization rules. In those years, naturally, the presence of a spin or an intrinsic magnetic moment of the particle or, especially, spin-orbit interaction, or interaction with the nuclear spin, was not supposed.

In 1916, Sommerfeld, within the framework of relativistic approaches, derived a formula for the energy levels of a hydrogen-like atom, without taking into account the spin [2]. Sommerfeld proceeded from the model of the Bohr atom and used the relativistic relation between the momentum $\mathbf{p}$ and the energy E of a free particle with the mass m.

$$E^2 - (\mathbf{p}c)^2 = \left(mc^2\right)^2, \tag{1}$$

where c is the speed of light.

In an external field with a four-dimensional potential $(\varphi, \mathbf{A})$, it was supposed that for a particle with the charge q this relation can also be used if we subtract the components of the four-dimensional momentum of the field $(q\varphi, q\mathbf{A})$ from the expression for the generalized particle momentum:

$$(E - q\varphi)^2 - (\mathbf{p}c - q\mathbf{A})^2 = \left(mc^2\right)^2. \tag{2}$$

In the case of the Coulomb potential $\varphi = Z|e|/r$, where e is the charge of electron, r is the distance from the nucleus, and Z is an atomic number, we obtain in spherical coordinates

$$p_r{}^2 + r^2 p_\varphi{}^2 = p_r{}^2 + \frac{L^2}{r^2} = \frac{\left(E + Ze^2/r\right)^2 - \left(mc^2\right)^2}{c^2} \tag{3}$$

where L is the angular momentum. The Bohr-Sommerfeld quantization conditions take the form

$$\begin{aligned} \oint p_\varphi d\varphi &= \hbar n_\varphi, \\ \oint p_r dr &= \oint \sqrt{\frac{\left(E + Ze^2/r\right)^2 - \left(mc^2\right)^2}{c^2} - \frac{L^2}{r^2}} dr = \hbar n_r, \end{aligned} \tag{4}$$

where n_φ and n_r are the orbital and radial quantum numbers, respectively. For the energy levels, Sommerfeld obtained the formula

$$E_{n,l} = \frac{mc^2}{\sqrt{1 + \dfrac{(Z\alpha)^2}{\left(n - \dfrac{(Z\alpha)^2}{l + 1/2 + \sqrt{(l+1/2)^2 - (Z\alpha)^2}}\right)^2}}}, \tag{5}$$

where the principal quantum number $n = n_r + l + 1 = 1,\ 2,\ 3,\ \ldots, l = 0,\ 1\ ,\ 2,\ 3,\ \ldots,\ n - 1$, and $\alpha = 1/137.036$ is the fine structure constant. However, in a paper published in 1916 [3], Sommerfeld '*made a fortunate mistake*' [4] and the derived formula was presented in the following form

$$E_{n,l} = \frac{mc^2}{\sqrt{1 + \frac{(Z\alpha)^2}{\left(n - \frac{(Z\alpha)^2}{l+1+\sqrt{(l+1)^2-(Z\alpha)^2}}\right)^2}}}. \tag{6}$$

The formula (6) perfectly described all the peculiarities of the structure of the spectrum of hydrogen and other similar atoms with the limiting for those years accuracy of measurements, and there was no doubt about the correctness of the formula itself. Therefore, the Sommerfeld formula was perceived as empirical, and instead of the quantum number l, a '*mysterious*' internal quantum number with half-integer values $j = 1/2,\ 3/2,\ 5/2,\ \ldots,\ n + 1/2$ was introduced, and formula (6) was used in the representation

$$E_{n,j} = \frac{mc^2}{\sqrt{1 + \frac{Z\alpha^2}{\left(n - \frac{Z^2\alpha^2}{j+1/2+\sqrt{(\ j+1/2)^2-Z^2\alpha^2}}\right)^2}}}, \tag{7}$$

where $n = n_r + j + 1/2 = 1,\ 2,\ 3,\ \ldots,\ j = 1/2,\ 3/2,\ 5/2,\ \ldots,\ n + 1/2$, and l possess the values $l = 0$ at $j = 1/2$ and $l = j \pm 1/2$ for others. This formula coincides with the result of an exact solution of the relativistic Dirac equations in 1928 [5] for a particle with the spin 1/2 with the classical expression for the potential energy of an immobile charge in the Coulomb field of a nucleus with an atomic number Z in the form $U(r) = Ze^2/r$.

Formula (7) also indicated a strange limitation of value the charge of a nucleus with the atomic number $Z < 137$, above which the formula is losing its meaning. It was also evident that within the framework of the approaches outlined, the strong and gravitational interactions, the motions of the planets are not described. The problem $Z < 137$ or $\alpha > 1$ remains the unresolved problem of relativistic quantum mechanics. Expanding the formula (7) over the order of powers $Z\alpha^2$ in the Taylor series, with an accuracy of expansion up to the terms by the powers $Z\alpha^6$, we obtain

$$E_{n,j} = mc^2 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right) + \ldots \tag{8}$$

In 1925–1926, Schrödinger worked on the derivation of the equation for the wave function of a particle describing the De Broglie waves [6]. The derivation of the equation also was based on the relativistic relation (1) between the momentum $\mathbf{p}$ and the energy E of the particle, which he presented with the help of the operators of squares of energy and momentum in the form of an equation for the wave function

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2\Psi - c^2\left(-i\hbar\frac{\partial}{\partial \mathbf{r}}\right)^2\Psi = \left(mc^2\right)^2\Psi \tag{9}$$

Like Sommerfeld, Schrödinger used the following representation for a particle in an external field

$$\left(i\hbar\frac{\partial}{\partial t} - q\varphi\right)^2\Psi - c^2\left(-i\hbar\frac{\partial}{\partial \mathbf{r}} - \frac{q}{c}\mathbf{A}\right)^2\Psi = \left(mc^2\right)^2\Psi \tag{10}$$

In the case of stationary states of a charged particle in the field of the Coulomb potential for a hydrogen atom it was necessary to solve the equation

$$\frac{d^2\psi}{d\mathbf{r}^2}+\frac{2m}{\hbar^2}\left(\frac{E^2-m^2c^4}{2mc^2}-\frac{E}{mc^2}q\varphi(\mathbf{r})+\frac{q^2}{2mc^2}\varphi^2(\mathbf{r})\right)\psi=0 \tag{11}$$

As can be seen, the quadratic expression of potential energy $q^2\varphi^2(\mathbf{r})/2mc^2$ is present in the equation with a positive sign and in the case of attracting fields, the solutions lead to certain difficulties. When approaching the singularity point, due to the negative sign, the attractive forces increase and the presence of the singularity leads to known limitations on the magnitude of the interactions (**Figure 1**).

Next, the wave vector k is represented as

$$k_1=\frac{1}{\hbar c}\sqrt{E^2-(mc^2)^2},\qquad k_2=\frac{1}{\hbar c}\sqrt{(E-U)^2-(mc^2)^2} \tag{12}$$

and when considering the problem of the passage of a particle with energy E through a potential barrier $U=q\varphi(\mathbf{r})$ (**Figure 2**), the height of which is greater than the doubled resting energy of the particle $U>2mc^2$, the transmission coefficient becomes unity, regardless of the height of the barrier (Klein paradox) [7].

Another difficulty is that, as the solution of the particle problem in a potential well shows, at a sufficient depth, a particle with a wavelength $\lambda\!\!\!^{-}=\hbar/mc$ can have bound states (can be localized) in a well width narrower than the wavelength of the particle $d<\lambda\!\!\!^{-}/2$ (**Figure 3**), which contradicts the fundamental principle of quantum mechanics—the Heisenberg's uncertainty principle.

Also, the solution of the problem of a hydrogen-like atom is limited by the value of the ordinal number of the atomic nucleus $Z\leq 68$ (for the Dirac equation, the restriction of the atomic number is $Z\leq 137$). The same in relativistic mechanics—when considering strong interactions, the solution of the Hamilton-Jacoby relativistic equation indicates the so-called "*particle fall on the center*" [8].

In order to get rid of the quadratic term or reverse its sign, in recent years it has been proposed to represent potential energy in the Klein-Gordon and Dirac equations as the difference of squares from the expressions of scalar and vector potentials

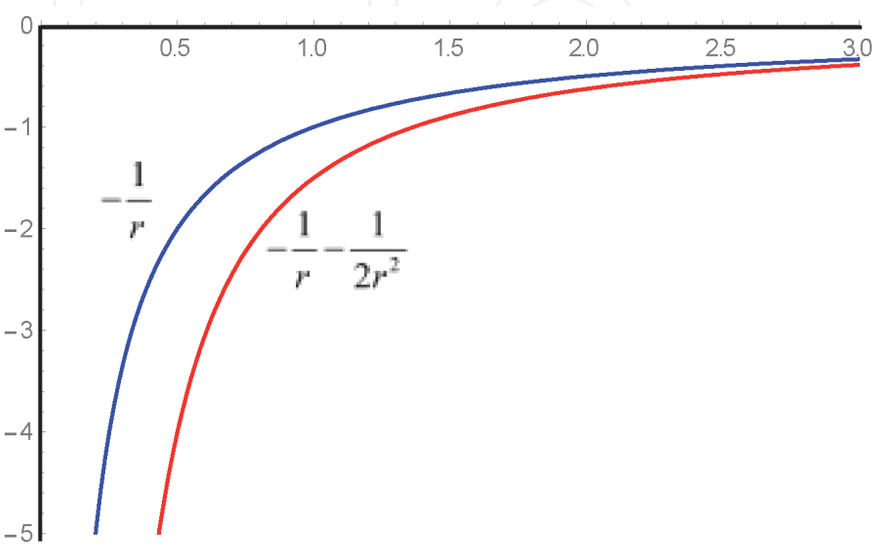

Figure 1.
The sample dependency of the attractive field potential $-1/r$ and potential interaction energy $-1/r-1/2r^2$ in the Klein-Gordon equations.

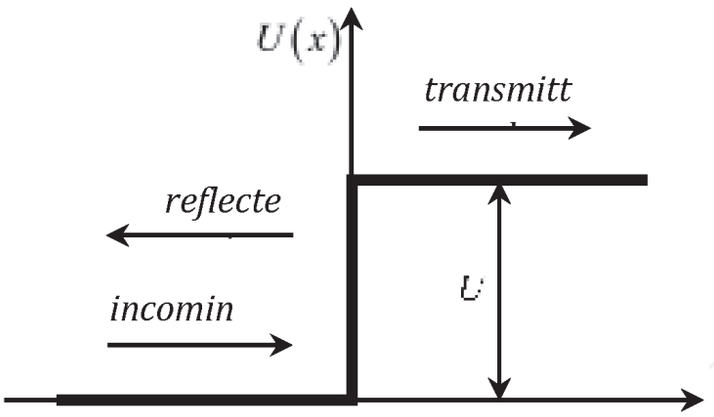

Figure 2.
Passage of a particle through a potential barrier U.

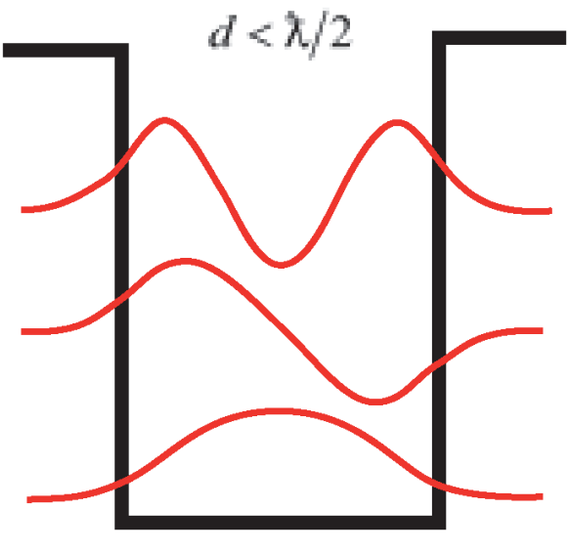

Figure 3.
A particle with a wavelength ƛ can be localized in a well width d < ƛ/2.

(S-wave equation) [9–11]. Such a mathematical formalism corrects the situation, but from a physical point of view such representations are in no way justified, and the fields corresponding to such pseudo-potentials do not exist in nature.

Things are even worse with the presence of a quadratic term of the vector field, because of the sign of which we obtain non-existent states in nature and solutions that contradict experience.

$$\frac{d^2\psi}{d\mathbf{r}^2} - 2i\frac{q}{\hbar c}\mathbf{A}(\mathbf{r})\cdot\frac{d\psi}{d\mathbf{r}} + \frac{2m}{\hbar^2}\left(\frac{E^2 - m^2c^4}{2mc^2} - \frac{q^2}{2mc^2}\mathbf{A}^2(\mathbf{r})\right)\psi = 0 \tag{13}$$

According to the solutions of the equations of quantum mechanics and Hamilton-Jacoby, it turns out that a charged particle in a magnetic field, in addition to rotating in a circle, also has radial vibrations—Landau levels [12] (even in the case of zero orbital momentum).

$$\frac{\hbar^2}{2M}\left(R'' + \frac{1}{\rho}R' - \frac{m^2}{\rho^2}\right) + \left(E - \frac{p_z^2}{2M} - \frac{M\omega_H^2}{8}\rho^2 - \frac{\hbar\omega_H m}{2}\right)R = 0. \tag{14}$$

$$E = \hbar\omega_H\left(n_\rho + \frac{|m| + m + 1}{2}\right) + \frac{p_z^2}{2M}$$

Over these 90 years, especially in very accurate cyclotron resonance experiments, none has detected the electron radial vibrations and the Landau levels.

Solving this equation, Schrödinger, like Sommerfeld, received the formula (5), which described the structure of the hydrogen spectrum not exactly. Moreover, from the solution of the problem for a particle in a potential well, it turns out that a particle with a wavelength $\lambda\!\!\!^{-} = \hbar/mc$ has bound states (is placed) in a well of arbitrary size and, in particular, much smaller than $\lambda\!\!\!^{-}/2$. This fact contradicts the fundamental principle of the quantum (wave) theory, the principle of uncertainty.

In 1925 Schrödinger sent this work to the editors of 'Annalen der Physik' [13], but then took the manuscript, refused the relativistic approaches and in 1926 built a wave equation based on the classical Hamiltonian expression, the Schrödinger equation [14].

$$\mathrm{H} = \frac{\mathbf{p}^2}{2m} + U; \quad \rightarrow \quad i\hbar\frac{\partial}{\partial t}\Psi = \left(\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial \mathbf{r}}\right)^2 + U\right)\Psi \tag{15}$$

Equation described the spectrum of the hydrogen atom only qualitatively, however, it did not have any unreasonable restrictions or singular solutions in the form of the Sommerfeld-Dirac formula. Klein [15], Fock [16] and Gordon [17] published the relativistic equation based on the wave equation for a particle without spin in 1926; it is called the Klein-Fock-Gordon equation.

With the discovery of the spin, the situation changed drastically, and in 1926 Heisenberg and Jordan [18] showed that, within the Pauli description of the spin of an electron, half the energy of the spin-orbit interaction is equal to a term with a power of α^4 in the Taylor series expansion of the Sommerfeld formula *equation reference goes here*.

Why exactly the half, Thomas tried to explain this in 1927 by the presence of a relativistic precession of an electron in the reference frame of motion along the orbit [19]. The energy of the Thomas precession is exactly equal to half the value of the energy of the spin-orbit interaction with the inverse (positive) sign, which should be added to the energy of the spin-orbit interaction. However, the incorrect assumption that the Thomas precession frequency is identical in both frames of reference and the absence of a common and correct derivation for non-inertial (rotating) frames of reference raised doubts about the correctness of such approaches. The reason for the appearance of half the energy of the spin-orbit interaction in the Sommerfeld formula is still under investigation and is one of the unresolved problems in modern physics.

On the other hand, both in the derivation of the Sommerfeld formula and at the solution of the Klein-Fock-Gordon equation for the hydrogen atom problem [20], neither the spin nor the spin-orbit interaction energy was taken into account initially. Therefore, the obtained fine splitting can in no way be owing to the spin-orbit interaction. This is a relativistic but purely mechanical effect, when the mass (inertia) of a particle is already depends on the velocity of motion along the orbit (of the angular momentum), because of which the radial motion of the electron changes, and vice versa. Just this dependence, which results in the splitting of the energy levels of the electron, and to the impossibility of introducing only one, the principal quantum number. Nevertheless, even with this assumption, the order of splitting of the levels according to formula (8) contradicts to the logic; it turns out to be that the greater the orbital angular momentum, the lesser the energy of the split level.

The matrix representation of the second-order wave Eq. (9) by a system of equations of the first order is the Dirac construction of the relativistic electron equation [21] (the Dirac matrices are the particular representation of the

Clifford-Lipschitz numbers [22]). In the standard representation the Dirac equation for a free particle has the form [23].

$$\begin{aligned} \hat{\varepsilon}\,\phi - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\chi &= mc\,\phi, \\ -\hat{\varepsilon}\,\chi + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\varphi &= mc\,\chi, \end{aligned} \tag{16}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

are the Pauli matrices (the unit matrix in the formulas is omitted).

For a particle in an external field, Eq. (16) is usually written in the form

$$\begin{aligned} \left(\hat{\varepsilon} - \frac{q}{c}\varphi\right)\phi - \boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}\right)\chi &= mc\,\phi, \\ -\left(\hat{\varepsilon} - \frac{q}{c}\varphi\right)\chi + \boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}\right)\phi &= mc\,\chi, \end{aligned} \tag{18}$$

where for an invariant representation in the case of a free particle, the equations are composed for the difference between the generalized momentum and the momentum of the field.

In the case of the potential energy of an immobile charge in a Coulomb field, we obtain the Sommerfeld-Dirac formula as a result of an exact solution of this particular equation. There, again, although for a system with spin 1/2 the energy of the spin-orbit interaction is not taken into account initially, but the half is obtained from the exact solution of the hydrogen atom problem.

More accurate measurements of Lamb in 1947 and subsequent improvements in the spectrum of the hydrogen atom revealed that, in addition to the lines with the maximum j, all the others are also split and somewhat displaced (the Lamb shift). To harmonize the results of the theory with more accurate experimental data on the spectrum of the hydrogen atom, one had to propose other solutions and approaches than were laid down by the derivation of the Dirac equation.

The new theoretical approaches had yield nothing and only supplemented the theory with the illogical and non-physical proposals to overcome the emerging singularity of solutions: the renormalization, the finite difference of infinities with the desired value of the difference, and so on. The accounting for the size of the nucleus corrected only the Z value into the bigger value, but did not solve the $Z > 137$ problem. An incredible result was also obtained for the hydrogen atom problem that the electron is located, most likely, at the center of the atom, that is, in the nucleus.

The results of solution of the problem for a particle in a potential well both in the case of the Klein-Fock-Gordon equation and of the Dirac equation contradict to the basic principle of quantum mechanics, to the uncertainty principle. From the solutions, it turns out to be that a particle can be in a bound state in a well with any dimensions, in particular, with the size much smaller than the wavelength of the particle itself, $A = \hbar/mc$ [23].

Despite Dirac himself proposed a system of linear first-degree relativistic equations in the matrix representation that described the system with spin 1/2, the contradictions did not disappear, and he himself remained unhappy with the results of his theory. As Dirac wrote in 1956 [24], the development of relativistic electron theory can now be considered as an example of how incorrect arguments sometimes

lead to a valuable result. In the 70s, it became clear that the relativistic theory of quantum mechanics does not exist, and new, fundamental approaches and equations should be sought for constructing a consistent theory of relativistic quantum mechanics. And in the 80s, Dirac already spoke about the insuperable difficulties of the existing quantum theory and the need to create a new one [25].

The reason for the failure of these theories is quite simple—it is in the ignoring of the dependence of the interaction energy with the field on the velocity of the particle. The generalized momentum of the system, the particle plus the external field, is the sum of the relativistic expression for the mechanical momentum of the particle and the field momentum in the case of interaction with the immobile particle

$$\mathbf{P} = (\varepsilon,\ \mathbf{p}) = \frac{1}{c}\left(\frac{mc^2}{\sqrt{1-\beta^2}} + q\varphi,\quad \frac{mc^2}{\sqrt{1-\beta^2}}\boldsymbol{\beta} + q\mathbf{A},\quad \mathbf{P}^2 \neq inv\right), \tag{19}$$

which is not an invariant representation of the particle velocity. To construct some invariant from such a representation, an 'invariant' relation was used in all cases in the form of a difference between the generalized momentum of the system and the field momentum in the case of interaction with the immobile particle

$$(\varepsilon - q\varphi,\ \mathbf{p} - q\mathbf{A}) = \frac{1}{c}\left(\frac{mc^2}{\sqrt{1-\beta^2}},\quad \frac{mc^2}{\sqrt{1-\beta^2}}\boldsymbol{\beta}\right),\quad (\varepsilon - q\varphi)^2 - (\mathbf{p} - q\mathbf{A})^2 = (mc)^2 \tag{20}$$

Obviously, the permutation of the components of the generalized momentum for the construction of the invariant does not solve the posed problem. The statement that the expression (20) is the mechanical momentum of a particle and therefore is an invariant is unproven and it is necessary to apprehend the formula (20) as an empirical. Therefore, at high velocities or strong interactions, an unaccounted dependence of the energy of particle interaction with the field on the velocity of the particle motion, which results to the erroneous results or the impossibility of calculations.

In [26], an invariant representation of the generalized momentum of the system was suggested, where the dependence of the interaction energy of the particle with the field on the velocity was taken into account:

$$\mathbf{P} = (\varepsilon,\ \mathbf{p}) = \frac{1}{c}\left(\frac{mc^2 + q\varphi + q\boldsymbol{\beta}\cdot\mathbf{A}}{\sqrt{1-\beta^2}},\quad \frac{(mc^2 + q\varphi)\boldsymbol{\beta} + q\mathbf{A}_{\parallel}}{\sqrt{1-\beta^2}} + q\mathbf{A}_{\perp}\right) \tag{21}$$

$$\mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = \frac{(mc^2 + q\varphi)^2 - (q\mathbf{A})^2}{c^2}, \tag{22}$$

which is the four-dimensional representation of the generalized momentum of the system based on the expression for the generalized momentum of an immobile particle in a state of rest

$$\mathbf{P}_0 = (\varepsilon_0,\ \mathbf{p}_0) = \frac{1}{c}(mc^2 + q\varphi,\ q\mathbf{A}) \tag{23}$$

whose invariant is always equal to the expression (19) regardless of the state of the system.

The application of variational principles to construct the relativistic and quantum theory was based on the principles of construction the mechanics with the help of the Lagrangian of the system [27], which originally was not intended for relativistic approaches. The Lagrangian construction is parametric with the one time variable $\tau = ct$, singled out from the variables of the four-dimensional space (the rest are represented by the dependence on this variable τ) and contains the total differential with respect to this variable, the velocity of the particle. Such a construction is unacceptable because of the impossibility to apply the principle of invariance of the representation of variables and the covariant representation of the action of the system.

In [28], to construct the relativistic theory on the basis of variational principles, the canonical (non- parametric) solutions of the variational problem for canonically defined integral functionals have been considered and the canonical solutions of the variational problems of mechanics in the Minkowski spaces are written. Because of unifying the variational principles of least action, flow, and hyperflow, the canonically invariant equations for the generalized momentum are obtained. From these equations, the expressions for the action function and the wave function are obtained as the general solution of the unified variational problem of mechanics.

Below, we present the generalized invariance principle and the corresponding representation of the generalized momentum of the system, the equations of relativistic and quantum mechanics [29], give the solutions of the problems of charge motion in a constant electric field, the problems for a particle in a potential well and the passage of a particle through a potential barrier, the problems of motion in an exponential field (Morse), the problems of charged particle in a magnetic field, and also the problems of a hydrogen atom are given.

2. Principle of invariance

2.1 Generalization of the principle of invariance

The principle of invariance of the representation of a generalized pulse is applicable also in the case of motion of a particle with the velocity $\mathbf{v}$ and in the case of a transition to a reference frame moving with the velocity $\mathbf{V}$.

The four-dimensional momentum of a particle $\mathbf{P}$ with the rest mass m moving with the velocity $\boldsymbol{\beta} = \mathbf{v}/c$ is represented in the form.

$$\mathbf{P} = (\varepsilon,\ \mathbf{p}) = \left(\frac{mc}{\sqrt{1-\beta^2}},\quad \frac{mc}{\sqrt{1-\beta^2}}\boldsymbol{\beta}\right), \mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = (mc)^2 \tag{24}$$

This is the property of invariance of the representation of the four-dimensional momentum $\mathbf{P}$ in terms of the velocity of the particle $\boldsymbol{\beta} = \mathbf{v}/c$.

If to consider the representation of the four-dimensional momentum of an immobile particle with a mass m by transition into the reference frame moving with the velocity $\boldsymbol{\beta}' = \mathbf{V}/c$, for the four-dimensional particle momentum $\mathbf{P}$ we have.

$$\mathbf{P} = (\varepsilon,\ \mathbf{p}) = \left(\frac{mc}{\sqrt{1-\beta^2}},\quad \frac{mc}{\sqrt{1-\beta^2}}\boldsymbol{\beta}'\right), \mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = (mc)^2. \tag{25}$$

This is a property of invariance of the representation of the four-dimensional momentum $\mathbf{P}$ through the velocity of the reference system $\boldsymbol{\beta}' = \mathbf{V}/c$.

For an invariant of the system I, we have

$$I^2 = \mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = (\varepsilon_0)^2 = (mc)^2. \tag{26}$$

At $\boldsymbol{\beta} = \boldsymbol{\beta}' = 0$, we obtain

$$\mathbf{P} = (\varepsilon, \ \mathbf{p})\big|_{\boldsymbol{\beta}=\boldsymbol{\beta}'=0} = \varepsilon_0(1, \ 0) = mc(1, \ 0). \tag{27}$$

Thus, the generalized momentum of a particle has an invariant representation on the particle velocity **v** and the velocity of the reference system **V**. This property should be considered because of the general principle of the relativity of motion. Accordingly, the generalized momentum of the particle **P** is an invariant regardless of the state of the system.

If a charged particle is in an external electromagnetic field with potentials $(\varphi, \ \mathbf{A})$, then the stationary charge sees the field exactly with such potentials. If the charge has a nonzero velocity **v**, then it will interact with the field differently. To determine the interaction for a charge moving with the velocity **v**, one can start from the principle of the relativity of motion. The effective values of the force or interaction with the field of the charge moving with the velocity v are the same as in the case when the charge is immobile, and the field moves with the velocity $-\mathbf{v}$ (in the laboratory frame of reference).

The fact that the interaction of a charged particle with a field depends on the speed of motion is evidently represented in the formula for the Liénard-Wiechert potential [8].

More clearly, this can be demonstrated by an example of the Doppler effect for two atoms in the field of a resonant radiation, when one of the atoms is at rest and the other moves with the velocity **v** (**Figure 4**).

The atom, which is at rest, absorbs a photon, and the moving one does not absorb or interacts weakly with the field, because of the dependence of the interaction on the velocity of the atom. It is also known that the acting field for an atom moving with the velocity v corresponds to the interaction with the field moving with the velocity $-\mathbf{v}$.

2.2 Invariant representation of the generalized momentum

Thus, for a moving charge, the effective values of the potentials $(\varphi', \ \mathbf{A}')$ (in the laboratory frame of reference) can be written in the form [8]

$$(\varphi', \mathbf{A}') = \left(\frac{\varphi + \boldsymbol{\beta}\cdot\mathbf{A}}{\sqrt{1-\beta^2}}, \quad \mathbf{A}_\perp + \frac{\mathbf{A}_\parallel + \varphi\boldsymbol{\beta}}{\sqrt{1-\beta^2}}\right). \tag{28}$$

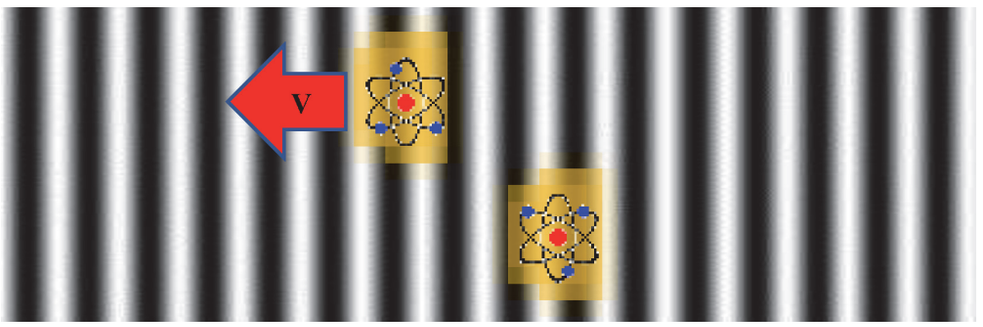

Figure 4.
Two atoms in the field of a resonant radiation.

If one represents the generalized momentum of the particle in the form

$$\mathbf{P} = \frac{1}{c}\left(\frac{mc^2}{\sqrt{1-\beta^2}} + q\varphi', \quad \frac{mc^2}{\sqrt{1-\beta^2}}\boldsymbol{\beta} + q\mathbf{A}'\right), \tag{29}$$

where φ' and $\mathbf{A}'$ already effective values of the interaction potentials of the particle moving with velocity $\mathbf{v}$ in a field with the potentials φ and $\mathbf{A}$, we obtain

$$\mathbf{P} = \frac{1}{c}\left(\frac{mc^2 + q\varphi + q\boldsymbol{\beta}\cdot\mathbf{A}}{\sqrt{1-\beta^2}}, \quad \frac{(mc^2 + q\varphi)\boldsymbol{\beta} + q\mathbf{A}_{\parallel}}{\sqrt{1-\beta^2}} + q\mathbf{A}_{\perp}\right). \tag{30}$$

The expression (30) can be represented in the form

$$\mathbf{P} = \left(\frac{mc^2 + q\varphi + q\boldsymbol{\beta}\cdot\mathbf{A}}{c\sqrt{1-\beta^2}}, \ \frac{mc^2 + q\varphi + q\boldsymbol{\beta}\cdot\mathbf{A}}{c\sqrt{1-\beta^2}}\boldsymbol{\beta} + \frac{q}{c}\mathbf{A} - \frac{q}{c}\frac{1}{1+\sqrt{1-\beta^2}}(\mathbf{A}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\right) \tag{31}$$

or

$$\mathbf{P} = \left(\varepsilon, \ \varepsilon\boldsymbol{\beta} + \frac{q}{c}\mathbf{A} - \frac{q}{c}\frac{1}{1+\sqrt{1-\beta^2}}(\mathbf{A}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\right). \tag{32}$$

This transformation can be presents in matrices form

$$\{\varepsilon', \ \mathbf{p}'\} = \{\varepsilon, \ \mathbf{p}\} + \hat{\mathrm{T}}\{\varepsilon, \ \mathbf{p}\} \tag{33}$$

where a Lorentz transformation have a form

$$\hat{1} + \hat{\mathrm{T}} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} +$$

$$(\gamma-1)\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} + \gamma\begin{Vmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 \\ \beta_3 & 0 & 0 & 0 \end{Vmatrix} + (\gamma-1)\begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_1\beta_1}{\beta^2} & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_1\beta_3}{\beta^2} \\ 0 & \frac{\beta_2\beta_1}{\beta^2} & \frac{\beta_2\beta_2}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} \\ 0 & \frac{\beta_3\beta_1}{\beta^2} & \frac{\beta_3\beta_2}{\beta^2} & \frac{\beta_3\beta_3}{\beta^2} \end{Vmatrix} \tag{34}$$

The matrices of the invariant representation of a four-dimensional vector, which preserve the vector module in four-dimensional space, form the Poincare group (inhomogeneous Lorentz group). In addition to displacements and rotations, the group contains space-time reflection representations $\hat{\mathrm{P}}$, $\hat{\mathrm{T}}$ and inversion $\hat{\mathrm{P}}\hat{\mathrm{T}} = \hat{\mathrm{I}}$.

For the module I of the four-dimensional vector of the generalized momentum $\mathbf{P}$, we have

$$I^2 = \mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = \frac{(mc^2 + q\varphi)^2 - (q\mathbf{A})^2}{c^2}, \tag{35}$$

which is the four-dimensional representation of the generalized momentum of the system on the basis of the expression of the generalized momentum of a particle in the state of rest

$$\mathbf{P}_0 = (\varepsilon_0,\ \mathbf{p}_0) = \frac{1}{c}(mc^2 + q\varphi,\ q\mathbf{A}), \tag{36}$$

whose invariant is defined by the expression (30).

Thus, the generalized momentum of the particle in an external field is not only invariant relative to the transformations at the transition from one reference system to another but also has an invariant representation in terms of the velocity of motion of the particle (30); at each point of space, the value of the invariant I is determined by the expression (35). This property has not only the representation of the proper momentum of the particle (the mechanical part), but also the generalized momentum of the particle in general.

Let us generalize this result to the case of representation of the generalized momentum of any systems and interactions, arguing that, regardless of the state (the motion) of the system, the generalized four-dimensional momentum always has an invariant representation

$$\mathbf{P} = (\varepsilon,\ \mathbf{p}) \ \Rightarrow\ \mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = {\varepsilon_0}^2 - {\mathbf{p}_0}^2 = \pm I^2 = \text{inv}, \tag{37}$$

where ε и p are the energy and momentum of the system, respectively, and the invariant is determined by the modulus of sum of the components of the generalized momentum of the system ε_0 and $\mathbf{p}_0$ at rest. If the particles interact with the field in the form $\varepsilon_0 + \alpha\varphi$, the invariants of the generalized momentum of the system are represented by the expressions [25].

$$\begin{aligned}
{\mathbf{P}_+}^2 &= (\varepsilon_0 + \alpha\varphi)^2 - (\alpha\mathbf{A})^2 = {\varepsilon_0}^2 + 2\varepsilon_0\alpha\varphi + (\alpha\varphi)^2 - (\alpha\mathbf{A})^2,\\
{\mathbf{P}_-}^2 &= (\alpha\varphi)^2 - (\varepsilon_0\mathbf{n} + \alpha\mathbf{A})^2 = -{\varepsilon_0}^2 - 2\varepsilon_0\alpha\mathbf{n}\cdot\mathbf{A} + (\alpha\varphi)^2 - (\alpha\mathbf{A})^2,\\
{\mathbf{P}_0}^2 &= (\varepsilon_0 + \alpha\varphi)^2 - (\varepsilon_0\mathbf{n} + \alpha\mathbf{A})^2 = 2\varepsilon_0\alpha(\varphi - \mathbf{n}\cdot\mathbf{A}) + (\alpha\varphi)^2 - (\alpha\mathbf{A})^2.
\end{aligned} \tag{38}$$

Let us represent the expression for the invariant $\varepsilon^2 - p^2$ (35) in the following form

$$\varepsilon^2 = \frac{E^2}{c^2} = \mathbf{p}^2 + \frac{(mc^2 + q\varphi)^2 - (q\mathbf{A})^2}{c^2} = \mathbf{p}^2 + m^2c^2 + 2mq\varphi + \frac{q^2}{c^2}(\varphi^2 - \mathbf{A}^2) \tag{39}$$

and divide it by $2m$. Grouping, we obtain the Hamiltonian H of the system in the form

$$\mathrm{H} = \frac{\varepsilon^2 - m^2c^2}{2m} = \frac{E^2 - m^2c^4}{2mc^2} = \frac{\mathbf{p}^2}{2m} + q\varphi + \frac{q^2}{2mc^2}(\varphi^2 - \mathbf{A}^2), \tag{40}$$

that is, we obtain the formula for the correspondence between the energy of the system E and the energy of the system in the classical meaning H. The correspondence in the form $\mathrm{H} = \mathbf{p}^2/2m + U(\tau, r)$ [26] will be complete and accurate if we determine the potential energy of interaction U and the energy of system in the classical meaning as

$$U = q\varphi + \frac{q^2}{2mc^2}\left(\varphi^2 - \mathbf{A}^2\right), \quad \mathrm{H} = \frac{E^2 - m^2c^4}{2mc^2} \;\Rightarrow\; E = \pm mc^2\sqrt{1 + \frac{2\mathrm{H}}{mc^2}}. \tag{41}$$

For example, the potential energy U of the electron in the field of the Coulomb potential $\varphi = Ze/r$ and in a homogeneous magnetic field $\mathbf{B}$ with the vector potential $\mathbf{A} = [\mathbf{r} \times \mathbf{B}]/2$ is

$$U = -e\varphi + \frac{e^2}{2mc^2}\left(\varphi^2 - \mathbf{A}^2\right) = -\frac{Ze^2}{r} + \frac{1}{2mc^2}\frac{Z^2e^4}{r^2} - \frac{e^2B^2}{8mc^2}r^2\sin^2\theta. \tag{42}$$

Note, whatever is the dependence of the potential φ, the possible minimum potential energy $U_{\min} = -mc^2/2$, and the potential energy as a function of the vector potential is always negative. The hard constraint of the classical potential energy value $U_{\min} = -mc^2/2$, which does not depend on the nature of the interactions, results in the fundamental changes in the description of interactions and the revision of the results of classical mechanics. At short distances, the origination of repulsion for attraction forces caused by the uncertainty principle is clearly reflected in the expression for the potential energy of the particle.

Many well-known expressions of the potential energy of interaction with attractive fields have a repulsive component in the form of half the square of these attractive potentials—Kratzer [30], Lennard-Jones [31], Morse [32], Rosen [33] and others. Expression (41) justifies this approach, which until now is phenomenological or the result of an appropriate selection for agreement with experimental data.

The Hamiltonian H can be called the energy and its value remains constant in the case of conservation of energy E, but the value of Hand its changes differ from the true values of the energy E and changes of its quantity. Thus, the classical approaches are permissible only in the case of low velocities, when $\mathrm{H} \ll mc^2$ and the energy expression can be represented in the form

$$E = mc^2\sqrt{1 + \frac{2\mathrm{H}}{mc^2}} \approx mc^2 + \mathrm{H}. \tag{43}$$

3. Equations of relativistic mechanics

3.1 Canonical Lagrangian and Hamilton-Jacoby equation

Let us use the parametric representation of the Hamilton action in the form [28].

$$S = -\int_{t_1,\ \mathbf{r}_1}^{t_2,\ \mathbf{r}_2}(\varepsilon dt - \mathbf{p}\cdot d\mathbf{r}) = -\int_{\mathbf{R}_1}^{\mathbf{R}_2}\mathbf{P}\cdot d\mathbf{R} = -\int_{\mathbf{R}_1}^{\mathbf{R}_2}\mathbf{P}\cdot\frac{d\mathbf{R}}{ds}ds = -\int_{\mathbf{R}_1}^{\mathbf{R}_2}(\mathbf{P}\cdot\mathbf{V})ds \to \min, \tag{44}$$

where ds is the four-dimensional interval and $\mathbf{V}$ is the four-dimensional generalized velocity.

The functional that takes into account the condition of the invariant representation of the generalized momentum $\mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = I^2 = inv$, can be composed by the method of indefinite Lagrange coefficients in the form

$$S = \int_{s_1}^{s_1}\left(-\mathbf{P}\cdot\mathbf{V} + \frac{\mathbf{P}^2 - I^2}{2\lambda}\right)ds = \int_{s_1}^{s_1}\left(\frac{(\mathbf{P} - \lambda\mathbf{V})^2 + \lambda^2 - I^2}{2\lambda}\right)ds \to \min, \tag{45}$$

where $\lambda = \lambda(s)$ is the given parameter, determined by the condition of invariance of the representation. Because λ and I are given and they do not depend on the velocity, we have an explicit solution in the form

$$\mathbf{P} - \lambda \mathbf{V} = 0, \qquad \lambda = \pm I(\tau, \mathbf{r}), \tag{46}$$

where the four-dimensional momentum is represented in the form

$$\mathbf{P} = I\mathbf{V} = \sqrt{\varepsilon^2 - \mathbf{p}^2}\left(\frac{1}{\sqrt{1-\eta^2}}, \quad \frac{\boldsymbol{\eta}}{\sqrt{1-\eta^2}}\right). \tag{47}$$

Thus, the action is represented in the form

$$S = \int_{s_1}^{s_2} I ds = \int_{s_1}^{s_2} \sqrt{\varepsilon^2 - \mathbf{p}^2} ds = \int_{\tau_1}^{\tau_2} \sqrt{\varepsilon^2 - \mathbf{p}^2}\sqrt{1-\eta^2} d\tau \tag{48}$$

and the canonical Lagrangian of the system is given by

$$\mathrm{L} = I\sqrt{1-\eta^2} = \sqrt{\varepsilon^2 - \mathbf{p}^2}\sqrt{1-\eta^2}. \tag{49}$$

The correctness of the presented parametrization is confirmed by the obtained expressions for the generalized momentum and energy from the Lagrangian of the system in the form

$$\begin{aligned} \varepsilon &= \boldsymbol{\eta}\frac{\partial \mathrm{L}}{\partial \boldsymbol{\eta}} - \mathrm{L} = \frac{I}{\sqrt{1-\eta^2}} = \frac{\sqrt{\varepsilon^2 - \mathbf{p}^2}}{\sqrt{1-\eta^2}}, \\ \mathbf{p} &= \frac{\partial \mathrm{L}}{\partial \boldsymbol{\eta}} = \frac{I}{\sqrt{1-\eta^2}}\boldsymbol{\eta} = \varepsilon\boldsymbol{\eta}, \end{aligned} \tag{50}$$

which coincide with the initial representations of the generalized momentum and energy. Accordingly, the Lagrange equation of motion takes the form

$$\frac{d\mathbf{p}}{d\tau} = -\frac{I}{\varepsilon}\frac{\partial I}{\partial \mathbf{r}}. \tag{51}$$

If we multiply Eq. (50) by $\mathbf{p} = \varepsilon\boldsymbol{\eta}$ scalarly, after reduction to the total time differential, we obtain,

$$\frac{d\varepsilon^2}{d\tau} = \frac{\partial I^2}{\partial \tau}. \tag{52}$$

If the invariant is clearly independent of time, then the energy ε is conserved and the equation of motion is represented in the form of the Newtonian equation

$$\frac{d\boldsymbol{\eta}}{d\tau} = -\frac{I}{\varepsilon^2}\frac{\partial I}{\partial \mathbf{r}}. \tag{53}$$

For a particle in an external field we have

$$\mathrm{L} = -\frac{1}{c}\sqrt{(mc^2 + q\varphi)^2 - (q\mathbf{A})^2}\sqrt{1 - \eta^2/c^2}. \tag{54}$$

Using the explicit form of the generalized momentum (32) with the accuracy of the expansion to the power of β^2, we obtain the equation of motion in the form

$$\frac{d}{d\tau}\left(\varepsilon - \frac{q}{2c}\mathbf{A}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta} = q\mathbf{E} + q[\boldsymbol{\beta}\times\mathbf{B}] - \frac{\partial}{\partial\mathbf{r}}\left(\frac{q}{c}\mathbf{A}\cdot\boldsymbol{\beta} + \frac{q^2}{2mc^2}\left(\varphi^2 - \mathbf{A}^2\right)\right), \tag{55}$$

where the velocity-dependent components of the force are present. In particular, the velocity-dependent force is present in the Faraday law of electromagnetic induction [34], which is absent in the traditional expression for the Lorentz force.

The Hamilton-Jacobi equation is represented in the form

$$\left(\frac{\partial S}{\partial\tau}\right)^2 - \left(\frac{\partial S}{\partial\mathbf{r}}\right)^2 = \frac{\left(mc^2 + q\varphi\right)^2 - (q\mathbf{A})^2}{c^2} \tag{56}$$

and it reflects the invariance of the representation of the generalized momentum. The well-known representations of the Hamilton-Jacobi Eq. (8) also contain the differential forms of potentials—the components of the electric and the magnetic fields.

3.2 Motion of a charged particle in a constant electric field

Let us consider the motion of a charged particle with the mass m and charge –q in the constant electric field between the plane electrodes with the potential difference U and the distance l between them. For one-dimensional motion, taking the cathode location as the origin and anode at the point $x = l$, from (56) we have

$$\left(\frac{\partial S}{\partial\tau}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 = \frac{\left(mc^2 + qU(1 - x/l)\right)^2}{c^2}. \tag{57}$$

Let us represent the action S in the form

$$S = -Et + f(x), \tag{58}$$

where $E = mc^2 + qU$ is an the electron energy at the origin on the surface of the cathode under voltage –U; as a result, from (57) we obtain

$$S = -Et + \frac{1}{c}\int\sqrt{E^2 - \left(mc^2 + qU - qU\frac{x}{l}\right)^2}\,dx. \tag{59}$$

We find the solution from the condition $\partial S/\partial E = \text{const}$. As a result of integration, we obtain

$$t = \frac{l}{c}\left(1 + \frac{1}{\alpha}\right)\arccos\left(1 - \frac{\alpha}{1+\alpha}\frac{x}{l}\right), \qquad \alpha = qU/mc^2 \tag{60}$$

or

$$x = l\frac{1+\alpha}{\alpha}\left(1 - \cos\left(\frac{\alpha}{1+\alpha}\frac{ct}{l}\right)\right), \qquad t \le \frac{l}{c}\left(\frac{1+\alpha}{\alpha}\right)\arccos\left(\frac{1}{1+\alpha}\right). \tag{61}$$

The well-known solution in the framework of the traditional theory [8] is the following:

$$t = \frac{l}{\alpha c}\sqrt{\left(1+\alpha\frac{x}{l}\right)^2 - 1} \text{ or } x = l\frac{(\alpha ct/l)^2}{1+\sqrt{1+(\alpha ct/l)^2}}, \qquad t \le \frac{l}{c}\sqrt{1+\frac{2}{\alpha}}. \tag{62}$$

In the ultrarelativistic limit $qU \gg mc^2$, the ratio of the flight time of the gap between the electrodes $(x = l)$ is equal to $\pi/2$ according to formulas (60) and (62) (**Figure 5**).

The electron velocity $v = dx/dt$ when reaching the anode is

$$v = c\sqrt{1 - 1/(1+\alpha)^2}. \tag{63}$$

3.3 Problem of the hydrogen-like atom

Let us consider the motion of an electron with the mass m and charge $-e$ in the field of an immobile nucleus with the charge Ze. Then the problem reduces to an investigation of the motion of the electron in the centrally symmetric electric field with the potential $-Ze^2/r$.

Choosing the polar coordinates (r, φ) in the plane of motion, we obtain the Hamilton-Jacobi equation in the form

$$\left(\frac{\partial S}{\partial \tau}\right)^2 - \left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \frac{\left(mc^2 - Ze^2/r\right)^2}{c^2} = 0. \tag{64}$$

Let us represent the action S in the form

$$S = -Et + M\varphi + f(r), \tag{65}$$

where E and M are the constant energy and angular momentum of the moving particle, respectively. As a result, we obtain

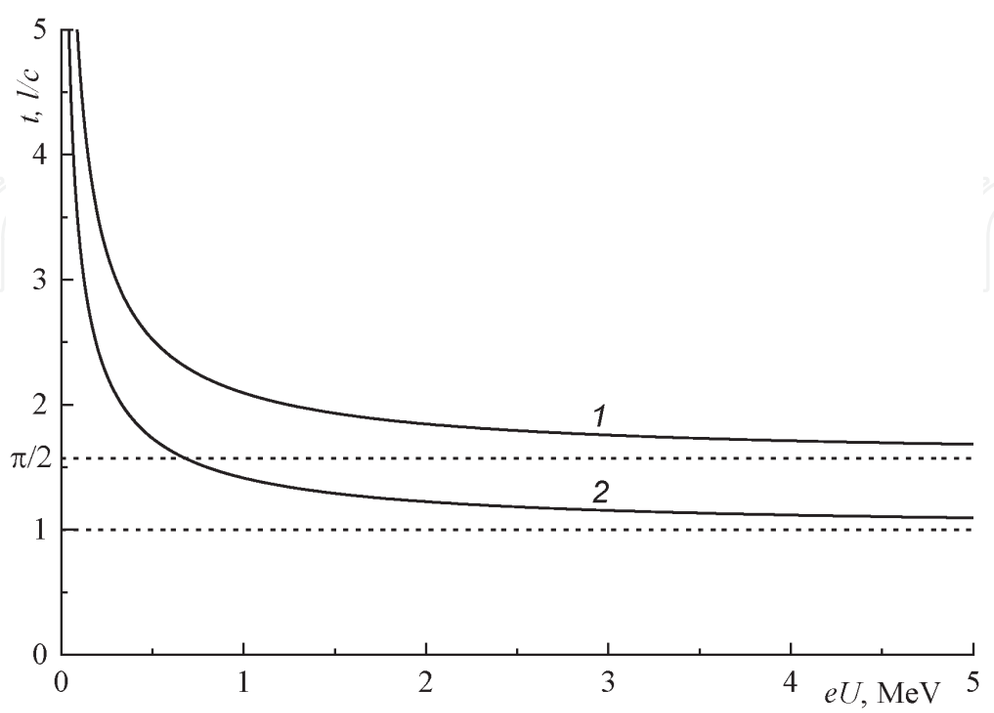

Figure 5.
Dependence of the flight time of the gap between the electrodes on the applied voltage according to the formula (60) and (curve 1) and (62) (curve 2) in l/c units.

$$S = -Et + M\varphi + \frac{1}{c}\int \sqrt{E^2 - (mc^2)^2 + 2mc^2\frac{Ze^2}{r} - \frac{M^2c^2 + \left(Ze^2\right)^2}{r^2}}dr. \tag{66}$$

We find trajectories from the condition $\partial S/\partial M = \text{const}$, with use of which we obtain,

$$\varphi = \int \frac{Mc}{\sqrt{E^2 - (mc^2)^2 + 2mc^2\frac{Ze^2}{r} - \frac{M^2c^2+Ze^2}{r^2}}}d\frac{1}{r}, \tag{67}$$

which results in the solution

$$r = \frac{(Mc)^2 + \left(Ze^2\right)^2}{mc^2Ze^2}\frac{1}{1 + \sqrt{\left(\frac{E}{mc^2}\right)^2\left(1 + \left(\frac{Mc}{Ze^2}\right)^2\right) - \left(\frac{Mc}{Ze^2}\right)^2}\cos\left(\varphi\sqrt{1 + \left(\frac{Ze^2}{Mc}\right)^2}\right)}. \tag{68}$$

The coefficient of the repulsive effective potential is essentially positive, that is, $M^2c^2 + \left(Ze^2\right)^2 > 0$ therefore, any fall of the particle onto the center is impossible. The minimum radius $r_{\min} = r_0(Z+1)$, where $r_0 = e^2/mc^2$ is the classical radius of an electron.

The secular precession is found from the condition

$$\varphi\sqrt{1 + \left(Ze^2/Mc\right)^2} = 2\pi, \tag{69}$$

whence, we obtain

$$\Delta\varphi = 2\pi - \frac{2\pi}{\sqrt{1 + \left(Ze^2/Mc\right)^2}} \approx \pi\left(\frac{Ze^2}{Mc}\right)^2, \tag{70}$$

that has the opposite sign as compared with the solution in [8]. The reason for the discrepancy of the sign is the unaccounted interaction of the self-momentum with the rotating field, that is, the spin-orbit interaction.

4. Equations of the relativistic quantum mechanics

Using the principle of the invariant representation of the generalized momentum

$$\mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = I^2 = \text{inv}, \tag{71}$$

it is possible to compose the corresponding equation of the relativistic quantum mechanics by representing the energy and momentum variables by the corresponding operators $\hat{\varepsilon} = i\hbar\partial/\partial\tau$ and $\hat{\mathbf{p}} = -i\hbar\partial/\partial\mathbf{r}$:

$$\begin{aligned}(\hat{\varepsilon})^2\Psi - (\hat{\mathbf{p}})^2\Psi = \left(i\hbar\frac{\partial}{\partial\tau}\right)^2\Psi - \left(-i\hbar\frac{\partial}{\partial\mathbf{r}}\right)^2\Psi = \\ \left(\varepsilon^2 - \mathbf{p}^2\right)\Psi + i\hbar\left(\frac{\partial\varepsilon}{\partial\tau} + \text{div}\mathbf{p}\right) = I^2\Psi + i\hbar\left(\frac{\partial\varepsilon}{\partial\tau} + \text{div}\mathbf{p}\right),\end{aligned} \tag{72}$$

and

$$(\hat{\varepsilon}\Psi)^2 - (\hat{\mathbf{p}}\Psi)^2 = \left(i\hbar\frac{\partial\Psi}{\partial\tau}\right)^2 - \left(-i\hbar\frac{\partial\Psi}{\partial\mathbf{r}}\right)^2 = \left(\varepsilon^2 - \mathbf{p}^2\right)\Psi^2 = I^2\Psi^2. \tag{73}$$

The case of conservative systems, when any energy losses or sources in space are absent, corresponds to the relation $\partial\varepsilon/\partial\tau + \mathrm{div}\mathbf{p} = 0$. In this way,

$$\begin{cases} \dfrac{\partial^2\Psi}{\partial\tau^2} - \dfrac{\partial^2\Psi}{\partial\mathbf{r}^2} = -\dfrac{I^2}{\hbar^2}\Psi \\ \left(\frac{\partial\Psi}{\partial\tau}\right)^2 - \left(\frac{\partial\Psi}{\partial\mathbf{r}}\right)^2 = -\dfrac{I^2}{\hbar^2}\Psi^2. \end{cases} \tag{74}$$

For the charged particle in an external field with an invariant in the form of (30), the equations will take the form

$$\begin{cases} \dfrac{\partial^2\Psi}{\partial\tau^2} - \dfrac{\partial^2\Psi}{\partial\mathbf{r}^2} = -\dfrac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi \\ \left(\frac{\partial\Psi}{\partial\tau}\right)^2 - \left(\frac{\partial\Psi}{\partial\mathbf{r}}\right)^2 = -\dfrac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi^2. \end{cases} \tag{75}$$

For stationary states we obtain

$$\begin{cases} \dfrac{\partial^2\Psi}{\partial\mathbf{r}^2} + \dfrac{E^2 - \left(mc^2 + q\varphi\right)^2 + \left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi = 0 \\ \left(\frac{\partial\Psi}{\partial\mathbf{r}}\right)^2 + \dfrac{E^2 - \left(mc^2 + q\varphi\right)^2 + \left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi^2 = 0. \end{cases} \tag{76}$$

Rewriting the equations taking into account the formulas of the classical correspondence (40), we will obtain the equations for the wave function in the traditional representation

$$\begin{gathered} \Delta\Psi + \frac{2m}{\hbar^2}(\mathrm{H} - U)\Psi = 0, \\ \left(\frac{\partial\Psi}{\partial\mathbf{r}}\right)^2 + \frac{2m}{\hbar^2}(\mathrm{H} - U)\Psi^2 = 0, \end{gathered} \tag{77}$$

the first of which formally coincides with the Schrödinger equation for the wave function of stationary states.

For the action function S associated with the wave function by the representation $\Psi = A\ exp\ (-iS/\hbar)$ or $S = i\hbar\ln\Psi + i\hbar\ln A$, we will obtain

$$\begin{cases} \dfrac{\partial^2 S}{\partial\mathbf{r}^2} = 0 \\ \left(\frac{\partial S}{\partial\mathbf{r}}\right)^2 - \dfrac{E^2 - \left(mc^2 + q\varphi\right)^2 + \left(q\mathbf{A}\right)^2}{c^2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 S}{\partial\mathbf{r}^2} = 0 \\ \left(\frac{\partial S}{\partial\mathbf{r}}\right)^2 - 2m(\mathrm{H} - U) = 0, \end{cases} \tag{78}$$

which represents the exact classical correspondence instead of the quasiclassical approximation [12]. Note, the equations similar to (78) also follow from the Eq. (46) in [12] if we demand for an exact correspondence and equate to zero the real and imaginary parts.

4.1 Particle in the one-dimensional potential well

Let us consider the particle of mass m in a one-dimensional rectangular potential well of the form

$$V(x) = \begin{cases} 0, & 0 \geq x \geq a \\ -V_0, & 0 \leq x \leq a. \end{cases} \tag{79}$$

From the first equation of system (70) we have

$$\frac{d^2\Psi}{dx^2} + \frac{E^2 - (mc^2 + V(x))^2}{\hbar^2 c^2}\Psi = 0. \tag{80}$$

Then, $U_0 = -V_0 + {V_0}^2/(2mc^2)$ corresponds to the potential energy of the particle in the well in the classical meaning. In the latter case, it is known [12] that the bound state with the energy $H = 0 (E = mc^2)$ arises under the conditions

$$U_0 = -\frac{\pi^2\hbar^2}{2ma^2}n^2 = -V_0 + \frac{{V_0}^2}{2mc^2} \geq -\frac{mc^2}{2}, \quad a \geq \frac{\pi\hbar}{mc}n = \frac{\lambda}{2}n, \tag{81}$$

$$E_n = mc^2\left(1 - \sqrt{1 - \frac{\pi^2\hbar^2}{m^2c^2a^2}n^2}\right) = mc^2\left(1 - \sqrt{1 - \left(\frac{\lambda}{2a}n\right)^2}\right)$$
$$= mc^2\frac{\left(\frac{\lambda}{2a}n\right)^2}{1 + \sqrt{1 - \left(\frac{\lambda}{2a}n\right)^2}}, \tag{82}$$

where $\lambda = 2\pi\lambda\!\!\!^{-} = 2\pi\hbar/mc = h/mc$. Maximum depth of the classic well is equal to $U_0 = -mc^2/2$ at $V_0 = mc^2$. The condition for the existence of the bound state with an energy H = 0 $(E = mc^2)$ in a potential well of size a is expressed by the relation

$$a = \lambda n/2, \qquad n = 1, \ 2, \ 3 \ldots \tag{83}$$

In the three-dimensional case, the bound state with the energy H = 0 $(E = mc^2)$ arises under the same conditions [23] for a spherical well with a diameter d and depth V_0 with the $d = \lambda n/2, \ n = 1, 2, 3 \ldots$.

The solution of this simple example is fundamental and accurately represents the uncertainty principle $\Delta x \Delta p \geq \hbar/2$. It clearly represents the wave property of the particle, clearly showing that the standing wave exists only at the condition $a \geq \lambda/2$ when the geometric dimensions of the well are greater than half the wavelength of the particle.

4.2 Penetration of a particle through a potential barrier

Let us consider the problem of penetration of a particle through the rectangular potential barrier [23] with the height V_0 and width a. Then, $U_0 = V_0 + {V_0}^2/(2mc^2)$ corresponds to the potential energy of the particle in the well in the classical meaning, and H = $\left(E^2 – m^2c^4\right)/2mc^2$ corresponds to the energy. Substituting these expressions into the solution of the Schrödinger equation for the rectangular potential barrier, we obtain for the transmission coefficient D of the particle penetrating through the potential barrier at $E > \left|V_0 + mc^2\right|$

$$D = \left[1 + \frac{\left(\left(\frac{V_0}{mc^2}+1\right)^2 - 1\right)^2}{4\left(\left(\frac{E}{mc^2}\right)^2 - 1\right)\left(\left(\frac{E}{mc^2}\right)^2 - \left(\frac{V_0}{mc^2}+1\right)^2\right)} \sin^2\left(\frac{a}{\lambda\!\!\!^{-}}\sqrt{\left(\frac{E}{mc^2}\right)^2 - \left(\frac{V_0}{mc^2}+1\right)^2}\right)\right]^{-1} \tag{84}$$

and at $E < \left|V_0 + mc^2\right|$

$$D = \left[1 + \frac{\left(\left(\frac{V_0}{mc^2}+1\right)^2 + 1 - 2\left(\frac{E}{mc^2}\right)^2\right)^2}{4\left(\left(\frac{E}{mc^2}\right)^2 - 1\right)\left(\left(\frac{V_0}{mc^2}+1\right)^2 - \left(\frac{E}{mc^2}\right)^2\right)} \sinh^2\left(\frac{a}{\lambda\!\!\!^{-}}\sqrt{\left(\frac{V_0}{mc^2}+1\right)^2 - \left(\frac{E}{mc^2}\right)^2}\right)\right]^{-1} \tag{85}$$

where $\lambda\!\!\!^{-} = \hbar/mc$ is the de Broglie wavelength of the particle. As can be seen, the barrier is formed only in the energy range $-2mc^2 > V_0 > mc^2$.

For the problem of the passage of a particle with energy E through a potential barrier U (**Figure 2**) the wave vector k is represented as

$$k_1 = \frac{1}{\hbar c}\sqrt{E^2 - (mc^2)^2}, \qquad k_2 = \frac{1}{\hbar c}\sqrt{E^2 - (mc^2 + U)^2} \tag{86}$$

and if the particle energy does not exceed the potential barrier, then the transmission coefficient is zero, regardless of the height of the barrier and not have. In this case, there is no contradiction similar to the Klein paradox.

4.3 Charged particle in a magnetic field

The vector potential of a uniform magnetic field **A** along the **z** axis direction in the cylindrical coordinate system (ρ, φ, z) has components $A_\varphi = H\rho/2$, $A_\rho = A_z = 0$ and Eq. (76) takes the form

$$\frac{\hbar^2}{2M}\left(R'' + \frac{1}{\rho}R'\right) + \left(E - \frac{\hbar^2 m^2}{2M}\frac{1}{\rho^2} + \frac{M\omega_H^2}{8}\rho^2 - \frac{p_z^2}{2M}\right)R = 0, \tag{87}$$

where m – angular quantum number, M – mass of electron, H– magnetic field value, $\omega_H = eH/Mc$. In this case, the equation below differs from the known [12] one by the absence of the field linear term $\hbar\omega_H m/2$ and the sign of a quadratic term $M\omega_H^2\rho^2/8$.

In this form, the Eq. (87) does not have a finite solution depending on the variable ρ and, provided R = const, we have

$$\begin{aligned} &R'' + \frac{1}{\rho}R' = 0, \\ &\left(E - \frac{\hbar^2 m^2}{2M\rho^2} + \frac{M\omega_H^2}{8}\rho^2 - \frac{p_z^2}{2M}\right)R = 0. \end{aligned} \tag{88}$$

Or

$$E - \frac{\hbar^2 m^2}{2M}\frac{1}{\rho^2} + \frac{M\omega_H^2}{8}\rho^2 - \frac{p_z^2}{2M} \equiv 0. \tag{89}$$

From (89) we have for the energy levels

$$W = Mc^2\sqrt{1 + m^2\left(\frac{\lambda\!\!\!^{-}}{\rho}\right)^2 - \left(\frac{\rho}{2\rho_H}\right)^2 + \left(\frac{p_z}{Mc}\right)^2}. \tag{90}$$

where $\rho_H = c/\omega_H$ (magnetic event horizon), and ρ as a constant parameter.

If an electron is excited by a magnetic field from a state of rest, then $W = Mc^2$ and from (88) we obtain

$$-\frac{\hbar^2 m^2}{2M\rho^2} + \frac{M\omega_H^2}{8}\rho^2 = 0, \tag{91}$$

or

$$m\hbar\omega_H = M(\rho\omega_H)^2/2 = \frac{Mc^2}{2}\left(\frac{\rho}{\lambda\!\!\!^{-}_H}\right)^2 \tag{92}$$

From (92) for a magnetic flux quantum we have

$$\frac{e}{hc}H\pi\rho^2 = \frac{e}{hc}\Phi = m, \qquad \Delta\Phi = \frac{hc}{e}. \tag{93}$$

We get the same results when solving the Hamilton-Jacobi equation.

4.4 Particle in the field with Morse potential energy

We determine the energy levels for a particle moving in a field with a potential $\varphi(x) = -\varphi_0 e^{-x/d}$.

According to (41), for the potential energy of interaction $V(x)$ with the field $\varphi(x)$ we obtain the expression of the potential Morse energy (**Figure 6**)

$$V(x) = -q\varphi_0 e^{-x/d} + \frac{1}{2mc^2}\left(q\varphi_0 e^{-x/d}\right)^2 = mc^2\left(-\frac{q\varphi_0 e^{-x/d}}{mc^2} + \frac{1}{2}\left(\frac{q\varphi_0}{mc^2}e^{-x/d}\right)^2\right). \tag{94}$$

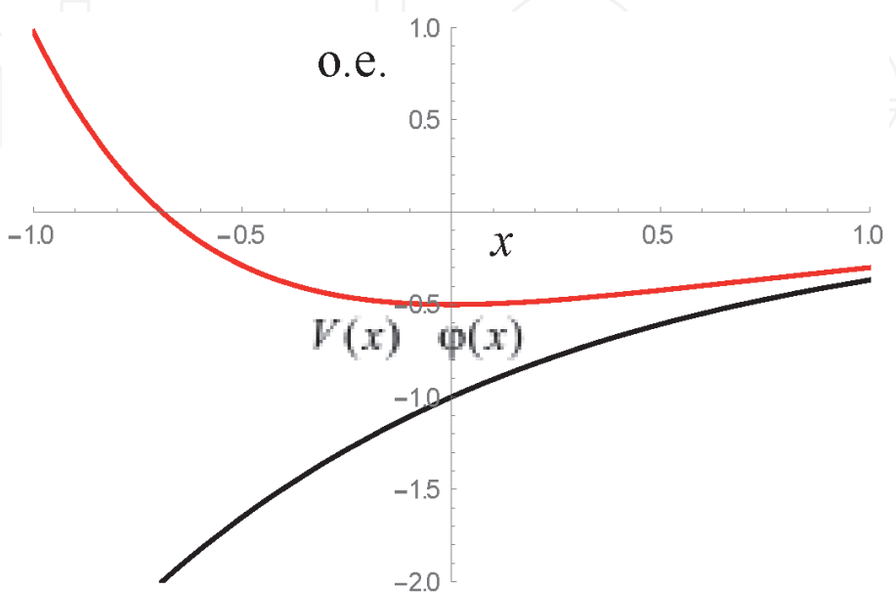

Figure 6.
The exponential potential of the field $\varphi(x)$ and Morse potential energy of interaction $V(x)$.

Schrödinger equation takes the form

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\left(E-mc^2\left(-\frac{q\varphi_0}{mc^2}e^{-x/d}+\frac{1}{2}\left(\frac{q\varphi_0}{mc^2}e^{-x/d}\right)^2\right)\right)\psi=0. \tag{95}$$

Following the procedure for solving Eq. (95) in [12], introducing a variable (taking values in the interval [0,∞]) and the notation

$$\xi=2d\frac{q\varphi_0}{mc^2}\frac{\lambda\!\!\!^{-}}{d}e^{-x/d},\qquad s=\frac{d}{\lambda\!\!\!^{-}}\sqrt{-\frac{2E}{mc^2}},\qquad n=\frac{d}{\lambda\!\!\!^{-}}-\left(s+\frac{1}{2}\right), \tag{96}$$

We get

$$\frac{d^2\psi}{d\xi^2}+\frac{1}{\xi}\frac{d\psi}{d\xi}+\left(-\frac{1}{4}+\frac{n+s+1/2}{\xi}-\frac{s^2}{\xi^2}\right)\psi=0. \tag{97}$$

Given the asymptotic behavior of function ψ for $\xi\to\infty$ and $\xi\to 0$, after substituting $\psi=e^{-\xi/2}\xi^s w(\xi)$ we obtain

$$\xi w''+(2s+1-\xi)w'+nw=0 \tag{98}$$

equation of degenerate hypergeometric function (Kummer function).

$$w={}_1F_1(-n,2s+1,\xi) \tag{99}$$

A solution satisfying the finiteness condition for $\xi=0$ and when $\xi\to\infty$ the$w$ turns to infinity no faster than a finite degree ξ is obtained for a generally positive n. Moreover, the Kummer function ${}_1F_1$ reduces to a polynomial.

In accordance with (96) and (99), we obtain values for energy levels W (**Figure 7**)

$$W=mc^2\sqrt{1-\left(1-\frac{\lambda\!\!\!^{-}}{2d}(2n+1)\right)^2}. \tag{100}$$

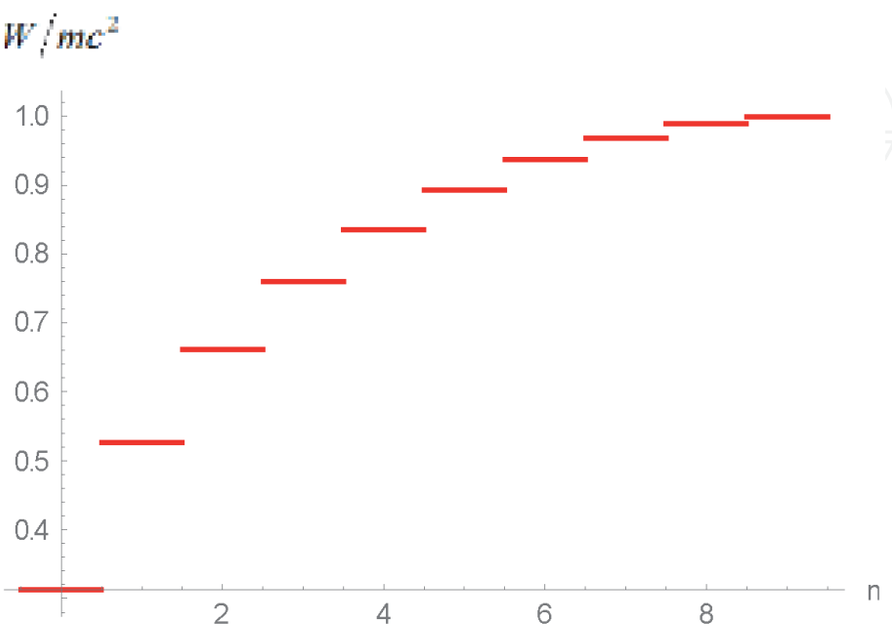

Figure 7.
The dependence of the energy of particle W on the quantum number n(100) at d = 10ƛ in units of mc².

For the binding energy in the ground state W_0 for $n = 0$ of (100) we have (**Figure 8**).

$$W_0 = mc^2 - mc^2\sqrt{1 - \left(1 - \frac{\lambda\!\!\!^{-}}{2d}\right)^2} \tag{101}$$

Because parameter s is determined to be positive (96) $s = d/\lambda\!\!\!^{-} - n - 1/2 \geq 0$ and $n \leq d/\lambda\!\!\!^{-} - 1/2$, then at $n = 0$ the minimum value is $d = \lambda\!\!\!^{-}/2$, which reflects the Heisenberg uncertainty principle. The maximum binding energy of a particle $mc^2 - W$ is limited from above by a value mc^2 regardless of the nature and magnitude of the interaction (**Figure 9**).

The interaction constant $q\varphi_0/mc^2$ (97) does not have any limitation on the value and is not included in the expression for energy levels (100) and only determines the spatial properties of the wave function (99) through variable ξ (**Figures 9** and **10**).

We emphasize that despite the fact that the potential energy for a stationary particle $V(x)$ has a depth of $mc^2/2$, the maximum binding energy for a moving

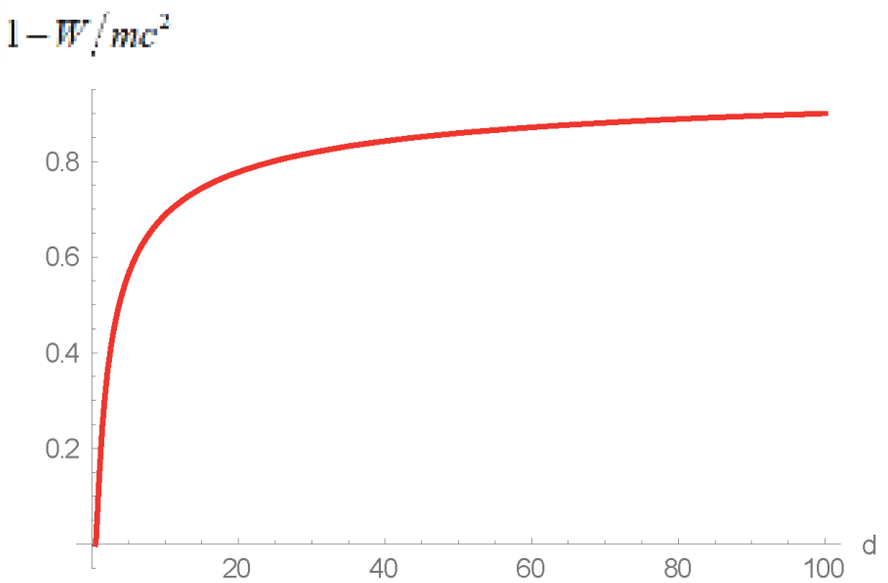

Figure 8.
The dependence of the binding energy of the ground state $mc^2 - W_0$ (101) on the size $d \geq \lambda\!\!\!^{-}/2$ in units of mc^2.

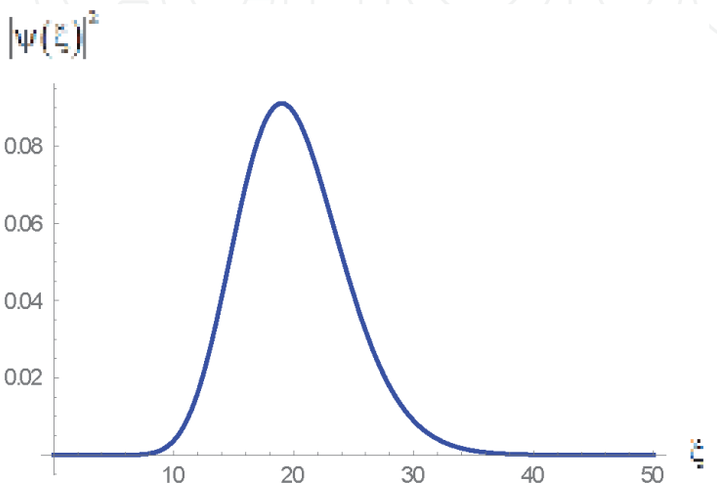

Figure 9.
Dependency of function $|\psi(\xi)|^2$ at $d = 10\lambda\!\!\!^{-}$ and $n = 0$.

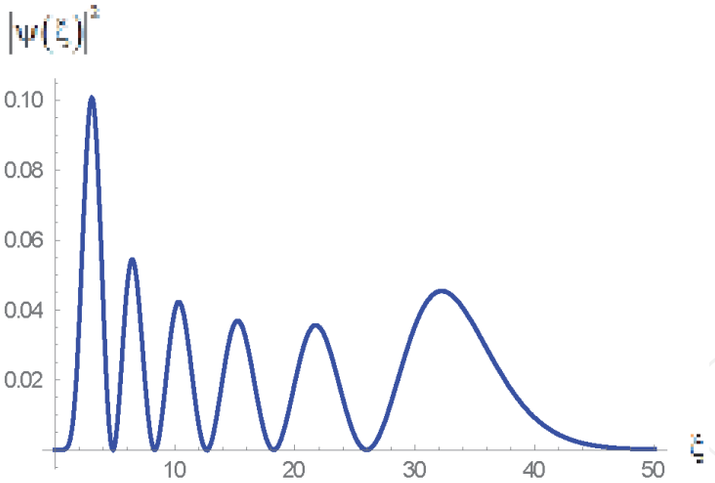

Figure 10.
Dependency of function $|\psi(\xi)|^2$ at $d = 10\lambda$ and $n = 5$.

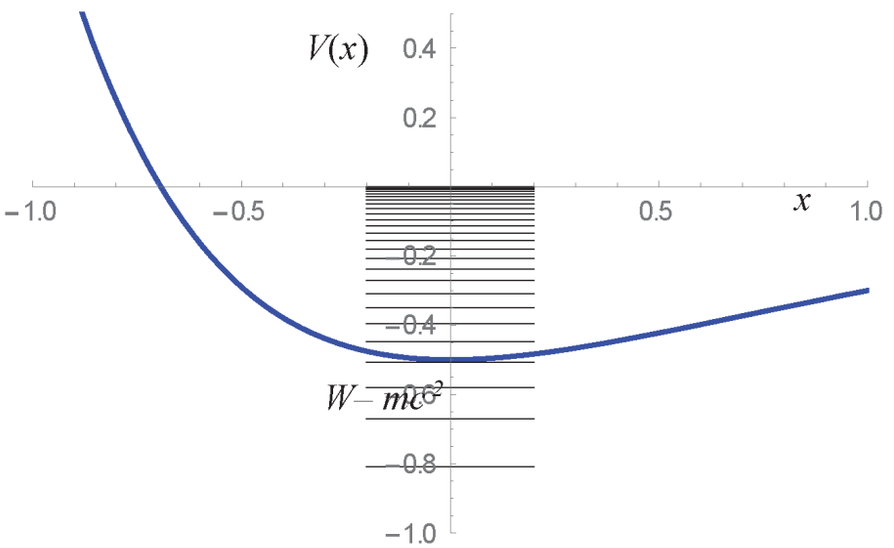

Figure 11.
The dependency of the potential energy of the interaction of Morse $V(x)$ and energy levels of the particle $W - mc^2$ at $d = 27\lambda$ in units of mc^2.

particle in the ground state is equal to mc^2 (**Figure 11**), which is a relativistic effect of the particle's motion in the ground state - in the ground state, the particle not at rest.

4.5 Problem of the hydrogen-like atom

The motion of a charged particle in the Coulomb field can be described as a motion in the field of an atomic nucleus (without the spin and magnetic moment) with the potential energy $-Ze^2/r$.

In spherical coordinates, Eq. (70) for the wave function takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial \varphi^2} + \frac{1}{\hbar^2c^2}\left(E^2 - \left(mc^2 - \frac{Ze^2}{r}\right)^2\right)\Psi = 0. \tag{102}$$

Separating the variables

$$\Psi = \Phi_m(\varphi) Y_{l,\ m}(\theta) R_{\ n_R,\ l}(r) \tag{103}$$

and introducing the notations [12]

$$\begin{aligned} &\alpha = \frac{e^2}{\hbar c}, \qquad \rho = \frac{mZe^2}{\hbar^2}\frac{2r}{N} = Z\alpha\frac{mc}{\hbar}\frac{2r}{N}, \qquad M^2 = \hbar^2 l(l+1), \\ &\mathrm{H}_n = \frac{{E_n}^2 - m^2c^4}{2mc^2} = -\frac{mZ^2e^4}{\hbar^2}\frac{1}{2N^2} = -mc^2Z^2\alpha^2\frac{1}{2N^2}, \\ &s(s+1) = l(l+1) + Z^2\alpha^2 \ \Rightarrow\ s = -1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2} \end{aligned} \tag{104}$$

(only the positive root is taken for s), for stationary states we have

$$\begin{aligned} &\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi, \\ &\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right) - \frac{m^2}{\sin^2\theta}Y = -l(l+1)Y, \\ &\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} - \frac{s(s+1)}{\rho^2}R = -\left(\frac{n_\mathrm{r}}{\rho} - \frac{1}{4}\right)R, \end{aligned} \tag{105}$$

where $m = \pm 0,\ \pm 1,\ \pm 2,\ \ldots, l = 0,\ 1,\ 2,\ 3,\ \ldots, |m| < l$ and $s = -1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2}$.

The solution of Eq. (88) formally coincides with the well-known Fuse solution for the molecular Kratzer potential in the form $U = \frac{A}{r^2} - \frac{B}{r} = \frac{Z^2e^4}{2mc^2}\frac{1}{r^2} - Ze^2\frac{1}{r}$ at the condition, that $n - s - 1 = n_\mathrm{r}$ must be a positive integer or zero. According to (87), we obtain the energy levels

$$\begin{aligned} &\mathrm{H}_{n,j} = -mc^2\frac{Z^2\alpha^2}{2\left(n_\mathrm{r} + 1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2}\right)^2}, \\ &E_{n,j} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n_\mathrm{r} + 1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2}\right)^2}}, \end{aligned} \tag{106}$$

where the radial quantum number $n_\mathrm{r} = 0, 1, 2, \ldots$. Introducing the principal quantum number $n = n_\mathrm{r} + l + 1/2$, $l < n$ $(n = 1, 2, 3, \ldots)$, we finally obtain

$$E_{n,j} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n + \frac{Z\alpha^2}{l+1/2+\sqrt{(l+1/2)^2+Z^2\alpha^2}}\right)^2}}. \tag{107}$$

For the ground state with the $l = 0$ and $n = 1$, we have

$$E_0 = \frac{mc^2}{\sqrt{1/2 + \sqrt{1/4 + Z^2\alpha^2}}}, \quad s = \frac{Z^2\alpha^2}{1/2 + \sqrt{1/4 + Z^2\alpha^2}} \tag{108}$$

without any restrictions for the value of Z. In this case, $1–s > 0$ and there is no fall of the particle on the center [8], and the probability of finding the particle at the center (in the nucleus) is always equal to zero.

In this case, the obtained fine splitting is in no way connected with the spin-orbit interaction and is due to the relativistic dependence of the mass on the orbital and radial velocity of motion, which results to the splitting of the levels.

4.6 Dirac equations

In the standard representation, the Dirac equations in compact notation for a particle have the form [21].

$$\begin{aligned} \hat{\varepsilon}\phi - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\chi &= mc\phi, \\ \hat{\varepsilon}\chi + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\phi &= mc\chi. \end{aligned} \tag{109}$$

In addition, for the particle in an external field they can be represented in theorm

$$\begin{aligned} \hat{\varepsilon}\phi - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\chi &= \left(mc + \frac{q}{c}\varphi\right)\phi + \frac{q}{c}\boldsymbol{\sigma}\cdot\mathbf{A}\chi, \\ \hat{\varepsilon}\chi + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\phi &= \left(mc + \frac{q}{c}\varphi\right)\chi - \frac{q}{c}\boldsymbol{\sigma}\cdot\mathbf{A}\phi. \end{aligned} \tag{110}$$

By writing the wave equations for the wave functions, we obtain

$$\begin{aligned} \left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\mathbf{r}^2}\right)\phi &= -\frac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{\hbar^2c^2}\phi - \frac{q}{\hbar c}\boldsymbol{\sigma}\cdot(\mathbf{B} - i\mathbf{E})\chi, \\ \left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\mathbf{r}^2}\right)\chi &= -\frac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{\hbar^2c^2}\chi + \frac{q}{\hbar c}\boldsymbol{\sigma}\cdot(\mathbf{B} - i\mathbf{E})\phi, \end{aligned} \tag{111}$$

where we used the properties of the Pauli matrices. It is easy to verify that the functions ϕ and χ differ only in the constant phase $\phi = \chi e^{\pm i\pi} = -\chi$ and the equations can be completely separated and only one equation can be used, bearing in mind that (111) can be of a variable sign

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\mathbf{r}^2}\right)\Psi = -\frac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi \pm \frac{q}{\hbar c}\boldsymbol{\sigma}\cdot(\mathbf{B} - i\mathbf{E})\Psi. \tag{112}$$

In the case of a stationary state, the standard representation of the wave Eq. (110) has the form

$$\begin{aligned} \left(\varepsilon - mc - \frac{q}{c}\varphi\right)\varphi &= \boldsymbol{\sigma}\cdot\left(\mathbf{p} + \frac{q}{c}\mathbf{A}\right)\chi, \\ \left(\varepsilon + mc + \frac{q}{c}\varphi\right)\chi &= \boldsymbol{\sigma}\cdot\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)\varphi. \end{aligned} \tag{113}$$

4.7 Dirac equations solution for a hydrogen-like atom

For a charge in a potential field with the central symmetry [23], we have

$$\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \begin{pmatrix}\dfrac{f(r)}{r}\Omega_{jlm}\\ \\ (-1)^{1+l-l'}\dfrac{g(r)}{r}\Omega_{\,jl'm}\end{pmatrix}. \tag{114}$$

After substituting (96) into (95), we obtain

$$\begin{cases} f' + \dfrac{\chi}{r} f - \left(\varepsilon + mc - \dfrac{Ze^2}{c}\dfrac{1}{r}\right) g = 0 \\ g' - \dfrac{\chi}{r} g + \left(\varepsilon - mc + \dfrac{Ze^2}{c}\dfrac{1}{r}\right) f = 0, \end{cases} \qquad \begin{cases} j = |l \pm 1/2|, \quad j_{\max} = l_{\max} + 1/2 \\ \chi = -1, \quad l = 0 \\ \chi = \pm(j + 1/2). \end{cases} \tag{115}$$

Let us represent the functions f and g in the form

$$\begin{aligned} f &= \sqrt{mc + \varepsilon} e^{-\rho/2} \rho^{\gamma} (Q_1 + Q_2), \\ g &= \sqrt{mc - \varepsilon} e^{-\rho/2} \rho^{\gamma} (Q_1 - Q_2), \end{aligned} \tag{116}$$

where

$$\rho = 2\lambda r/\hbar, \qquad \lambda = \sqrt{(mc)^2 - \varepsilon^2}, \qquad \gamma = \sqrt{\chi^2 + Z^2\alpha^2}, \qquad \alpha = \frac{e^2}{\hbar c}. \tag{117}$$

Substituting (116) into the Eq. (117), for the sum and difference of the equations we have

$$\begin{aligned} &\rho Q_1{}' + (\gamma - Z\alpha mc/\lambda) Q_1 + (\chi - Z\alpha\varepsilon/\lambda) Q_2 = 0, \\ &\rho Q_2{}' + (\gamma + Z\alpha mc/\lambda - \rho) Q_2 + (\chi + Z\alpha\varepsilon/\lambda) Q_1 = 0. \end{aligned} \tag{118}$$

Close to $\rho = 0$, the system of equations always has a solution, because

$$\gamma^2 - (Z\alpha mc/\lambda)^2 = \chi^2 - (Z\alpha\varepsilon/\lambda)^2. \tag{119}$$

Then

$$Q_2 = -\frac{\gamma - Z\alpha mc/\lambda}{\chi - Z\alpha\varepsilon/\lambda} Q_1 = -\frac{\chi + Z\alpha\varepsilon/\lambda}{\gamma + Z\alpha mc/\lambda} Q_1. \tag{120}$$

Forming equations of the second order and solving with respect to Q_1 and Q_2, we obtain

$$\begin{aligned} &\rho Q_1{}'' + (2\gamma + 1 - \rho) Q_1{}' - (\gamma - Z\alpha mc/\lambda) Q_1 = 0, \\ &\rho Q_2{}'' + (2\gamma + 1 - \rho) Q_2{}' - (\gamma + 1 - Z\alpha mc/\lambda) Q_2 = 0. \end{aligned} \tag{121}$$

With allowance for (121), the solution of these equations is

$$\begin{aligned} Q_1 &= A F(\gamma - Z\alpha mc/\lambda,\ 2\gamma + 1,\ \rho), \\ Q_2 &= -A \frac{\gamma - Z\alpha mc/\lambda}{\chi - Z\alpha\varepsilon/\lambda} F(\gamma + 1 - Z\alpha mc/\lambda,\ 2\gamma + 1,\ \rho), \end{aligned} \tag{122}$$

where $F(\alpha, \beta, z)$ is the degenerate hypergeometric function and A is the normalization constant of the wave function. The function $F(\alpha, \beta, z)$ reduces to a polynomial, if the parameter α is equal to an integer negative number or zero. Therefore, finite solutions for the functions f and g are

$$\gamma - \frac{Z\alpha mc}{\lambda} = -n_{\mathrm{r}}. \tag{123}$$

From expressions (117), we obtain

$$f = A\sqrt{mc+\varepsilon}e^{-\rho/2}\rho^{\gamma-1}\left(F(-n_{\mathrm{r}}, \quad 2\gamma+1, \quad \rho) + \frac{n_r}{\chi - Z\alpha\varepsilon/\lambda}F(1-n_{\mathrm{r}}, \quad 2\gamma+1, \quad \rho)\right),$$
$$g = A\sqrt{mc-\varepsilon}e^{-\rho/2}\rho^{\gamma-1}\left(F(-n_{\mathrm{r}}, \quad 2\gamma+1, \quad \rho) - \frac{n_r}{\chi - Z\alpha\varepsilon/\lambda}F(1-n_{\mathrm{r}}, \quad 2\gamma+1, \quad \rho)\right), \tag{124}$$

where $n_r = 0, 1, 2, \ldots$ is the radial quantum number. For the energy levels, we obtain from the condition (117)

$$\frac{\varepsilon_{p,\chi}}{mc} = \sqrt{1 - \frac{Z^2\alpha^2}{\left(n_{\mathrm{r}} + \sqrt{\chi^2 + Z^2\alpha^2}\right)^2}} \tag{125}$$

and taking into account the obtained values of χ, we finally have

$$E_{n,j} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n_{\mathrm{r}} + \sqrt{(j+1/2)^2 + Z^2\alpha^2}\right)^2}} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n + \frac{z_{\alpha^2}}{j+1/2+\sqrt{(j+1/2)^2+Z^2\alpha^2}}\right)^2}}, \tag{126}$$

where the principal quantum number $n = n_r + j + 1/2$. Besides $j = n - 1/2$, all other levels with $j < n - 1/2$ are degenerated twice in the orbital angular momentum $l = |j \pm 1/2|$. The ground state energy for $n = 1$ and $j = 1/2$ is

$$E_0 = \frac{mc^2}{\sqrt{1+Z^2\alpha^2}} \tag{127}$$

without any limitations for the value of Z. In this case $\gamma - 1 = -1 + \sqrt{1+Z^2\alpha^2} > 0$, and no falling of particle on the center is observed, and the probability to find the particle in the center (in the nucleus) is always equal to zero.

In the resulting formula (126), the order of sequence of the fine splitting levels is inverse relative to the order of sequence in the well-known Sommerfeld-Dirac formula. If to compare the expansions in a series in the degree of the fine-structure constant of two formulas

$$\frac{E_n}{mc^2} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)\alpha^2 + \left(\frac{1}{8} + \frac{3}{8n^4} - \frac{1}{2n^3(j+1/2)}\right)\alpha^4, \frac{\Delta E_{3/2,1/2}}{mc^2} = \frac{\alpha^4}{32}, \tag{128}$$

$$\frac{E_n}{mc^2} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)\alpha^2 + \left(-\frac{3}{8} - \frac{1}{8n^4} + \frac{1}{2n^3(j+1/2)}\right)\alpha^4, \frac{\Delta E_{3/2,1/2}}{mc^2} = -\frac{\alpha^4}{32}, \tag{129}$$

then the difference will be equal to

$$\frac{\Delta E_n}{mc^2} = \frac{\alpha^4}{2} - \frac{\alpha^4}{2n^4} - \frac{\alpha^4}{n^3(j+1/2)}, \tag{130}$$

where the last term is the expression for the spin-orbit interaction energy. Thus, to obtain the true value of the energy levels of the hydrogen atom, it is necessary to add the energy of the spin-orbit interaction in formula (126) in the form (130). This is completely justified, because such an interaction was not initially included in Eq. (115) and was not reflected in the final result.

5. Conclusion

The principle of invariance is generalized and the corresponding representation of the generalized momentum of the system is proposed; the equations of relativistic and quantum mechanics are proposed, which are devoid of the above-mentioned shortcomings and contradictions. The equations have solutions for any values of the interaction constant of the particle with the field, for example, in the problem of a hydrogen-like atom, when the atomic number of the nucleus $Z > 137$. The equations are applicable for different types of particles and interactions.

Based on the parametric representation of the action and the canonical equations, the corresponding relativistic mechanics based on the canonical Lagrangian is constructed and the equations of motion and expression are derived for the force acting on the charge moving in an external electromagnetic field.

The matrix representation of equations of the characteristics for the action function and the wave function results in the Dirac equation with the correct enabling of the interaction. In this form, the solutions of the Dirac equations are not restricted by the value of the interaction constant and have a spinor representation by scalar solutions of the equations for the action function and the wave function.

The analysis of the solutions shows the full compliance with the principles of the relativistic and quantum mechanics, and the solutions are devoid of any restrictions on the nature and magnitude of the interactions.

The theory of spin fields and equations for spin systems will be described in subsequent works.

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