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Chapter

Integral Geometry and Cohomology in Field Theory on the Space-Time as Complex Riemannian Manifold

Francisco Bulnes

Abstract

The study of the relationships between the integration invariants and the different classes of operators, as well as of functions inside the context of the integral geometry, establishes diverse homologies in the dual space of the functions. This is given in the class of cohomology of the integral operators that give solution to certain class of differential equations in field theory inside a holomorphic context. By this way, using a cohomological theory of appropriate operators that establish equivalences among cycles and cocycles of closed submanifolds, line bundles and contours can be obtained by a cohomology of general integrals, useful in the evaluation and measurement of fields, particles, and physical interactions of diverse nature that occurs in the space-time geometry and phenomena. Some of the results applied through this study are the obtaining of solutions through orbital integrals for the tensor of curvature $R^{\mu\nu}$, of Einstein's equations, and using the imbedding of cycles in a complex Riemannian manifold through the duality: line bundles with cohomological contours and closed submanifolds with cohomological functional. Concrete results also are obtained in the determination of Cauchy type integral for the reinterpretation of vector fields.

Keywords: complex Riemannian manifold, cocycles, cohomology, cohomology of cycles, geometrical integration, integral curvature, integration invariants, integral operators

AMS Classification: 32A19 32C26 32C30 23C15 58 K70 14 J17

1. Introduction

Obtaining an integral cohomology of general integral operators that determine complex analytic solutions through classes of cohomology born of the $\bar{\partial}$-cohomology is necessary to use a holomorphic language with the purpose of obtaining the holomorphic forms that involve exact forms. In fact, this methodology is a way of so many perspectives that suggest the use of complex hyperholomorphic functions in approaching functions in complex analysis, although using fibrations on some quaternionic algebra. The holomorphic forms required in this language, are good to express the integral of complex vector fields as integral of line, which have more than enough lines and hyperplanes, respectively, in $\mathbb{CP}^n$ and $\mathbb{C}^n$, visualizing these

fields like holomorphic sections of complex holomorphic bundles of fibrations $X \to M$.

But the $\bar{\partial}$-cohomology exists naturally in coverings of Stein $X \to M$, like holomorphic forms. Then, the integral can be expressed on spaces M_δ, and Δ_z, [1, 2], that are given as lines and hyperplanes of $\mathbb{CP}^n$ and $\mathbb{C}^n$, and that as such they are integral orbital of the complex manifolds $M = G/L$ and $\Delta = \Gamma/\Sigma$, belonging to a $\bar{\partial}$-cohomology in holomorphic language.

The cohomologies of functionals and functions, respectively, that they can built through the complex cohomology of hyperspaces are generalizable for vector fields in the same sense of the coverings of Stein and therefore of the $\bar{\partial}$-cohomology.

The following question arises, how to establish isomorphisms of cohomological classes for functions, functional, and vector fields inside the holomorphic context possible? How to determine a cohomological theory of integral operators that establish equivalences among these objects and the geometric objects of closed submanifolds, bundles of lines, and Feynman diagrams? How everything can decrease to a single cohomology of general integrals on contours or a cohomology of generalized functionals?

Before giving an answer to the previous questions, we give some preliminary definitions that we will use to fix concepts and outlines of the wanted general theory.

Let M, be a complex Riemannian manifold and be a sheaf of germs of holomorphic sections of a vector holomorphic sheaf.

Definition 1.1. We say that a space $H^\bullet(M, \mathfrak{J})$, is an integral cohomology (not in the sense of the set $\mathbb{Z}$, but yes in the sense of the integrals of partial differential equations) of those $\bar{\partial}$-equations, if this is a class of solutions or general integrals of these equations in M [1, 3].

Definition 1.2. An integral as generalized solution of a $\bar{\partial}$-equation is a realization of an irreducible representation of a $\bar{\partial}$-cohomology of complex closed submanifolds [2–4].

If the irreducible representations are unitary, then we have a complex L^2- cohomology or $\bar{\partial}$-cohomology with coefficients in L^2. The corresponding integral operators to their integral cohomology are those of the complex Fourier analysis, which in the complex geometrical context (geometrical analysis) could be integrals constructed through integral transforms as the Hilbert transforms and other [3].

In the case of a real reductive Lie group, the generalized integrals come to be determined by their orbital integrals. Let G, be a real form of $G^\mathbb{C}$, and P, their parabolic subgroup. The generalized integrals in G, are the integrals on open orbits of the generalized flag manifold $G^\mathbb{C}/P$. For this way, if $G = U(n, 1)$, and the generalized flag manifold is then $\mathbb{P}^n$, the open orbit is the group of positive lines $\mathbb{P}^+$, which is an $U(n, 1)-$orbit in $\mathbb{P}^n(\mathbb{C})$. The integrals are of John type [3, 5, 6]:

$$\varphi(P) = \int_{L^+ \subset \mathbb{P}^n(\mathbb{C})} f, \qquad (1)$$

The general integral in this case is given by the twistor transform [7] on the corresponding homogeneous bundle of lines, that is to say:

$$\mathcal{T} : H^1(\mathbb{P}^+, O(-n-2)) \to H^1(\mathbb{P}^{+*}, O(n-2)), \qquad (2)$$

Using the twistor transform like intertwining operator of induced tempered representations on a $\bar{\partial}$-cohomology, we have representations of $SU(2, 2)$ that are orbits of a fundamental unitary $(g, K)-$module of the electrodynamics [8]
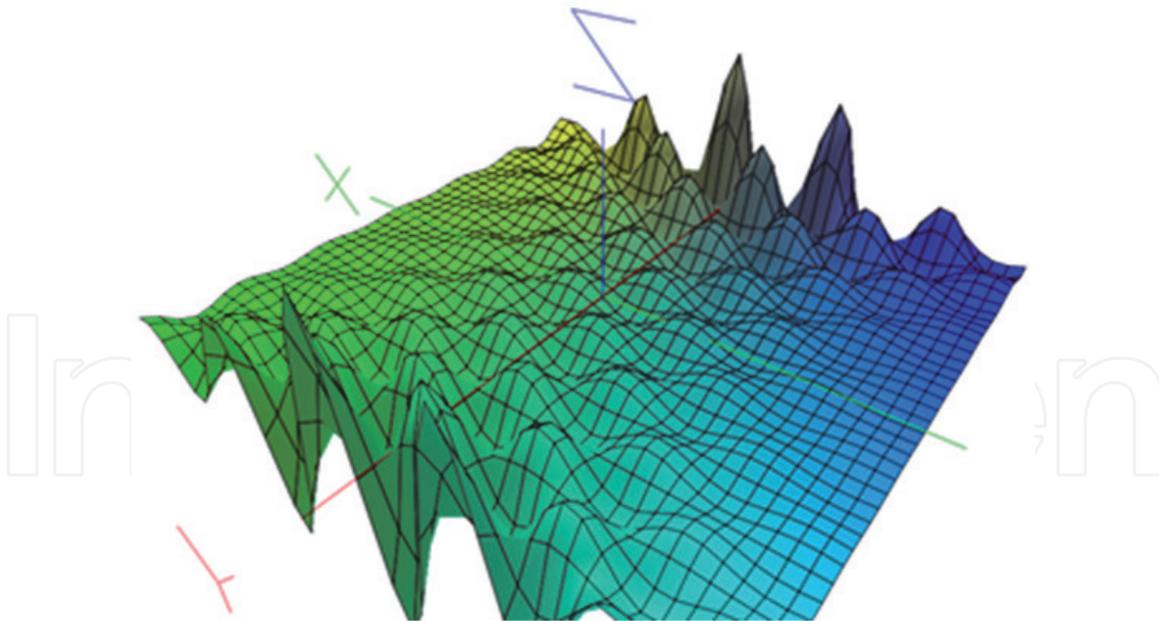
Figure 1.
Electromagnetic waves in conformal actions of the group $\mathrm{SU}(2, 2)$ *on a two-dimensional flat model of the space-time [9–11].*

(see **Figure 1**). Then, it is possible to assign a vector bundle of lines with a unitary representation, where it can be classified.

The concepts of general integral and generalized integral are different because one refers to the whole class of cohomology of solutions of those $\overline{\partial}$-equations about a complex analytic manifold, and the other refers to the classes of cohomology of solutions on cycles or cocycles of the complex Riemannian manifold [1, 2].

Another example in the recovery of a space of functions mainly the space M, is the recovery of real functions in the space $\mathbb{R}^n$, through values of certain integral operators. Such is the case of the formula to $f(x)$, recovered on $\mathbb{R}^n$,

$$f(x) = c \int_{\Gamma} \left\{ \int_{-\infty}^{\infty} f^{\wedge}(\xi, p)[p - (\xi, x)]^{-n} dp \right\} \omega(\xi) \qquad (3)$$

where the integral on p, is understood in terms of its regularization (role that carries out the Hilbert transform). The constant c depends on the dimension parity of the space $\mathbb{R}^n$, where it was carrying the tomography [6].

To answer the first question, we need a structure of complexes that induce isomorphisms in integral cohomology.

Definition 1.3. A covering of Stein is a set of manifolds of Stein[1] M_δ and Δ_z, of the corresponding fibers $X \to M$ and $X \to \Delta$, of the double fibration [1].

Let us consider the complexes given in Ref. [1], and let us consider the structure defined by a covering of Stein given by the set of open $\{M_\delta\}$, and $\{\Delta_z\}$, in the topology

$$\tau_X = \{(z, \xi) \in M \times \Xi | X \subset M \times \Xi (\Leftrightarrow M_\delta \cap \Xi_z)\}, \qquad (4)$$

---

[1] A Stein manifold is an open orbit of a semi-simple Lie group G, in a generalized flag manifold whose nilpotent radical is opposite to the parabolic subgroup P, of G [12]. A definition of the Stein manifold that uses the Hermitian structure of a complex holomorphic manifold is:

Let be $G = G_0$, a real form of $G_{\mathbb{C}}$, and $F_D = G/H$, an open orbit in the flag manifold $F = G_{\mathbb{C}}/P = G_0/U$. F_D is Stein if H is compact (or equivalently, if F_D is Hermitian symmetric). Likewise, a Stein manifold is a Hermitian symmetric flag domain.

Then, a complex in X is the space such that Ω_h^r, for any complex Ω^r, in a corresponding long succession is given as follows:

$$\{\Omega_h^r\} \subseteq \{\Omega^r\} \tag{5}$$

(i.e., to say, all the subcomplexes Ω_h^r, of the complex Ω^r). Then, the integral operator cohomology $H^{\bullet}(M, \mathfrak{J})$, in a complex manifold M, is that whose complexes conform a holomorphic structure that induces (in the corresponding integral manifold) a generalized according structure of integral submanifolds.

The integral submanifolds represent solutions of those $\bar{\partial}$-equations in cycles of M. The integral submanifolds are the corresponding cocycles of M, under the integral operators of $H^{\bullet}(M, \mathfrak{J})$.

For example, if we take the complex manifold M, like a manifold of rational curves E_z, of a twistor manifold $\mathcal{I}$ [where should understand each manifold $\mathcal{I}$, as the manifold of integral submanifolds (locally)], then its structure comes from a structure projective of their line on E_z, guided according to the vectors in T_zM.

These correspond to sections of a normal bundle $\mathcal{N}E_z$, to the curve E_z (infinitesimal deformations to the curve), that is to say, these conform the holomorphic structure that will induce the corresponding structure (that is to say in the corresponding integral manifold). In this case, the generalized structure of integral submanifolds is the $V^{(k)}$-conformal integrable structure given by $\mathcal{I}$. The integral cohomology in this case is given by the family of rational curves.

The twistor content in this case helps and is necessary to establish the deformation of the integral curves of the vector sheaf of lines O(k). In such case, the integral cohomology is $H^{\bullet}(M, \mathfrak{J}) = H^0(\mathcal{I}, O(k))$.

This example is interesting not only for the fact of the definition of the integral cohomology, which defines, for this way, a class of integrals for M, but also for the fact of satisfaction of the integrability condition for the equation of the tensor of Weyl $W_{ij} = 0$, where $H^0(\mathcal{I}^+, O(k))$ (respectively, $H^0(\mathcal{I}, O(k))$) are the solutions or integral of $W^+ = 0$ (respectively, $W^- = 0$ [13, 14].

2. Duality: line bundles with cohomological contours and closed submanifolds with cohomological functional

We consider the following result on integral cohomology for integral geometry.

Proposition 2.1. In the integral cohomology $H^{\bullet}(M, \mathfrak{J})$, on complex manifolds, the following statements are equivalent:

a. The open M_δ, and Δ_z, are $G-$orbits opened up in X, and their integrals are generalized integral for M,

b. Exists an integral operator T, such that $H^{\bullet}(M, \mathfrak{J}) \cong_T \ker\{D\text{-equations}\}$,

c. $M_\delta = \cup_M \underline{\pi}/\underline{z}$, and $\Delta_z = \cup_\Delta \underline{z}/\underline{\pi}$, where $H^{\bullet}(M, \mathfrak{J}) = H^{n-1}(U, \rho^{-1}O(V))$.

Proof. The integrals on the open $G-$ orbits satisfy the $G-$invariant integration:

$$\int_{G/H} f d\Phi(\phi^r) = \int_{G/H} (f \circ \Phi^{-1})\phi^r, \tag{6}$$

For the theorem of Buchdall Eastwood [12], we have that the orbits generalized in X, give us a new cohomological class that is related to the previous for an integral operator T, defined for

$$H^\bullet(M, \mathfrak{J}(\nu)) \to H^\bullet(\Xi, \tau^{-1}\mathfrak{J}(\nu)), \tag{7}$$

and such that $H^\bullet(\Xi, \tau^{-1}\mathfrak{J}(\nu)) \cong_T \ker\{D\text{-equations } G/H\}$, has more than enough. By the theorem II.1 [1][2], the G−orbits are K−orbits in X. Then (i) == > (ii). Now for the theorem II.2 [1],[3] we have that each canonical fibration of a flag manifold will give a G−orbit in Z, for some internal symmetrical G−space M.

In particular, $\ker\{D\text{-equations}\}$, takes place the correspondence with the cycles of $H^\bullet(M, \mathfrak{J})$. In fact, $\ker\{D\text{-equations}\}$, is similar to a compact number of components on which G, acts transitively, and these belong together to the cocycles of $H^\bullet(\Xi, \tau^{-1}\mathfrak{J}(\nu))$.

But $\ker\{D\text{-equations}\}$ only exists as integral of those $\bar{\partial}$-equations in M (with M, integrable) if $|R^M_{\mathcal{I}}(j) = 0$. This $g_{\mathbb{C}}$ establishes a generalized structure of M, which underlies in its composition (in fact $[m^{+,}, m^+] \subset m$, for integrability). Then, $\forall z \in M$ and $\sigma \in \Sigma\left(\cong \sum_i V_z\right)$ exist $z \in F$, such that $T_z F = \sigma T_w F \in \sum_i(V_w), \forall w \in F$. Then, $M = \cup_{\sigma \in \Sigma} V_\sigma$, and $\Xi = \cup_{\gamma \in \Sigma} V_\gamma$, then for n-dimensional planes of a Grassmann manifold, $G_{1,n}$ had that $M_\delta = \cup_M \underline{\pi}/\underline{z}$, and $\Delta_z = \cup_\Delta \underline{z}/\underline{\pi}$, which defines cycles in $H^{n-1}(U, \rho^{-1} O(V))$, with $U \subset M$. Then (ii) implies (iii).

However, fixed G exists alone a finite number of flag manifolds of certain biholomorphism of this type. These are in bijective correspondence with the conjugated classes of parabolic subalgebras of $g_{\mathbb{C}}$, and each flag manifold admits a finite number of canonical fibrations.

Then, $\ker\{D\text{-equations}\}$, is made of a finite number of G−orbits, all which are closed and (iii) == > (ii). Then since each one of these G−orbits exists like an K−orbit of the space of classes G/K, with Nijenhuis null curvature tensor, then each flag submanifold is an K−orbit of the vector holomorphic G−bundle of the 2n− dimensional irreducible symmetrical Riemannian manifold J(M).

Its integrals are orbital, and their extensions to M_δ, and Δ_z, are generalized integrals (since they are integral of line along the fibers of M_δ, and Δ_z, respectively) for which it is continued. Then, (ii) implies (i).◆.

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[2] **Theorem.** The K− invariance given for the G−structure $S_G(M)$, of complex holomorphic M, is induced to each closed submanifold given for the flag manifolds of the corresponding vector holomorphic G−bundle. Furthermore, the given integral cohomology on such complex submanifolds is equivalent to the integral cohomology on submanifolds of a maximum complex torus.

[3] **Theorem.** Let be M = G/K, an internal symmetric simply connected Riemannian manifold and of compact type. Then

$$Z = \{R^M_{\mathcal{I}}(j) \in \text{End}(\wedge^2 T * (M) \otimes_{\mathbb{R}} E) | R^M_{\mathcal{I}}(j) = 0\},$$

It consists of a finite number of connected components on each a G, that acts transitively. Further, any flag G−manifold is realized as a such orbit for some M.

The requirement of the transitive action of G, on the orbits is, for example, indispensable to the spatial isotropy hypothesis in the constructions of an integral cohomology to the space–time curvature, since the curvature integrals must be determined on G−invariant orbits and will be calculated for reduction of the corresponding holonomy group on K− invariant orbits.

Proposition 2.2. The $(n-1) - \bar{\partial} -$ complex cohomology with coefficients in a complex holomorphic bundle of M, is a cohomology of hyperlines and hyperplanes[4].

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[4] We can have a little digression with certain details on the complex Radon transform using submanifolds in the space $\mathbb{CP}^1$, to the $\bar{\partial}-$ cohomology. Let be M, a complex holomorphic manifold (or complex Riemannian manifold [15]). We consider its corresponding reductive homogeneous space determined for the flag manifold $F = G_{\mathbb{C}}/P$, with P, a parabolic subgroup of $G_{\mathbb{C}}$. We consider the open orbit given for the Stein manifold $F_D = G/H$ (as was defined in the footnote 1) with H, a compact subgroup of the real form G_0, of $G_{\mathbb{C}}$.

Let be $M \cong \mathbb{C}^n$, and we consider concave linearly domains D, in $\mathbb{C}^n$ (or so better in $\mathbb{PC}^n$ [16]). D, has structure of complex vector space. Let be $D_1 = \xi^n/\overline{D}$, a holomorphic convex linear domain conformed for holomorphic hyperplanes $\pi_i(D)$, $i = 1, 2, 3, \ldots$, in D_1. Let be $H^1(D)$, the complex holomorphic functions space defined on D_1. Let $D \to M$, be a fiber vector bundle seated in the complex holomorphic manifold M. Let $A^{p,q}(D)$, be the (p, q)-forms space on M, with values in D (that is to say, the global sections space of the fiber tangent bundle $\wedge^{p,q}T*(M) \otimes D$).

Of this way, the bi-graded algebra is the space

$$\mathcal{A}(D) = \oplus_{n+m=p} A^{n,m}(D),$$

Let $\bar{\partial}$-be the scalar operator on the complex manifold M, with values on the vector bundle of global sections $\mathcal{E}(\wedge^{p,q}T*(M) \otimes D)$, that is to say, the differential operator

$$\bar{\partial} : \mathcal{E}(\wedge^{p,q}T*(M) \otimes D) \to \mathcal{E}(\wedge^{p,q+1}T*(M) \otimes D),$$

The operator $\bar{\partial}$, complies certain anticommutative properties [1]. Now, we consider the Radon transform on the complex holomorphic functions H(D), and its analogous for the complex holomorphic functionals $H(D*)$, through the corresponding diagram:

$$D \xrightarrow{f} H(D) \xrightarrow{R_{\bar{\partial}}} L(D)$$
$$\downarrow \text{Functional} \downarrow \text{Functional} \downarrow \text{Functional} \quad,$$
$$D* \xrightarrow{\varphi} H(D*) \xrightarrow{R_{\bar{\partial}}*} L(D*)$$

and we consider an $\bar{\partial}$-operator in D. The before diagram represents a first cohomological advance on the relation between functionals of the $(n-1)-$dimensional $\bar{\partial}$- cohomology with coefficients in a complex vector bundle Ω^n (holomorphic complex bundle) and the integration of the cohomology on hyperplanes in D, which an integral geometry is equivalent to consider an adequate Radon transform in D. Likewise, and using the satellites $\bar{\partial} R$, and $\bar{\partial} R^{-1}$, of the before diagram and composing the diagram with the correspondences to R, on D, and $D*$, we can have:

$$\bar{\partial} R(f) = R\bar{\partial}(f),$$

The details of the demonstration of this identity can be seen in Ref. [1].

Due to that R, is injective, this is equivalent to have the exact succession of Radon transform images:

$$0 \to R_0 A^{0,0}(L(D)'') \to RA^{0,1}(H(D)'') \to R*A^{0,1}(L(D)') \to 0,$$

or in equivalent way, for the exact succession:

$$0 \to R_0^{-1} B^{0,0}(L(D)'') \to R^{-1}B^{0,1}(H(D)'') \to R*^{-1}B^{0,1}(L(D)') \to 0,$$

Here the $''$ and $'$ denote the projectivizing of spaces for R. Then the Radon transform can be generalized to the $\bar{\partial}$-cohomology classes on the complex spaces D, respectively D_1, in $\mathbb{C}^n$, as the mapping:

$R_{\bar{\partial}} : \bar{\partial}$-cohomology of dimension $n \to \bar{\partial}$-cohomology of dimension (n-1),

whose restriction to a domain D, is the mapping:

$$R_{\bar{\partial}}|_D : H(D) \to L(D),$$

which satisfies the diagram of correspondences for functionals of a (n-1)-dimensional cohomology with coefficients in a holomorphic vector bundle $\mathcal{E} \to M$.

The natural question arises, what relation there is between the two different corresponding objects to functionals, that is to say, cohomology and functions?

The relation is an integral relation of certain $\bar{\partial}$-cohomology, which comes defined for the Radon transform of the Dolbeault cohomology $R_{\bar{\partial}}$.

The Radon transform can be viewed as the mapping of cohomological spaces:

$$R_{\bar{\partial}} : H^{p,n}(D, V) \to H^{p,n-1}(D, V),$$

Therefore, it is enough to demonstrate that $R_{\bar{\partial}}(H^{p,n}(D, V))$, is the q*th*-projection (n-1)-dimensional of $H^{p,n}(D, V)$, in $H^{0,q}(D, V) = H^{0,n}(D, V)$. Remember that the Radon transform in the complex context D, is the continuous and analytic mapping [17]:

Their demonstration is a simple consequence of the digression in part II of Ref. [1], on some basic integral $\bar{\partial}$−cohomologies on n−dimensional complex spaces (see **Figure 2**), of this same philosophical dissertation of integral operators published in 2007.

Proposition 2.3. The integrals of contour are generalized function in a cohomology of contours (cohomological functional).

We define the following concept.

Definition 2.1. Cohomological function of a cohomology $H^{\bullet}(M - SingM, \Omega^r)$,[5] is an integral cohomology of the form $H^{\bullet}(\mathcal{M} - Sing\,M, \mathbb{C})$, where $\mathcal{M}$, can be understood as a twistor space corresponding to M. (see **Figure 3**).

For example, this class belongs to the Feynman integrals.

We consider p and differential q forms of the cohomologies on the complex manifolds X, and Y, respectively, $\alpha \in H^p(X, S)$, and $\beta \in H^q(Y, S)$.

We consider their cup product given for $\alpha \cup \beta \in H^{p+q}(X \cap Y, S \otimes T)$, and the connecting map in the succession of Mayer-Vietoris:

$$\partial* : H^{p+q}(X \cap Y, S \otimes T) \to H^{p+q+1}(X \cup Y, S \otimes T), \tag{8}$$

$$R : H(D) \to L(D),$$

with rule of correspondence for a complex coordinates system $z_1, z_2, \ldots, z_n$,

$$f^{\wedge}(\zeta_1, \zeta_2, \ldots, \zeta_n) = \left(\frac{i}{2}\right)^{n-1} \int_D f(z_1, z_2, \ldots, z_n)\delta(s-[(\zeta_1, \zeta_2, \ldots, \zeta_n), (z_1, z_2, \ldots, z_n)]) \bullet$$

$$< dz_1, dz_2, \ldots, dz_n, d\bar{z}_1, d\bar{z}_2, \ldots, d\bar{z}_n),$$

$\forall f \in H(D)$.

We consider $\bar{\partial}$, complex scalar mapping defined to Dolbeault cohomology. Let $D* = \mathcal{L}(H(D), \mathbb{C})$, be the set of hyperplanes corresponding to D. Let

$$ev_f : \mathcal{L}(H(D), \mathbb{C}) \to \mathbb{C},$$

the evaluation of $f \in H(D)$, in the complex hyperplane $\pi(z)$, of D, with rule of correspondence:

$$\pi(z)(f) = <\pi(z), f>,$$

Due to that $\bar{\partial}R(f) = R\bar{\partial}(f), \forall f^{\wedge} \in L(D)$, we have that:

$$\bar{\partial}\left(f^{\wedge}(z)\right) = \bar{\partial}(f^{\wedge}) \otimes z + f^{\wedge}\bar{\partial}(z) = \bar{\partial}(z) \otimes R(f) \in A^{p,q-1}(V),$$

where in particular the exterior algebra $\wedge^{0,q}(T*(M) \otimes V_{\mathbb{C}})$, is generated for elements $\pi(z) \wedge \bar{\partial}R(f)(z))$. Then

$$R_{\bar{\partial}}(f) \otimes ev_f = <\pi(z)f, \bar{\partial}(RF)>,$$

Therefore p = 0. Then $R_{\bar{\partial}}(f) \otimes ev_f \in A^{0,q}(V)$. Then $H^{0,q}(\mathbb{C}^n/D, V) = H^{0,q}(D_1, V)$, which is a Dolbeault cohomology. Of this form, it can be established that through the hyperspaces geometry can be determined a certain class of holomorphic complex functions using an appropriate $\bar{\partial}$- cohomology. Also Gindikin generalizes this idea using the (n-1-q)-$\bar{\partial}$-cohomology determining the Radon transform on hyperplanes defined as linearly concave domains of dimension q, first to the real case and after to the complex case [16]. For example, a good modern research to the respect is the followed residual $\bar{\partial}$-cohomology and the complex Radon transform on subvarieties of $\mathbb{CP}^1$ [16].

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[5] Here, $\mathcal{M}$, is the product of all the twistor spaces and Sing M, is the union of all the subspaces on which the $(Z^{\alpha}W_{\alpha})^{-1}$, and $(Z^{\alpha}A_{\alpha})^{-1}$, are singular factors. Its differential form is integrated over a contour, which can be traditional contour, for example with cohomology space $H_{3v}(\mathcal{M} - Sing\,M, \mathbb{C})$.
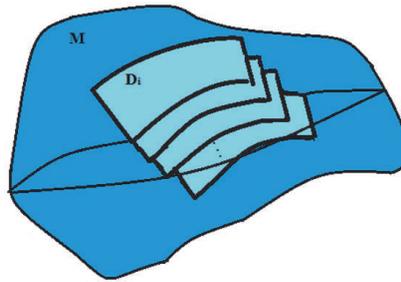
Figure 2.
Convex domains conformed for holomorphic hyperplanes $\pi_i(D)$.
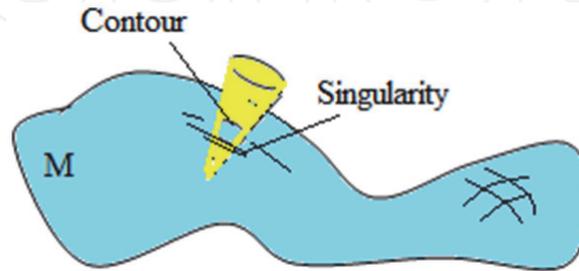
Figure 3.
Cohomology of contours isomorphic to $H^\bullet(M - \text{Sing } M, \Omega^r)$.

We consider for the inner product of α, and β, the relation is

$$\alpha \bullet \beta = \partial * (\alpha \cup \beta), \tag{9}$$

This description of the inner product has been used in a new development of the cohomology for twistor diagrams foreseen in Refs. [14, 18]. This new method is almost opposed to the procedure that we want to use in the unification of contour integrals on diagrams, in respect of the Feynman integral, although also proper to the Conway integrals, Cauchy integrals,[6] and some integral transforms as the Hilbert transforms.

We want to assemble a Feynman diagram for applications of the product "cup." The interior edges of a Feynman diagram are taken again as elements of groups H^0 (such extra elements have to be abandoned in a cohomology, for example, $H^\bullet(M, \tau^{-1} \mathfrak{J}(\nu))$), and the interior edges form the fields (assuming that they are elementary states) in several cohomology groups H^1.

Let denote $\mathcal{M}$, for Π, and sing $M = \ell$. Likewise, if f, is one of these elements of H^1, this new procedure determines an element of the cohomology $H^f(\Pi\text{-}\ell', \Omega^d)$, where ℓ', is the union of all the subspaces defined by internal edges, always with the subspaces $\mathbb{CP}^1$, on whose elementary states f, are singular.

Then for Proposition 2.1 (b), the following mapping exists

$$H^f(\Pi\text{-}\ell', \Omega^d) \to H^{f+d}(\Pi\text{-}\ell', \mathbb{C}), \tag{10}$$

_____

[6] For example, to this case for holomorphic functions, we have the generalized Cauchy formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z\text{-}a)^{n+1}} dz.$$

using the description of Dolbeault of the first group, forgetting the bi-graduation (d, f), (d, f) and reminding only the total grade $d + f$. A description of Cĕch of this mapping is used for the evaluation of twistor cohomology. In our case, we will only use the duality of Poincaré to know in what moment of the evaluation of an element of $H^{f+d}(\Pi\text{-}\ell', \mathbb{C})$, one can need a contour in $H_{f+d}(\Pi\text{-}\ell', \Omega^d)$. This can define in a more general sense the cohomological functional. Likewise, the mapping Eq. (10) is an example of the cohomological functional.

This contour "cohomologic" is easy to relate it with a traditional in $H_d(\Pi\text{-}\ell, \mathbb{C})$, due to that the following mapping exists

$$H_{f+d}(\Pi\text{-}\ell', \Omega^d) \to H_d(\Pi\text{-}\ell', \mathbb{C}), \tag{11}$$

given for iteration of the constant mapping of Mayer-Vietoris (in homology) f-times; one for each field.

For example, for diagram, product can be demonstrated that $H_8(\Pi\text{-}\ell, \mathbb{C})$, and that the image of the generator of this group low two mappings of Mayer-Vietoris as is the usual in the physical contour for the product of diagrams given. This affirms that only exists a cohomological contour for the product climb (as is expected) and suggests a method for contours that verifies and observes which belong to cohomologics.

Definition 2.2. (Hyperfunction). A hyperfunction on $\mathbb{R}^n$, is an element of the $(n-1)\text{-}\overline{\partial}$ — cohomology $H^{(n-1)}(M, \mathfrak{J})$, with $M = \mathbb{C}^n/\mathbb{R}^n$.

Proposition 2.4. The general integrals of line are functional on arches γ, in geometry of conformal generalized structure.

Proof: Consider a vector holomorphic G-invariant sheaf and their corresponding bundle of lines associated with those $(r, 0)$-forms on the corresponding topological vector space. Then, the integrals on the fibers of the vector holomorphic sheaf are the integrals of line on the cycles of the sections X, of the vector sheaf, given by $\int_\gamma X \bullet \delta$, $\forall$ $\delta \in \Omega^r$ [where Ω^r is a complex defined in Eq. (5)]. Then the holomorphic structure that constitutes these complexes induces (in the corresponding integral manifold) a conformal generalized structure of integral submanifolds where the arches γ, are local parts of integral curves of the fibers of the vector sheaf of lines. In other words, $\forall \gamma \in \sum_i(V_z)$ exists locally an integral submanifold S, with $z \in S$, such that $T_z S = \gamma$, and $T_w S \in \sum_i(V_w)$, $\forall w \in S$. Then the integral of line can be re-written in this conformal generalized structure as

$$\int_\gamma X \bullet \delta = \int_{T_z S} f \bullet \delta, \quad \forall \quad \delta \in \Omega^r, f \in \mathcal{T}, \tag{12}$$

where $\mathcal{T}$, is the tube domain (in the local structure where the integral submanifold S, exists) $\mathcal{T} = \mathbb{R}^n + iV$, where V, is a cone, not necessarily convex (that has applicability on the fibers of the sheaf of lines). The idea is to define the expression $f \bullet \delta$, inside the context of the integral of line in such case that the values of f, on the arch γ, are values off, a hyperfunction represented this like a variation of holomorphic functions $f(z|\delta)$, in a submanifold of Stein M_δ, such that $M_\delta \supset \mathcal{T}$.

Then, the sesquilinear coupling of the hyperfunction corresponding to f, and the function f itself, is an integral of contour, and for Proposition 2.3, a generalized functional in the cohomology $H^1(\Pi\text{-}\ell, \mathbb{C})$. Indeed, let be $\mathcal{T} = \mathbb{R}^n + iV$, the tube domain where the cone V, is not necessarily convex. This cone $V = \cup_{\gamma \in \Sigma} V_\gamma$, in the conformal generalized structure where the V_γ, are the convex maximal sub-cones in V. Considers our manifold, complex Riemannian manifold. The idea is that a holomorphic form required in this language is a good expression to write the

integral of complex vector fields as an integral of line through more than enough bundles of hyperlines and hyperplanes. As for example, we have more than enough hyperlines and hyperplanes, respectively, in $\mathbb{R}\mathbb{P}^n$, and $\mathbb{C}^n$, visualizing these fields like holomorphic sections of complex holomorphic bundles of fibers $X \to M$. In Δ, exists q-dimensional cycles such that $V = \cup_{\delta \in \gamma} V_\delta$. Let be $\mathcal{T}_\delta = \mathbb{R}^n + iV_\delta$, with covering of Stein $\mathcal{T} = \cup_{\delta \in \gamma} \mathcal{T}_\delta$. Let us consider the vector cohomology $H^{(q)}(\mathcal{T}, \mathfrak{J})$, using this covering. Then for proposition 2. 1, incise b), a canonical operator exists (of values frontier for f) defined for

$$H^{(q)}(\mathcal{T}, \mathfrak{J}) \to H^{(q)}(\mathbb{C}^n/\mathbb{R}^n, \mathfrak{J}), \tag{13}$$

Then, the integral can be expressed on spaces M_δ and Δ_z, which are affined to lines and hyperplanes $\mathbb{C}\mathbb{P}^n$ and $\mathbb{C}^n$ and that such are orbital integrals of the complex manifolds $M = G/L$ and $\Delta = \Gamma/\Sigma$, belonging to a $\bar{\partial}$-cohomology in holomorphic language.

In particular, if $f(z|\delta, d|\delta) \in \Omega_{\mathcal{T}}^q$ has regular values $\forall z \in \mathbb{R}^n$, then

$$\varphi(\mathbf{x}) = \int_\gamma \varphi(\mathbf{x}|\delta, d|\delta), \quad \forall \quad \mathbf{x} \in \mathbb{R}^n. \tag{14}$$

Then, in the integral submanifold M_δ, said integrals take the form

$$\int_\gamma X \bullet G = \int_\gamma \varphi(z|\delta, d|\delta) f(z) = \int_\gamma f(z|\delta). \tag{15}$$

However, these integrals are integral of contour belonging to a cohomology $H^1(\Pi\text{-}\ell, \mathbb{C})$ of cohomological functional. Then, the integral $\int_\gamma f(z|\delta)$ is a functional inside the integral cohomology $H^{(n-1)}(\mathbb{C}^n/\mathbb{R}^n, \mathfrak{J})$ (**Figure 4**).

The previous Propositions 2.3 and 2.4 establish that the structure of complexes for the integral operator cohomology does suitable to induce isomorfisms in other object classes of the manifold M, doing arise the question to some procedure that exists inside the relative cohomology on $\mathfrak{J}$: can we induce isomorfisms of integral cohomologies?

(a)   (b)
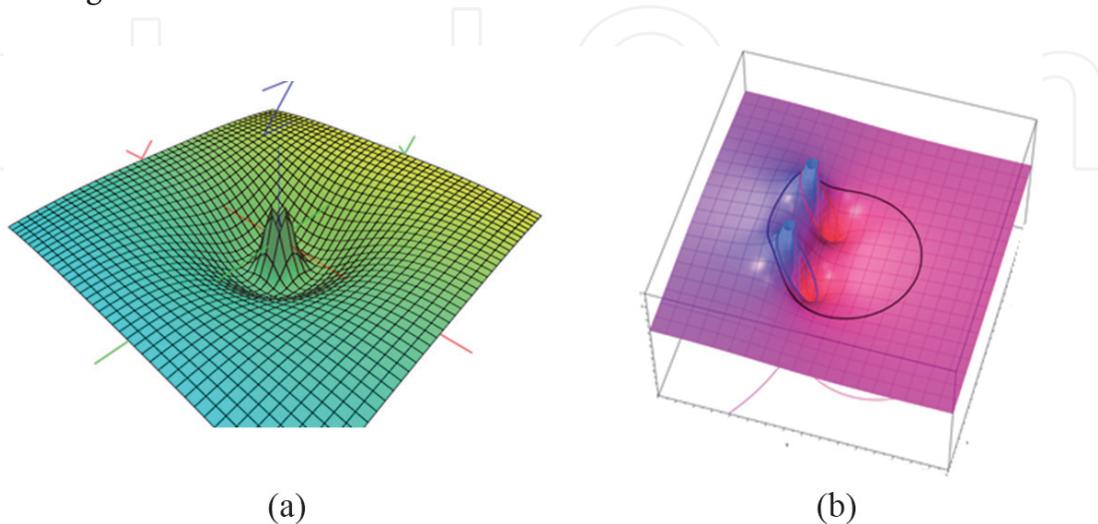
Figure 4.
(A). One state or source of a field. Its contour is well defined by only one Cauchy integral. (B). Two states or sources of a field. This represents the surface of the real part of the function $g(z) = \frac{z^2}{z^2 + 2z + 2}$. The moduli of these points are less than 2 and thus lie inside one contour. Likewise, the contour integral can be split into two smaller integrals using the Cauchy-Goursat theorem having finally the contour integral [19].

Now, we consider a closed subset (or relatively closed) F, of a space X, and a sheaf $\mathfrak{J}$, on X. In a way more than enough, we choose an open covering Y, of X, with a subcovering Y', of X/F.

A relative co-chain of Cěch is a co-chain of Cěch with regard to the covering Y, subject to the condition of annulling when we restrict to the subcovering Y'. Then, it had the exact succession of relative co-chains groups:

$$0 \to C_F^p(X, \mathfrak{J}) \to C^p(X, \mathfrak{J}) \to C_F^p(X/F, \mathfrak{J}) \to 0, \tag{16}$$

where $C_F^p(X, \mathfrak{J})$ is the group of relative co-chains. The inherent relative co-chain to a co-opposite operator of the ordinary co-chains and the limit on fine coverings of the homology of $C_F^*(X, \mathfrak{J})$ give the groups of relative cohomology $H_F^p(X, \mathfrak{J})$.

$$\oint_C g(z)dz = \oint_C \left(1 - \frac{1}{z-z_1} - \frac{1}{z - z_2}\right) dz = 0\text{-}2\pi i\text{-}2\pi i = -4\pi i.$$

This is a good example of traditional cohomological functional element of $H^f(\Pi\text{-}\ell', \Omega^r) = \mathbb{C}$.

In this case is not necessary to take the limit since ahead of time one has the relative theorem of Leray, which establish that if $H^p(U, \mathfrak{J}) = 0, p \geq 1$, for each set U, in the covering Y, then this covers enough to calculate the relative cohomology. The exact long succession cohomology of the exact short succession defined in Eq. (16) determines the exact succession of relative cohomology

$$0 \to H_F^0(X, \mathfrak{J}) \to H^0(X, \mathfrak{J}) \to C^0(X/F, \mathfrak{J}) \to H^1(X, \mathfrak{J}) \to \dots, \tag{17}$$

where the mappings of the cohomology on X, to the given on X/F, are restrictions.

Other important result on the relative cohomology is the split theorem, which establishes in shallow terms that the relative cohomology depends only on the immediate neighborhoods of the embedding of F, in X. With more precision, giving an open subset X, such that $(X/X') \cap F = \varnothing$, a canonical isomorfism exists if $H_F^n(X, \mathfrak{J}) = H_F^n(X', \mathfrak{J})$.

This is the form to induce isomorfisms. In our case, the covering Y, is a covering of Stein where the integral operator cohomology $H^\bullet(M, \mathfrak{J})$, should exist such as we wish. Why? Because the natural place, where a $\bar{\partial}$−cohomology exists, is in a covering of Stein and is because we want to obtain the solutions of $\bar{\partial}$−partial differential equations.

We apply the relative cohomology to cohomologies of contours because we want generalized function as solutions of the differential equations [5, 18].

We consider the following general procedure due to Baston [8] for the exhibition of all the cohomological functional on a collection of fields given. This procedure is required for the evaluation of boxes diagram, that is to say, the obtaining of the elementary states $\varphi^i(i = 1, 2, 3, 4, \dots)$ of the field through a local cohomology.

We consider a complex manifold given for $X \cup Y$, the closed subsets $F \subset X$, and $G \subset Y$, and elements $\alpha \in H^p(X\text{-}F, S)$, and $\beta \in H^q(Y\text{-}G, T)$. Then we can use the connecting mappings in the exact successions of relative cohomology

$$H^p(X, S) \to H^p(X\text{-}F, S) \xrightarrow{r} H_F^{p+1}(X, S) \to H^{p+1}(X, S), \tag{18}$$

and

$$H^q(Y, T) \to H^q(Y\text{-}G, T) \xrightarrow{r} H_G^{q+1}(Y, T) \to H^{q+1}(Y, T), \tag{19}$$

to obtain elements rα, and rβ. Then, the cup product on relative cohomology is defined as:

$$\cup : H^{p+q+1}(X \cup Y, S \otimes T) \xrightarrow{r} H^{p+q+1}(X \cup Y\text{-}F \cap G, S \otimes T)$$

$$\to H^{p+q+2}_{F \cap G}(X \cup Y, S \otimes T) \to H^{p+q+2}(X \cup Y, S \otimes T), \tag{20}$$

and this demonstrates that

$$\alpha \bullet \beta = r^{-1}(r\alpha \cup r\beta), \tag{21}$$

Due to that in the diagram boxes, the interactive vector fields φ, are given as elements of groups H^1, defined on different spaces, we need the vector product in relative cohomology:

$$\times : H^{p+1}_F(X, S) \otimes H^{q+1}_G(Y, T) \to H^{p+q+2}_{F \times G}(X \times Y, S \otimes T), \tag{22}$$

Likewise, for diagram box of four states, we have the cohomology of the left side of Eq. (22) that can be illustrated (**Figure 5**).

Strictly speaking $S \otimes T$, could be $\pi *_X S \otimes \pi *_Y T$. As before $r\alpha \times r\beta$, is in the image of the connecting mapping r, in:

$$H^{p+q+1}(X \cup Y\text{-}F \cap G, S \otimes T) \xrightarrow{r} H^{p+q+2}_{F \times G}(X \times Y, S \otimes T) \to H^{p+q+2}(X \times Y, S \otimes T), \tag{23}$$

with

$$\alpha \bullet \beta = r^{-1}(r\alpha \times r\beta)\left(\in \nu^{-1}O(p, q, r)\right), \tag{24}$$

The following technical question arises: how to relate contour cohomology as $H^{f+d}(\Pi\text{-}\ell', \mathbb{C})$, with an integral cohomology of vector fields?

Part of the replay to this question is found when are considered the complex components $F_i = P_i - U_i$, with $i = 1, 2, 3, 4, \ldots, f$; being $P_i, P, P*$, and U_i, open subsets of $\mathbb{P}_i$, belonging to the correct cohomology to the Penrose transform on $H^1(U, O(-\mathbf{r}))$.

The idea is to obtain an image of the vector field as element of a cohomology on homogeneous bundles of lines in each component of the field (that is to say, determine a cohomology for each line integral of each field component). Before-hand this is foreseen that will happen with the Penrose transform, which is an integral transform on the homogeneous bundles of lines.

(a)   (b)
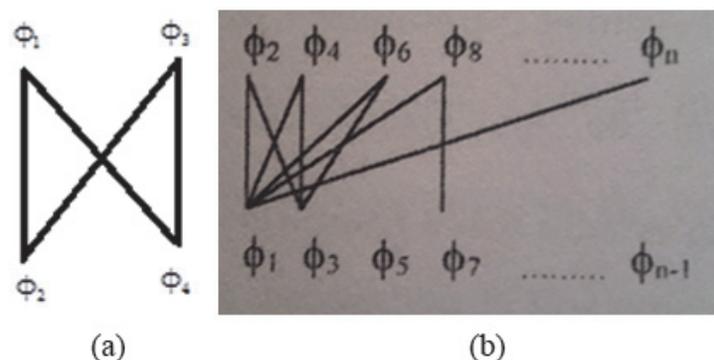
Figure 5.
Feynman boxes diagrams [1].

Let $F = F_1 \times \ldots \times F_f$. We denote for L_i, a projective line included in F_i, and let $L = L_1 \times \ldots \times L_f$. For f vector fields, we have an element in the q cohomological group $H^1(U_1, O(-r_1)) \otimes \cdots \otimes H^1(U_f, O(-r_f))$. For relative cohomology and projective twistor diagram results [18], the inner product for the line integrals for all these fields is not lost. Then for the Künneth formula to relative cohomology, we have:

$$H^1(U_1, O(-r_1)) \otimes \cdots \otimes H^1(U_f, O(-r_f)) \cong H_F^{2f}(\Pi, O(\mathbf{r})), \qquad (25)$$

where $\mathbf{r} = r_1 \times \ldots \times r_f$. Each linear continuous functional on these fields is thus an element of the compact relative cohomology group $H_F^{2f}(\Pi, \Pi\text{-}F, O(\mathbf{r}))$. We must establish that Eq. (25) and the group $H_F^{2f}(\Pi, \Pi\text{-}F, O(\mathbf{r}))$, are not in general dual.

Now well, considering this cohomology of vector fields, is necessary to decide how the interior of a diagram choose some of these functionals. We remember the interior of a diagram as the holomorphic nucleus $h \in H^{3f,q}(\Pi\text{-}\ell, O(\mathbf{r}))$. For example, in the scalar product (spin zero) $h = \frac{DW \wedge DZ}{(W_\alpha Z^\alpha)^2} \in H^{6,q}(\Pi\text{-}\ell, O(\text{-}2\text{-}2))$. While in the box $h = \frac{DW \wedge DZ \wedge DX \wedge DY}{(W_\alpha Z^\alpha W_\beta Y^\beta Z_\gamma X^\gamma Y_\delta Z^\delta)^2} \in H^{6,q}(\Pi\text{-}\ell_0, O(\text{-}2\text{-}2\text{-}2\text{-}2))$. Usually $q = 0$. In these cases h, can be determined for integration without the interior vertices of the twistor diagram, although it is not always easy. If $q \neq 0$, the determination of h in none time is clear. How to do about it?

We consider the complex cohomology, and also we consider an element $\alpha \in H_{\mathbb{C}}^{0,f\text{-}q}(\Pi - \ell, \Pi\text{-}\ell \cup F)$. Then $\alpha \cup h \in H_{\mathbb{C}}^{3f,f}(\Pi - \ell, \Pi\text{-}\ell \cup F; O(\text{-}\mathbf{r}))$. This is an induced mapping for the inclusion

$$i : H_{\mathbb{C}}^{3f,f}(\Pi - \ell, \Pi\text{-}\ell \cup F; O(\text{-}\mathbf{r})) \rightarrow H_{\mathbb{C}}^{3f,f}(\Pi, \Pi - F; O(\text{-}\mathbf{r})), \qquad (26)$$

where such $i(\alpha \cup h)$, is a chosen functional for the interior of the diagram (that is to say h) as required. However, as this was done through α, the results are hard to view α as a contour. For it, we first note that the embedding of the constant sheaf $\mathbb{C}$, in $O(\text{-}\mathbf{r})$, induces a mapping:

$$H_{\mathbb{C}}^{f\text{-}q}(\Pi - \ell, \Pi\text{-}\ell \cup F; \mathbb{R}) \rightarrow H_{\mathbb{C}}^{f\text{-}q}(\Pi - \ell, \Pi - \ell \cup F; O(\text{-}\mathbf{r})), \qquad (27)$$

and second, the cohomology groups $\alpha \in H_{5f+q}(\Pi - \ell, \Pi\text{-}\ell \cup F)$ and $H_{\mathbb{C}}^{f\text{-}q}(\Pi - \ell, \Pi\text{-}\ell \cup F; \mathbb{R})$ are isomorphic. Now, it is necessary to insist in that α is in the image of the mapping Eq. (27), which will produce a viewing as contour. Being α a contour, we call to $i(\alpha \cup h)$, the functional "associated with" the kernel h, and we remark strongly that this not exists if $F \subset \ell$, then $H_{5f+q}(\Pi - \ell, \Pi\text{-}\ell \cup F) = 0$, which is hoped. We can refer to this problem as impossible, since necessarily $\ell \neq F$, for the chosen fields in this cohomology, which are the most general possible. The idea is to wobtain an image of the vector field as an element of a cohomology on homogeneous bundles of lines in each component of the field. We note that our defined fields are generally perfect. In fact, if the vector fields are elemental states, then $F_i = L_i$, and F, is equal to a closed submanifold Λ (of real codimension 4f, with normal orientable bundle). Using the Thom isomorphism, we have:

$$H_{f+q}(\Lambda - \ell) \xrightarrow{\cong} H_{5f+q}(\Pi - \ell, \Pi\text{-}\ell \cup \Lambda) = 0, \qquad (28)$$

which is deduced that the viewed contours are given in $H_{f+q}(\Lambda - \ell)$. If the vector fields are not elemental states along $(\Pi - \ell, \Pi\text{-}\ell \cup F)$, then $(\Pi - \ell, \Pi\text{-}\ell \cup F)$, is homotopic to $(\Pi - \ell, \Pi\text{-}\ell \cup \Lambda)$, which establishes its generality in homology.
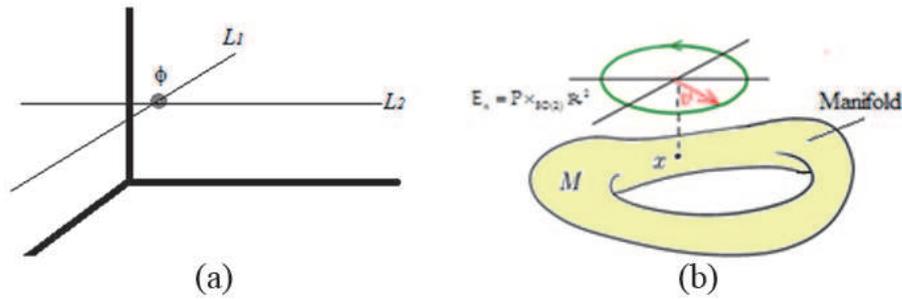
Figure 6.
(a). Field state in a cohomology $H^{3,0}(\mathbb{R}^3, \mathcal{L})$, *to the line bundle* $\mathcal{L}$. *(b). P, is a principal* $SO(2)-$*bundle.*
The fiber of the fiber bundle is the points $E_x = P \times_{SO(2)} \mathbb{R}^2$.

Likewise we have demonstrated that if$(\Pi - \ell, \Pi\text{-}\ell \cup F)$, is homotopic to
$(\Pi - \ell, \Pi\text{-}\ell \cup \Lambda)$, then the functionals on $H^1(U_1, O(-r_1)) \otimes \cdots \otimes H^1(U_f, O(-r_f))$,
associated with the kernel $h \in H^{3f,q}(\Pi\text{-}\ell, O(\text{-}r))$, are given for elements of the
homology group $H_{f+q}(\Lambda - \ell)$. Now, which of these contours are cohomological? A
classes of contours are the classic or traditional contours. However, realizing exten-
sions of these contour classes through twistor geometry, we can consider cohomo-
logical contours to all image elements of the generator of $H_d(\Pi\text{-}\ell'\prime, \mathbb{C})$, under two
mappings of Mayer-Vietoris. Likewise, the box nonprojective diagram also engages
three cohomological contours. Can this particular theory of contours to the spin
context be understood?

The response is yes, for example, of the foreseen construction given in
Figure 6b).

If $f \in C^1(\Omega) \cap C(\overline{\Omega}) \cap \ker D_{\pm a}(\Omega)$, with $f = K_{\pm a}\overline{f} \in H^0(\Pi - \wp, \mathbb{C})$, being $\wp = \{\pm a\}$, then $h = \frac{1}{(z \pm a)} \in H^1(\Pi - \wp, O(-2))$. Then an integral formula in
hypercomplex analysis of a vector field is an element of the integral cohomology
$H^1(\Pi - \wp, \mathbb{H})$. We can realize more work in this sense until we can arrive to the
Penrose transform on hypercomplex numbers.

3. The main conjecture and some notes of integral cohomology in low dimension of a complex Riemannian manifold

Using definitions and results exposed with before can be enunciated and
demonstrated the following conjectures:

Conjecture 3.1. The cohomology of closed submanifolds of co-dimensions
k-1, n-k, and n(k-1), can be represented and evaluated by a function cohomology.
The cohomology of contours is represented and evaluated by a complex functional
cohomology. The cohomology of line bundles is represented and evaluated by a
vector field cohomologies under the $\overline{\partial}$-cohomology corresponding.

There are indicium of that the differential operator class that accepts a scheme of
integral cohomology (integral cohomology) like due for the Penrose transform,
twistor transform, and so on is the class conformally invariant differential opera-
tors, of fact the Penrose transform generates these conformally invariant operators.
Some examples of these differential operators are for the massless field equations
(for flat versions and some curved versions [20]) and the conformally invariant
wave operator due to the mapping:

$$\Box + -R/6 : O[-1] \to O[-3], \tag{29}$$

or also the Einstein's operator

$$\nabla^{(A}_{\ (A'}\nabla^{B)}_{\ B')} + \Phi^{(AB))}_{(A'B')} : O[-1] \to O^{(AB))}_{(A'B')}[-1], \tag{30}$$

or the conformally invariant modification of the square of the wave operator $O[-1] \to O[-4]$, that is to say, the wave operator that involves in its term Ricci tensors:

$$\Box^2 : \phi \to \nabla_b \big[\nabla^b\nabla^a - 2R^{ab} + (2/3)\big(Rg^{ab}\big)\nabla_a\big], \tag{31}$$

Then, the integration of the partial differential equations corresponding to these linear invariant differential operators is realized due to integral transforms of the Penrose type since the irreducible unitary representation scheme to these operators is unitary representations of components of the group $SL(4,\mathbb{C})$, such as $SO(2n)$.

In fact, in the flat case, the invariant differential operator classifications were described to determine a problem of representation theory of Lie groups applied to the Lie group $SL(4,\mathbb{C})$ and its compact subgroups. Then, own vision to these operators through $SL(4,\mathbb{C})$ will be as equivariant operators between homogeneous vector bundles on M, considering to $SL(4,\mathbb{C})$ as homogeneous space or class space. The integrals in this case are realizations of these representations and are orbital integrals of the integral transform of the resolutions to these differential equations, which, in this concrete case, are the Penrose transform.

Then, the resolution problem of the partial differential equations is reduce to the use of representation theory, but for this case, no always can construct the curved analogues of conformally invariant differential operators of the flat space. This demonstrates that cannot be generated a curved analogous under an integral transform on homogeneous bundles of lines that are direct images of the operators D^n, of $\mathcal{G}$, of G/L. However, yes is possible to obtain a complete list under this procedure as a mapping of unitary modules.

Also, the scheme of the $\mathbb{H}$-modules in the quaternion analysis serves to compute and determine the properties of manifold through the scheme of fibers that can be in closed complex submanifolds. In fact, this is an alternative for the determination of vector fields through line bundles, which defined these as spin bundles.

Now well, cohomologically: How similar are these two methodologies for the study in field theory? Can the direct product of Lie groups $SU(2)T$, subjacent in the structure of a complex Riemannian manifold that models the space-time to its vector field study and its integration through the isometries of the space $\mathcal{L}(H)$ be carried? Which are the integration limitations for the integral transforms on homogenous bundles in global descriptions of the vector fields?

The first question is related to the double fibration that can be realized on some complex projective spaces and their quaternion equivalent. Since it always exists this bijection due to this double fibration with some corresponding homotopy group that is frequently given for spheres, some real and complex projective spaces that are necessarily identified with some *n*-dimensional sphere exist. Such is the case, for example, of the projective spaces $\mathbb{RP}^1 \cong S^1$, $\mathbb{HP}^1 \cong S^4$, or $\mathbb{CP}^7 \cong Spin(2,6)$. Then can be determined isomorphic cohomological spaces via some integral transform of the mentioned for the double fibrations. Some of these integrals result be of Feynman type due to the complex projective bundles are spin bundles in some sphere that determines some state space in quantum mechanics. For example, for the complex case, is had in an infinite succession of non-trivial bundles, the infinite set of bundles $S^1 \to \mathbb{P}^{2\ell-1}_k \to \mathbb{CP}$, $^{\ell-1}$ with $k \in \mathbb{Z}$, and $k \neq \ell$, which represents the infinite set of corresponding monopoles bundles to the case $\ell = 2$.

These with proper connections represent Dirac magnetic monopoles of charge k.

The constitutive integrals of these monopoles are Cauchy integrals that for diagrams of a cohomology $H^1(\Pi\text{-}\ell, \mathbb{C})$, these are reduced to integrals of Feynman type on the diagrams-boxes corresponding to the state monopoles vertices. These are identified for the factors $1/Z_\alpha W^\alpha$. Likewise, an integral of Cauchy type given for an integral for a ϕ^4-vertex representing the projective space $\mathbb{P}$, or its dual $\mathbb{P}^*$, comes given for

$$\mathcal{IF}(\phi^4) = \oint D^1(Z)f(Z_\alpha)g(W^\alpha)h(X_\alpha)j(Y^\alpha)/Z_\alpha W^\alpha X_\alpha W^\alpha X_\alpha Y^\alpha Z_\alpha Y^\alpha, \qquad (32)$$

which is not different to the Cauchy integral for a monopole in $z = z_0$, and representing the space $\mathbb{P}^1(\mathbb{R})$.

The response to the second question also is positive since it is possible to determine a cohomology of the space–time based on light geodesics as orbits of a complex torus T, when we consider our Universe as a complex hyperbolic manifold. The corresponding integral operators on the corresponding orbits result to be n- dimensional Fourier transforms $\mathcal{F}_n$, that can be calculated for the relation

$$f, = \mathcal{F}_n^{-1}\mathcal{F}_1\mathcal{R}f, \qquad (33)$$

in a n-dimensional manifold. The operator $\mathcal{R}$, is the Radon transform calculated on the corresponding cycles. It is well known that $\mathcal{F} \in \mathcal{L}(H)$, and that the integral cohomology given for $\mathcal{F}_n$, is the $\bar{\partial}$- cohomology of one codimensional submanifolds in M.

A response to the last question could be the limitations that are observed when it is wanted to extend the integration on the orbits of M, to a global integration of vector fields, since it is required the global integration of a vector field without the necessity of calculating previously the integrals on orbits of sections of a homogeneous bundle.

However, certain feasibility exists to obtain a methodology in this respect, generalizing, in some sense, the concept of conformal generalized structure on the manifold M.

The existing equivalences between twistor spaces, quaternion spaces, and Riemannian manifolds establish isomorphisms between different cohomology classes whose geometrical invariants are with similar invariant properties in such different cohomology classes. Likewise, we have, for example, a John integral on a complex bundle of lines F, which includes the same integration invariants with respect to the line bundle of the linear concave domains in the space $\mathbb{C}^n$ (respectively, $\mathbb{CP}^n$) for the integral of the Radon transform. The cohomology of the singularities in the description of the massless fields can be done through a twistor description of the fields using a relative cohomology of sheaves on the massless fields distributed on a real Minkowski space. Likewise, we can have other examples of equivalences for different cohomology classes.

Much results in complex analysis in $\mathbb{C}$, or $\mathbb{C}^2$, can be generalized on a context of analytic functions more extensive, using a holomorphic language of a $\bar{\partial}$- cohomology. Example of it is the use of hyperfunctions for generalizing some contour integrals. If $F \subset \mathbb{R}$, where F, is a closed interval and the hyperfunctions of F, are given for the quotient space $O(\mathbb{C}\text{-}F)/O(\mathbb{C})$, and iff, is an analytic complex function (analytic in a real sense), the sesquilinear coupling with a hyperfunction represented by the holomorphic function φ (which can be a hypercomplex function) on $(\mathbb{C}\text{-}F)$, is given for the contour integral:

$$(\phi, f) = \oint \varphi(z)f(z)dz, \qquad (34)$$

where the function f, must be extended holomorphically to a little portion of F, and the contour in $(\mathbb{C}\text{-}F)$, transits around of F, sufficiently near of the definition

domain off. This integral is not more different than the Cauchy integral, in fact, in certain sense, this is a generalization. Further, this is not more different than the John integrals, Conway integrals, and Penrose integrals on $\mathbb{R}$. The first two are used on the circle S^1, the third, for obtaining the harmonic functions of three and four variables determining the solution of the wave equation in $\mathbb{R}^4$.

The Penrose line integral in integral geometry has the interpretation as was mentioned in the Radon transform on lines of a flag manifold $F = \{L|L \subset \mathbb{R}^4\}$. All these integrals belong to a same cohomological class, which can be determined calculating the cohomology of $\mu^{-1}O(p, q, r)$, using the spectral sequence:

$$E^{p,q} = \Gamma(U, \nu^q_*(\mathbb{R}^p)),$$ (35)

Then it is possible to calculate the cohomology groups $H^{p,q}(U', \mu^{-1}O(p, q, r))$, in terms of $H^{p,q}(U', \nu^q_*(\mathbb{R}^p))$, taking the meaning of Eq. (35). These cohomology groups are sections of the sheaf with coefficients on the fibered bundle $O(p, q, r)$. The space $\mu^{-1}O(p, q, r)$, represents the inverse image of said sheaf. $U \subset M$, and $U' = \nu^{-1}(U) \subset F$, and $\mu^{-1}(U') \subset \mathbb{P}$, with μ, and ν, are the corresponding homomorphisms of the double fibration in integral geometry to relate objects in M, and $\mathbb{P}$, can be a projective twistor space and F, the flag manifold.

Finally, we can say that the descriptions in Section 3 are only few examples of our theory of integrals that we want to construct, and that are examples to enforce our conjecture on the integral geometry bases obtained from the geometry and analysis.

4. Conclusions

The idea to obtain an integral operator cohomology is develop a theory through integral invariants, that is to say, explore the complex Riemannian manifolds though the value of its integrals along the cycles and the corresponding cocycles (submanifolds, contours, vertices, edges, complexes, and so on) of the manifold. The duality between these cycles obeys to the spectral transformation that follows much of these integrals as solution of the corresponding differential equations. For example, in some case, it is used the tomography of Riemannian manifold whose cocycles are submanifolds. However, this idea can be generalized and induced beyond the tomography, for example, the integral transforms that generate differential operators with certain property of invariance inside the manifold and establish solution classes through these properties as the case to the conformally invariant differential operators. Then, the representation of objects, such as differential operators, functions, hyperfunctions, and fields, through integrals also appears in a natural way using the cohomology groups of its cocycles as first, second, ..., nth integrals for a problem of the differential or functional equations.

Likewise, much of these solutions are given through the integral transforms that search solution classes as equivalence classes in the dual problem. The inverse problems are developed in the geometrical analysis corresponding. The cohomological problem consists in developing a cohomology $H^\bullet(M, \mathfrak{J})$, the sufficiently general that means the solution to enlarge number of differential equations and that can be applied in the solution of the field equations in exploring the Universe.

The reinterpretation for physics phenomena in the case when said complex Riemannian manifold models the space-time, results interestingly, and let open the possibility of constructing an Universe theory that includes macroscopic and microscopic phenomena through a good integral theory.

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