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Chapter

Determination of the Properties of \((p, q)\)-Sigmoid Polynomials and the Structure of Their Roots

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Abstract

Nowadays, many mathematicians have great concern about \((p, q)\)-numbers, which are various applications, and have studied these numbers in many different research areas. We know that \((p, q)\)-numbers are different to \(q\)-numbers because of the symmetric property. We find the addition theorem, recurrence formula, and \((p, q)\)-derivative about sigmoid polynomials including \((p, q)\)-numbers. Also, we derive the relevant symmetric relations between \((p, q)\)-sigmoid polynomials and \((p, q)\)-Euler polynomials. Moreover, we observe the structures of appreciative roots and fixed points about \((p, q)\)-sigmoid polynomials. By using the fixed points of \((p, q)\)-sigmoid polynomials and Newton’s algorithm, we show self-similarity and conjectures about \((p, q)\)-sigmoid polynomials.

Keywords: \((p, q)\)-sigmoid numbers, \((p, q)\)-sigmoid polynomials, \((p, q)\)-Euler polynomials, roots structure, fixed point

1. Introduction

In 1991, Chakrabarti and Jagannathan [1] introduced the \((p, q)\)-number in order to unify varied forms of \(q\)-oscillator algebras in physics literature. Around the same time, Brodimas et al. and Arik et al. independently discovered the \((p, q)\)-number (see [2, 3]). Contemporarily, Wachs and White [4] introduced the \((p, q)\)-number in mathematics literature by certain combinatorial problems without any connection to the quantum group related to mathematics and physics literature.

For any \(n \in \mathbb{C}\), the \((p, q)\)-number is defined by

\[
|q| < 1.
\]

Thereby, several physical and mathematical problems lead to the necessity of \((p, q)\)-calculus. Based on the aforementioned papers, many mathematicians and physicists have developed the \((p, q)\)-calculus in many different research areas (see [1–21]).

Definition 1.1. Let \(z\) be any complex numbers with \(|z| < 1\). The two forms of \((p, q)\)-exponential functions are defined by
The useful relation of two forms of \((p, q)\)-exponential functions is taken by
\[
e_{p, q}(z) e_{p^{-1}, q^{-1}}(-z) = 1.
\]

In [9], Corcino created the theorem of \((p, q)\)-extension of binomials coefficients and found various properties which are related to horizontal function, triangular function, and vertical function.

**Definition 1.2.** Let \(n \geq k\). \((p, q)\)-Gauss Binomial coefficients are defined by
\[
\begin{aligned}
\binom{n}{k}_{p, q} &= \frac{n!_{p, q}}{k!_{p, q} \cdot (n-k)!_{p, q}}, \\
\binom{n}{k}_{p, q} &= \frac{n!_{p, q}}{k!_{p, q} \cdot (n-k)!_{p, q}}.
\end{aligned}
\]

In 2013, Sadjang [21] derived some properties of the \((p, q)\)-derivative, \((p, q)\)-integration and investigated two \((p, q)\)-Taylor formulas for polynomials.

**Definition 1.3.** We define the \((p, q)\)-derivative operator of any function \(f\), also referred to as the Jackson derivative, as follows:
\[
D_{p, q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,
\]
and
\[
D_{p, q} f(0) = f'(0).
\]

If \(t(x) = \sum_{k=0}^{n} a_k x^k\) then \(D_{p, q} t(x) = \sum_{k=0}^{n-1} a_{k+1} [k + 1]_{p, q} x^k\), since \(D_{p, q} x^n = [n]_{p, q} x^{n-1}\). This equation is equivalent to the \((p, q)\)-difference equation in \(q\) with known \(f\), \(D_{p, q} g(x) = f(x)\).

**Theorem 1.4.** This operator, \(D_{p, q}\), has the following basic properties:

(i) **Derivative of a product**
\[
D_{p, q} \left( f(x) g(x) \right) = f(px) D_{p, q} g(x) + g(qx) D_{p, q} f(x) = g(px) D_{p, q} f(x) + f(qx) D_{p, q} g(x).
\]

(ii) **Derivative of a ratio**
\[
D_{p, q} \left( \frac{f(x)}{g(x)} \right) = \frac{g(qx) D_{p, q} f(x) - f(qx) D_{p, q} g(x)}{g(px) g(qx)} = g(px) D_{p, q} f(x) - f(px) D_{p, q} g(x) g(px) g(qx).
\]

Let \(f\) be an arbitrary function. In [7], we note that the definition of \((p, q)\)-integral is
\[
f(x) d_{p, q} x = (p-q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left( \frac{q^k}{p^{k+1}} x \right).
\]
In 2016, Araci et al. [6] introduced a new class of Bernoulli, Euler and Genocchi polynomials based on the theory of $(p, q)$-number and found some properties including difference equations, addition theorem, recurrence relations were derived. We observe some special properties and roots structures of Bernoulli, Euler, and tangent polynomials (see [4, 11, 16–20]). In particular, roots structures and fixed points of tangent polynomials including $q$-numbers are shown in a different shape by [17].

Definition 1.5. $(p, q)$-Euler polynomials are defined by

$$\sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{x^n}{n!} = \frac{2}{e^{p(t)} + 1} e^{p(t)}.$$  (9)

Several studies have investigated the sigmoid function for various applications (see [11, 12, 15, 16]). For example, a variant sigmoid function with three parameters has been employed to explain hybrid sigmoidal networks [10] and sigmoid function has been defined using flexible sigmoidal mixed models based on logistic family curves for medical applications [11, 12, 15].

Definition 1.6. We define the sigmoid polynomials as follows:

$$\sum_{n=0}^{\infty} S_n(x) \frac{x^n}{n!} = \frac{1}{e^{-x} + 1} e^{x}.$$  (10)

One of the most widely used methods of solving equations is Newton’s method. This method is also based on a linear approximation of the function, but does so using a tangent to the curve. Starting from an initial estimate that is not too far from a root $x$, we extrapolate along the tangent to its intersection with the $x$-axis, and take that as the next approximation. This is continued until either the successive $x$-values are sufficiently close, or the value of the function is sufficiently near zero.

The calculation scheme follows immediately from the right triangle, which has the angle of inclination of the tangent line to the curve at $x = x_1$ as one of its acute angles:

$$\tan \theta = f'(x_1) = \frac{f(x_1)}{x_1 - x_2}, \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$  

We continue the calculation scheme by computing

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)},$$

or, in more general terms,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots.$$  

Newton’s algorithm is widely used because, at least in the near neighborhood of a root, it is more rapidly convergent than any of the methods so far discussed. The method is quadratically convergent, by which we mean that the error of each step approaches a constant $K$ times the square of the error of the previous step. The net result of this is that the number of decimal places of accuracy nearly doubles at each iteration. However, offsetting this is the need for two function evaluations at each step, $f(x_n)$ and $f'(x_n)$. We now use the result to show a criterion for convergence of
Newton’s method. Consider the form \(x_{n+1} = g(x_n)\). Successive iterations converge if \(|g'(x)| < 1\). Since
\[
   g(x) = x - \frac{f(x)}{f'(x)}, \quad g'(x) = 1 - \frac{f'(x)f''(x) - f(x)f'(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}.
\]
Hence if
\[
   \left| \frac{f(x)f''(x)}{(f'(x))^2} \right| < 1
\]
on an interval about the root \(r\), the method will converge for any initial value \(x_1\) in the interval. The condition is sufficient only, and requires the unusual continuity and existence of \(f'(x)\) and its derivatives. Note that \(f''(x)\) must not zero. In addition, Newton’s method is quadratically convergent and we can apply this method to polynomials.

Let \(f : D \to D\) be a complex function, with \(D\) as a subset of \(C\). We define the iterated maps of the complex function as the following:
\[
   f_r : z_0 \mapsto f(f(\cdots(f(z_0))\cdots))
\]
The iterates of \(f\) are the functions \(f, f \circ f, f \circ f \circ f, \cdots\), which are denoted \(f^1, f^2, f^3, \ldots\). If \(z \in \mathbb{C}\), and then the orbit of \(z_0\) under \(f\) is the sequence \(<z_0, f(z_0), f(f(z_0)), \ldots>\).

**Definition 1.7.** The orbit of the point \(z_0 \in \mathbb{C}\) under the action of the function \(f\) is said to be bounded if there exists \(M \in \mathbb{R}\) such that \(|f^n(z_0)| < M\) for all \(n \in \mathbb{N}\). If the orbit is not bounded, it is said to be unbounded.

**Definition 1.8.** Let \(f : D \to \mathbb{C}\) be a transformation on a metric space. A point \(z_0 \in D\) such that \(f(z_0) = z_0\) is called a fixed point of the transformation.

We know that the fixed point is divided as follows. Suppose that the complex function \(f\) is analytic in a region \(D\) of \(\mathbb{C}\), and \(f\) has a fixed point at \(z_0 \in D\). Then \(z_0\) is said to be:
- an attracting fixed point if \(|f'(z_0)| < 1\);
- a repelling fixed point if \(|f'(z_0)| > 1\);
- a neutral fixed point if \(|f'(z_0)| = 1\).

If \(z_0\) is an attracting fixed point of \(f\), then there exists a neighborhood of \(A\) such that if \(b \in A\) the orbit \(b\) converges to \(z_0\). Attractive fixed points of a function have a basin of attraction, which may be disconnected. The component which contains the fixed point is called the immediate basin of attraction. If \(z_0\) is a repelling periodic point of \(f\), then there is a neighborhood of \(N\) such that if \(b \in N\), there are points in the orbit of \(b\) which are not in \(N\). In the case of polynomials of degree greater than 0 and some rational functions, \(\infty\) is also called an attracting fixed point, as, for each such function, \(f_p\) there exist \(R > 0\) such that if \(|z| > R\) then \(f^n(z) \to \infty\) as \(n \to \infty\).

Based on the above, the contents of the paper are as follows. Section 2 checks the properties of \((p, q)\)-sigmoid polynomials. For example, we look for addition theorem, recurrence relation, differential, etc. and find the properties associated with the symmetric property and \((p, q)\)-Euler polynomials. Section 3 identifies the structure and accumulation of roots of \((p, q)\)-sigmoid polynomials based on the contents of Section 2 and checks the contents related to the fixed points. Also, we use...
Newton’s method to obtain a iterative function of \((p, q)\)-sigmoid polynomials to identify the domain leading to the fixed points.

2. Some properties and identities of \((p, q)\)-sigmoid polynomials

This section introduces about \((p, q)\)-sigmoid numbers and polynomials. From the generating function of these polynomials, we can observe some of the basic properties and identities of this polynomials. In particular, we can show the forms of \((p, q)\)-derivative, symmetric properties, and relations of \((p, q)\)-Euler polynomials for \((p, q)\)-sigmoid polynomials.

**Definition 2.1.** We define \((p, q)\)-sigmoid polynomials as following:

\[
\sum_{n=0}^{\infty} S_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{1}{e_{p,q}(-t) + 1} e_{p,q}(tx).
\]

In the Definition 2.1, if \(x = 0\) we can see that

\[
\sum_{n=0}^{\infty} S_{n,p,q}(0) \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} S_{n,q}(x) \frac{t^n}{[n]_{q}!} = \frac{1}{e_{q}(-t) + 1},
\]

so we can be called \(S_{n,p,q}\) is \((p, q)\)-sigmoid numbers. We note that \(e_{p,q}(0) = p^{n/2}\) because of the property for \((p, q)\)-exponential function. If \(p = 1\) in the Definition 2.1, then one holds

\[
\sum_{n=0}^{\infty} S_{n,1,q}(x) \frac{t^n}{[n]_{1,q}!} = \sum_{n=0}^{\infty} S_{n,q}(x) \frac{t^n}{[n]_{q}!} = \frac{1}{e_{q}(-t) + 1} e_{q}(tx),
\]

where \(S_{n,q}(x)\) is \(q\)-sigmoid polynomials. Moreover, if \(p = 1, q \to 1\) in the generating function of \((p, q)\)-sigmoid polynomials, we have

\[
\lim_{q^{-1} \to 1} \sum_{n=0}^{\infty} S_{n,1,q}(x) \frac{t^n}{[n]_{1,q}!} = \sum_{n=0}^{\infty} S_{n,q}(x) \frac{t^n}{n!} = \frac{1}{e^{-t} + 1} e^{tx},
\]

where \(S_{n}(x)\) is sigmoid polynomials (see [16]).

**Theorem 2.2.** Let be \(|q/p|< 1\). Then we get

\[
(i) \quad \binom{n}{2} p^{\binom{n}{2}} x^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} p^{\binom{n-k}{2}} S_{k,p,q}(x) + S_{n,p,q}(x),
\]

\[
(ii) \quad \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} p^{\binom{n-k}{2}} S_{k,p,q} + S_{n,p,q} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}
\]

**Proof.** (i) Consider that \(e_{p,q}(-t) \neq -1\). Then we can see

\[
\sum_{n=0}^{\infty} S_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} (e_{p,q}(-t) + 1) = e_{p,q}(tx).
\]
Using power series of \((p, q)\)-exponential function in the equation above (16) and Cauchy product, we can compare both-sides as following:

\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} p^n \frac{(n-k)}{p_{p,q}^n} \right) = \sum_{n=0}^{\infty} \frac{p^n}{n_{p,q}!},
\]

that is shown the required result of Theorem 2.2 (i).

(ii) This equation is a recurrence formulæ of \((p, q)\)-sigmoid numbers. We omit the proof of Theorem 2.2 (ii) since we can find the result for \((p, q)\)-sigmoid numbers to calculating the same method (i).

Based on the results from Definition 2.1 and Theorem 2.2, can be taken a few \((p, q)\)-sigmoid numbers and polynomials can be calculated by using computer. We can observe that some of the \((p, q)\)-sigmoid numbers are:

\[
S_{0,p,q}(x) = \frac{1}{2}, \\
S_{1,p,q}(x) = \frac{1}{4}(1 + 2x), \\
S_{2,p,q}(x) = \frac{1}{8}(-p + q + 2(p + q)x + 4px^2), \\
S_{3,p,q}(x) = \frac{1}{16}(q^4(1 + 2x) + 2p^2q(-1 + 2x^2) + 2pq^2(-1 + 2x^2) + p^3(1 - 2x + 4x^2 + 8x^3)), \\
S_{4,p,q}(x) = \frac{1}{32}(-3p^2q^2(1 + 2x) + q^6(1 + 2x) + 4p^3q^3x(-1 + x + 2x^2)) \\
+ \frac{1}{32}(p^4q^5(-1 + 2x)(-3 + 4x^2) + pq^6(-3 - 2x + 4x^2) + p^5q(3 - 2x + 8x^3)) \\
+ \frac{1}{32}(p^6(-1 + 2x(1 - 2x + 4x^2 + 8x^3)))
\]

\[\vdots\]

Example 2.3. Some of the \((p, q)\)-sigmoid polynomials are:

\[
S_{0,p,q}(x) = \frac{1}{2}, \\
S_{1,p,q}(x) = \frac{1}{4}(1 + 2x), \\
S_{2,p,q}(x) = \frac{1}{8}(-p + q + 2(p + q)x + 4px^2), \\
S_{3,p,q}(x) = \frac{1}{16}(q^4(1 + 2x) + 2p^2q(-1 + 2x^2) + 2pq^2(-1 + 2x^2) + p^3(1 - 2x + 4x^2 + 8x^3)) \\
S_{4,p,q}(x) = \frac{1}{32}(-3p^2q^2(1 + 2x) + q^6(1 + 2x) + 4p^3q^3x(-1 + x + 2x^2)) \\
+ \frac{1}{32}(p^4q^5(-1 + 2x)(-3 + 4x^2) + pq^6(-3 - 2x + 4x^2) + p^5q(3 - 2x + 8x^3)) \\
+ \frac{1}{32}(p^6(-1 + 2x(1 - 2x + 4x^2 + 8x^3)))
\]

\[\vdots\]

Theorem 2.4. Let \(k\) be a nonnegative integer. Then we obtain

\[
(i) \quad S_{n,p,q}(x) = \sum_{k=0}^{n} \binom{n}{k} p^n \frac{(n-k)}{p_{p,q}^n} S_{k,p,q} x^{n-k},
\]

\[
(ii) \quad S_{n,p,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} p^n \frac{(n-k)}{p_{p,q}^n} S_{k,p,q} y^{n-k}.
\]

Proof. (i) Using the definition of \((p, q)\)-exponential function, we can transform the Definition 2.1 as the follows:
\[
\sum_{n=0}^{\infty} S_{n,p,q}(x) \frac{t^n}{|n|_{p,q}} = \sum_{n=0}^{\infty} S_{n,p,q} \frac{t^n}{|n|_{p,q}} \sum_{n=0}^{\infty} p \binom{n}{2} x^n \frac{t^n}{|n|_{p,q}} \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} p \binom{n-k}{2} S_{k,p,q} x^{n-k} \right) \frac{t^n}{|n|_{p,q}}.
\]

(20)

Therefore, we complete the proof of the Theorem 2.4 (i) at once.

(ii) We also consider \((p, q)\)-sigmoid polynomials in two parameters. Then we derive

\[
\sum_{n=0}^{\infty} S_{n,p,q}(x,y) \frac{t^n}{|n|_{p,q}} = \frac{1}{e_{p,q}(-t) + 1} e_{p,q}(-t)e_{p,q}(ty)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} p \binom{n-k}{2} S_{k,p,q}(x) y^{n-k} \right) \frac{t^n}{|n|_{p,q}},
\]

where the required result (ii) is completed immediately.

\[\square\]

**Theorem 2.5.** Let \(|q/p| < 1\) and \(k\) be a nonnegative integer. Then we have

\[
\begin{align*}
(i) & \quad S_{n,p,q}(x) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{k-p} p \binom{n-k}{2} q \binom{k}{2} S_{n-p^{-1}q^{-1}}(-1)x^{n-k}, \\
(ii) & \quad S_{n,p,q} = (-1)^{p} (2) \binom{2}{2} q \binom{k}{2} S_{n-p^{-1}q^{-1}}(-1).
\end{align*}
\]

(22)

**Proof.** (i) Multiplying \(e_{p^{-1}q^{-1}}(t)\) in generating function of \((p, q)\)-sigmoid polynomials, we can investigate

\[
\sum_{n=0}^{\infty} S_{n,p,q}(x) \frac{t^n}{|n|_{p,q}} = \frac{1}{1 + e_{p^{-1}q^{-1}}(t)e_{p,q}(tx)}
\]

\[
= \sum_{n=0}^{\infty} S_{n,p^{-1}q^{-1}}(-1) \frac{(-t)^n}{|n|_{p^{-1}q^{-1}}} \sum_{n=0}^{\infty} p \binom{n}{2} x^n \frac{t^n}{|n|_{p,q}}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{k-p} p \binom{n-k}{2} q \binom{k}{2} S_{n-p^{-1}q^{-1}}(-1)x^{n-k} \right) \frac{t^n}{|n|_{p,q}}.
\]

(23)
Comparing the both-side in the equation above, (23), we find the required results (i).

(ii) Using the same method (i), we make the equation (ii), so we omit the proof of Theorem 2.5 (ii).

**Corollary 2.6.** From Theorem 2.5, one holds

\[
\sum_{k=0}^{n} \binom{n}{k} p \binom{n-k}{2} S_{k,p,q} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k p \binom{n-k}{2} q \binom{k}{2} S_{n,p,q} (-1).
\]

(24)

**Theorem 2.7.** For \(|q/p| < 1\), \((p,q)\)-derivative of \((p,q)\)-sigmoid polynomials is as the follows:

\[
\frac{D_{p,q} S_{n,p,q}(x)}{D_{p,q} x^n} = \frac{[n]_{p,q} S_{n-1,p,q}(px)}{1}.
\]

(25)

**Proof.** Using \((p,q)\)-derivative for \(S_{n,p,q}(x)\), we have

\[
\sum_{n=0}^{\infty} \frac{D_{p,q} S_{n,p,q}(x)}{D_{p,q} x^n} \frac{t^n}{[n]_{p,q} !} = \frac{1}{e_{p,q} (-1) + \frac{1}{D_{p,q} x^n}} \frac{t^n}{[n]_{p,q} !}.
\]

(26)

Here, we can note that \(D_{p,q} x^n = \frac{(px)^n - (qx)^n}{(p-q)x^n} = \frac{[n]_{p,q} S_{n-1,p,q}(px)}{[n]_{p,q}}\) (see [7]). From the equation above (26), we can transform the equation to

\[
\sum_{n=0}^{\infty} \frac{D_{p,q} S_{n,p,q}(x)}{D_{p,q} x^n} \frac{t^n}{[n]_{p,q} !} = \sum_{n=0}^{\infty} \binom{n}{k} [n-k]_{p,q} p \binom{n-k}{2} S_{k,p,q} x^{n-k} \frac{t^n}{[n]_{p,q} !}.
\]

(27)

Using the comparison of coefficients in the both-sides, we can find

\[
\frac{D_{p,q} S_{n,p,q}(x)}{D_{p,q} x^n} = \frac{[n]_{p,q} \sum_{k=0}^{n-1} \binom{n-1}{k} p \binom{n-1-k}{2} S_{k,p,q}(px)^{n-1-k}}{[n]_{p,q} !}.
\]

(28)

Applying the Theorem 2.4 (i) in the equation above, (28), we complete the proof of Theorem 2.7.

**Theorem 2.8.** Let \(|q/p| < 1\) and \(p \neq q\). Then we investigate
Determination of the Properties of \((p, q)\)-Sigmoid Polynomials and the Structure of Their Roots

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\[
D_{p,q}S_{n,p,q}(x) = \frac{S_{n,p,q}(px) - S_{n,p,q}(qx)}{(p - q)x}.
\]  

(29)

Proof. Applying \((p, q)\)-derivative in the \((p, q)\)-exponential function, we have

\[
D_{p,q} \sum_{n=0}^{\infty} S_{n,p,q}(x) \frac{t^n}{n!_{p,q}} = \sum_{n=0}^{\infty} D_{p,q}S_{n,p,q}(x) \frac{t^n}{n!_{p,q}}
\]

(30)

which is the required result.

Corollary 2.9. Comparing the Theorem 2.7 and Theorem 2.8, one holds

\[
[n]_{p,q}(p - q)xS_{n-1,p,q}(px) + S_{n,p,q}(qx) = S_{n,p,q}(px).
\]

(31)

Corollary 2.10. Putting \(x = 1\) in the Theorem 2.7 and Theorem 2.8, one holds

\[
[n]_{p,q}(p - q) \sum_{k=0}^{n-1} \binom{n-1}{k} p^{(n-k-1)} S_{k,p,q} = S_{n,p,q}(p) - S_{n,p,q}(q).
\]

(32)

Theorem 2.11. Let \(|q/p| < 1, p \neq q, \) and \(p, q \neq 0\). Then we obtain

\[
p(pqxDS_{n,p,q}(x) + D_{p,q}S_{n,p,q}(qx)) = q\left(D_{p,q}S_{n,p,q}(x) + pqxD_{p,q}S_{n,p,q}(x)\right).
\]

(33)

Proof. Using the Theorem 2.8 above, we have

\[
\sum_{n=0}^{\infty} D_{p,q}^{(2)}S_{n,p,q}(x) \frac{t^n}{n!_{p,q}} = \frac{1}{(p - q)(e_{p,q}(-t) + 1)} \left(\frac{e_{p,q}(ptx) - e_{p,q}(qtx)}{x}\right).
\]

(34)

Here, we can obtain that

\[
(i) \quad D_{p,q} e_{p,q}(ptx) = \frac{1}{pqx^2} (pxD_{p,q}e_{p,q}(ptx) - e_{p,q}(pqtx)D_{p,q}x)
\]

(35)

and

\[
(ii) \quad D_{p,q} e_{p,q}(qtx) = \frac{1}{pqx^2} (qxD_{p,q}e_{p,q}(qtx) - e_{p,q}(q^2tx)D_{p,q}x)
\]

(36)

Applying Eqs. (35) and (36) in the Eq. (34), we can catch the following equation:
Therefore, we can see that
\[
(p - q) pq x D_{p,q}^{(2)} S_{n,p,q}(x) = q D_{p,q}^{(2)} S_{n-1,p,q}(pq x) + p[n]_{p,q} S_{n-1,p,q}(px) - p D_{p,q}^{(2)} S_{n,p,q}(px) - q[n]_{p,q} S_{n-1,p,q}(qx),
\]
and this shows the required result at once.

Corollary 2.12. From the Theorem 2.11, one holds
\[
p^2 q x D_{p,q}^{(2)} S_{n,p,q}(x) + p[n]_{p,q} S_{n-1,p,q}(pq x) = q[n]_{p,q} S_{n-1,p,q}(p^2 x) + pq^2 x D_{p,q}^{(2)} S_{n,p,q}(x).
\]

Theorem 2.13. Let \(a, b\) be any positive integers. Then we find
\[
\sum_{k=0}^{n} \binom{n}{k}_{p,q} S_{n-k,p,q}(ax) S_{k,p,q}(by) a^n b^k = \sum_{k=0}^{n} \binom{n}{k}_{p,q} S_{n-k,p,q}(bx) S_{k,p,q}(ay) a^n b^k.
\]

Proof. Suppose the form \(A\) is as the following.
\[
A := \frac{\epsilon_{p,q}(tx) \epsilon_{p,q}(ty)}{(\epsilon_{p,q}(-\frac{t}{2}) + 1) (\epsilon_{p,q}(-\frac{t}{2}) + 1)}.
\]

The form \(A\) of the equation above (41) can be transformed as
\[
A := \sum_{n=0}^{\infty} \binom{n}{k}_{p,q} S_{n-k,p,q}(ax) S_{k,p,q}(by) a^n b^k,
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}_{p,q} S_{n-k,p,q}(ax) S_{k,p,q}(by) a^n b^k \right) \frac{r^n}{[n]_{p,q}!},
\]
or, equivalently,
\[
A := \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}_{p,q} S_{n-k,p,q}(bx) S_{k,p,q}(ay) a^n b^k \right) \frac{r^n}{[n]_{p,q}!}.
\]

Comparing the coefficients of \(r^n\) in both sides, we find the required result.

Corollary 2.14. Setting \(a = 1\) in the Theorem 2.13, one holds
\[
\sum_{k=0}^{n} \binom{n}{k}_{p,q} S_{n-k,p,q}(x) S_{k,p,q}(by) b^k = \sum_{k=0}^{n} \binom{n}{k}_{p,q} S_{n-k,p,q}(bx) S_{k,p,q}(y) b^{n-k}.
\]

Corollary 2.15. When \(p = 1\) and \(q = 1\) in the Theorem 2.13, one holds
\[
\sum_{k=0}^{n} \binom{n}{k} S_{n-k}(ax) S_{k}(by) a^n b^k = \sum_{k=0}^{n} \binom{n}{k} S_{n-k}(bx) S_{k}(ay) a^k b^{n-k},
\]
where \(S_n(x)\) is the sigmoid polynomials (see [16]).
**Theorem 2.16.** Let $a$, $b$ be any integers without $0$. Then we obtain

$$
\sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{(-1)^{k} S_{n-k, p, q}(ax) E_{k, p, q}(by)}{a^{n-k}b^{k}} = \sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{S_{n-k, p, q}(-bx) E_{k, p, q}(-ay)}{a^{n-k}b^{k}},
$$

(46)

where $E_{n, p, q}(x)$ is the $(p, q)$-Euler polynomials (see [10]).

**Proof.** We consider the form $B$ as

$$
B:= \frac{e_{p, q}(tx)e_{p, q}(ty)}{e_{p, q}(\frac{t}{2}) + 1}(e_{p, q}(\frac{t}{2}) + 1).
$$

(47)

The form $B$ of the equation above (47) can be transformed as

$$
B:= \frac{1}{2}\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{S_{n-k, p, q}(ax)e_{k, p, q}(by)}{a^{n-k}b^{k}}\cdot \frac{t^{n}}{|n|_{p, q}}.
$$

(48)

Also, we can transform the form $B$ such as

$$
B:= \frac{1}{2}\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{S_{n-k, p, q}(-bx)E_{k, p, q}(-ay)}{(-1)^{n}b^{n-k}a^{k}}\cdot \frac{t^{n}}{|n|_{p, q}}.
$$

(49)

Comparing the equation above (48) and (49), we derive the result of Theorem 2.16.

**Corollary 2.17.** Setting $a = 1$ in the Theorem 2.16, one holds

$$
\sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{(-1)^{k} S_{n-k, p, q}(x) E_{k, p, q}(by)}{b^{k}} = \sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{S_{n-k, p, q}(-bx) E_{k, p, q}(-y)}{b^{n-k}},
$$

(50)

where $E_{n, p, q}(x)$ is the $(p, q)$-Euler polynomials.

**Theorem 2.18.** Let $a$, $b$ be any integers without $0$. Then we have

$$
\sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{S_{n-k, p, q}(-ax) E_{k, p, q}(by)}{a^{n-k}b^{k}} = \sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{S_{n-k, p, q}(-bx) E_{k, p, q}(by)}{a^{n-k}b^{k}},
$$

(51)

where $E_{n, p, q}(x)$ is the $(p, q)$-Euler polynomials.

**Proof.** We set the form $C$ as

$$
C:= \frac{e_{p, q}(tx)e_{p, q}(ty)}{e_{p, q}(\frac{t}{2}) + 1}(e_{p, q}(\frac{t}{2}) + 1).
$$

(52)

In the form $C$ of the equation above (52), we can find that

$$
C:= \frac{1}{2}\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_{p, q} \cdot \frac{S_{n-k, p, q}(-ax) E_{k, p, q}(by)}{(-a)^{n-k}b^{k}}\cdot \frac{t^{n}}{|n|_{p, q}}.
$$

(53)
Comparing the both sides in the equation above (53) and (54), we complete the required result of Theorem 2.18.

Corollary 2.19. Putting \( a = -1 \) in the Theorem 2.18, one holds

\[
\sum_{k=0}^{n} \binom{n}{k} S_{n-k,p,q}(x) E_{k,p,q}(by) = \sum_{k=0}^{n} \binom{n}{k} S_{n-k,p,q}(-bx) E_{k,p,q}(-y),
\]

where \( E_{n,p,q}(x) \) is the \((p,q)-Euler\) polynomials.

3. Structure and various phenomena of roots of \( S_{n,p,q} \) using the computer

This section mentions the structure of roots of \( S_{n,p,q} \). Furthermore, based on the previous content, an example of \((p,q)-sigmoid\) polynomials is taken to identify the shape of the fixed points and the iterative function. And by applying it, we can find properties of self-similarity by using Newton’s method.

<table>
<thead>
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<th>( p = 0.5 )</th>
<th>( p = 0.2 )</th>
</tr>
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<td>(-1)</td>
<td>(-1)</td>
<td>(-0.9999)</td>
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<tr>
<td>(-0.390901 - 0.133409i)</td>
<td>(-0.319498 - 0.0466997i)</td>
<td>(-0.632357 - 0.0806528i)</td>
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<td>(-0.319498 + 0.0466997i)</td>
<td>(-0.632357 + 0.0806528i)</td>
</tr>
<tr>
<td>(-0.325772 - 0.249143i)</td>
<td>(-0.22888 - 0.212259i)</td>
<td>(-0.574469 - 0.251005i)</td>
</tr>
<tr>
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<td>(-0.22888 + 0.212259i)</td>
<td>(-0.574469 + 0.251005i)</td>
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<tr>
<td>(-0.229438 - 0.33589i)</td>
<td>(-0.149885 - 0.266689i)</td>
<td>(-0.458705 - 0.409513i)</td>
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<tr>
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<td>(-0.458705 + 0.409513i)</td>
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<tr>
<td>(-0.113312 - 0.387358i)</td>
<td>(-0.0593879 - 0.293992i)</td>
<td>(-0.102928 - 0.574072i)</td>
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<td>(0.518528 + 0.0823508i)</td>
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</table>

Table 1. Approximate zeros of \( S_{n,p,q}(x) \).
First, let us find an approximation of the root of $(p, q)$-sigmoid polynomials. At this time, $p$ and $q$ should not be the same value. If they have the same value, the denominator of $(p, q)$-sigmoid polynomials will be 0. Consider the roots of $S_{20, p, q}$. Once we fix the value of $q$ to 0.1 and switch $p$ into 0.9, 0.5, and 0.2, the following Table 1 can be found.

Here we can see that the approximate values of the roots change as the value of $p$ changes, as the roots always contain two real roots. Therefore, we can make the following assumptions.

**Conjecture 3.1. Roots of $S_{20, p, q}$ when $|p| < 1$, $n = 20$ and $q = 0.1$, always have two real roots.**

In the same way, we can find an approximation of the roots when the values of $q$ are changed in order of 0.9, 0.5, and 0.2 and $p$ is fixed at 0.1. In this case, when $p = 0.1$ and $q = 0.2$, the approximate roots of $S_{20, p, q}$ all possess real roots which can be confirmed through Mathematica ($x = -395824, -136973, -47951, -16837, -5912, -2076, -728, -255, -89, -31, -10, -3, -1, -0, 2, 19, 157, 1231, 9610, 71387$).

Table 1 can be illustrated as Figure 1. The figure on the left is when $p = 0.9$ and the figure on the right is when $p = 0.2$. Here we can see that the structure of the roots is changing. The following will identify the structure and the build-up of the roots of $S_{n, p, q}$. The structures of approximations roots in polynomials that combine the existing $(p, q)$-number can be found becoming closer to a circle as the $n$ increases and can be seen that a single root is continuously stacked at a certain point. Also, as the roots continue to pile up near at some point and the larger the $n$ becomes, it can be assumed that $S_{20, p, q}$ becomes closer to the circle. Actually, a picture of $S_{20, p, q}$ can be made using Mathematica.

![Figure 1](image1.png)

*Figure 1. Approximate value of zeros for $S_{20, p, q}(x)$ for $p = 0.9, 0.5, 0.2$.*

![Figure 2](image2.png)

*Figure 2. Zeros structures for $S_{n, p, q}(x)$ for $p = 0.9, 0.5, 0.2$ and $0 \leq n \leq 50$.*
The following Figure 2 shows the structure of the roots when $n$ is 0 to 50. When fixed at $q = 0.1$, the figure on the left is when $p = 0.9$ and the right is when $p = 0.2$.

Three-dimensional identification of Figure 2 shows the following Figure 3. Given the speculation, the roots are piling up near a point $(x = -1)$ and as the $p$ approaches 1, the rest of the roots become closer to a circle as the value of $n$ increases.

Based on the content above, we will now look at fixed points of $S_{n,p,q}$. At this point, the value of $q$ is fixed at 0.1 and $p$ is changed to 0.9, 0.5, and 0.2 respectively. This can be found as shown in the following Figure 4. Figure 4 shows a nearly circular appearance and always has the origin. Also, as the value of $p$ decreases, it can be seen that the distance from the origin and the roots increases.

Similarly, the fixed points in the 3D structure can be checked as shown in Figure 5.

Conjecture 3.2. $S_{n,p,q}$ may have one fixed point which is the origin and the rest of the fixed points appear in the form of a circle.

The following is a third polynomial of $S_{3,p,q}$, using a iterative function to find the approximate value of the fixed point. First, by iterating this third function five times for $p = 0.9, 0.5, 0.2$.

Figure 5. Scattering of fixed points for $S_{n,p,q}(x)$ for $p = 0.9, 0.5$ and $0 \leq n \leq 50$. 

14
times and getting the number of real roots, the value of fixed points will vary depending on the value of $p$. For $p = 0.9$ and $p = 0.5$, the number of the real roots of each of the five iterated function is 1, 1, 1, 1, 1, but for $p = 0.2, 3, 3, 3, 3$ appears. The structure of the approximate value of the actual fixed point is shown in Figure 6. The top part of Figure 6 is when $p = 0.9$ and the bottom figures represent $p = 0.2$. Typically, most of the roots structures of general polynomials using $q-$ numbers appear a circular shape, but it is difficult to find constant regularity in fixed points. However, sigmoid function including $(p,q)$-numbers might have a special property for fixed points. In other words, we can guess from Figure 6 that the structure of fixed points for $(p,q)$-sigmoid polynomials will become a circular shape if $n$ increases tremendously.

Let us look at the following by observing an application of an iterated $S_{3,p,0.1}$ using Figure 6. Let us try using the Newton’s method that we know well. Let us divide the values that go to the root of the tertiary function. First, fix $p$ at 0.9 and limit the range of values of $x$ and $y$ from $-4$ to $4$. Then the approximate root of $S_{3,0.9,0.1}$ becomes $-0.936067, 0.187169 - 0.256352i, 0.187169 + 0.256352i$. Also, if the values going to $-0.93606$ are shown in red, $0.187169 - 0.256352i$ in blue, $0.187169 + 0.256352i$ in yellow, it is shown as the left of Figure 7. The right side of Figure 7 is the picture that comes when $S_{3,0.9,0.1}$ are iterated twice.

The structure of the roots of $S_{n,p,q}$ appears to have one value near $-1$ and become a circular form as the $n$ increases. Also, as the value of $p$ increases, the

Figure 6.
Scattering of fixed points for $S_{3,p,0.1}(x)$ iterated 5-times.
diameter of the circle increases. A fixed point of $S_{n,p,q}$ can be seen to have a nearly constant form as $p$ is reduced, which can also confirm an increase in the radius.

4. Conclusion

Sigmoid function is a very important function in deep learning. In the current situation of artificial intelligence development, the properties and speculations of the sigmoid polynomials revealed in this paper in the area of using $(p,q)$-number could be an useful data in deep learning using activation functions. Through iterating $S_{n,p,q}$ with these properties, it can be assumed to have self-similarity and can be studied further to confirm new properties.

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Additional information

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Determination of the Properties of \((p, q)\)-Sigmoid Polynomials and the Structure of Their Roots

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