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# Boundary Element Model for Nonlinear Fractional-Order Heat Transfer in Magneto-Thermoelastic FGA Structures Involving Three Temperatures

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## Abstract

The principal objective of this chapter is to introduce a new fractional-order theory for functionally graded anisotropic (FGA) structures. This theory called nonlinear uncoupled magneto-thermoelasticity theory involving three temperatures. Because of strong nonlinearity, it is very difficult to solve the problems related to this theory analytically. Therefore, it is necessary to develop new numerical methods for solving such problems. So, we propose a new boundary element model for the solution of general and complex problems associated with this theory. The numerical results are presented graphically in order to display the effect of the graded parameter on the temperatures and displacements. The numerical results also confirm the validity and accuracy of our proposed model.

**Keywords:** boundary element method, fractional-order heat transfer, functionally graded anisotropic structures, nonlinear uncoupled magneto-thermoelasticity, three temperatures

## 1. Introduction

Functionally graded material (FGM) is a special type of advanced inhomogeneous materials. Functionally graded structure is a mixture of two or more distinct materials (usually heat-resisting ceramic on the outside surface and fracture-resisting metal on the inside surface) that have specified properties in specified direction of the structure to achieve a require function [1, 2]. This feature enables obtaining structures with the best of both material's properties, and suitable for applications requiring high thermal resistance and high mechanical strength [3–12].

Functionally Graded Materials have been wide range of thermoelastic applications in several fields, for example, the water-cooling model of a fusion reactor divertor is one of the most widely used models in industrial design, which is consisting of a tungsten (W) and a copper (Cu), that subjected to a structural integrity issue due to thermal stresses resulted from thermal expansion mismatch between the bond materials. Recently, functionally graded tungsten (W)–copper (Cu) has been developed by using a precipitation-hardened copper alloy as matrix

instead of pure copper, to overcome the loss of strength due to the softening of the copper matrix.

The carbon nanotubes (CNT) in FGM have new applications such as reinforced functionally graded piezoelectric actuators, reinforced functionally graded polyester calcium phosphate materials for bone replacement, reinforced functionally graded tools and dies for reduce scrap, better wear resistance, better thermal management, and improved process productivity, reinforced metal matrix functionally graded composites used in mining, geothermal drilling, cutting tools, drills and machining of wear resistant materials. Also, they used as furnace liners and thermal shielding elements in microelectronics.

There are many areas of application for elastic and thermoelastic functionally graded materials, for example, industrial applications such as MRI scanner cryogenic tubes, eyeglass frames, musical instruments, pressure vessels, fuel tanks, cutting tool inserts, laptop cases, wind turbine blades, firefighting air bottles, drilling motor shaft, X-ray tables, helmets and aircraft structures. Automobiles applications such as combustion chambers, engine cylinder liners, leaf springs, diesel engine pistons, shock absorbers, flywheels, drive shafts and racing car brakes. Aerospace applications rocket nozzle, heat exchange panels, spacecraft truss structure, reflectors, solar panels, camera housing, turbine wheels and Space shuttle. Submarine applications such as propulsion shaft, cylindrical pressure hull, sonar domes, diving cylinders and composite piping system. Biotechnology applications such as functional gradient nanohydroxyapatite reinforced polyvinyl alcohol gel biocomposites. Defense applications such as armor plates and bullet-proof vests. High-temperature environment applications such as aerospace and space vehicles. Biomedical applications such as orthopedic applications for teeth and bone replacement. Energy applications such as energy conversion devices and as thermoelectric converter for energy conservation. They also provide thermal barrier and are used as protective coating on turbine blades in gas turbine engine. Marine applications such as parallelogram slabs in buildings and bridges, swept wings of aircrafts and ship hulls. Optoelectronic applications such as automobile engine components, cutting tool insert coating, nuclear reactor components, turbine blade, tribology, sensors, heat exchanger, fire retardant doors, etc.

According to continuous and smooth variation of FGM properties throughout in depth, there are many laws to describe the behavior of FGM such as index [13], sigmoid law [14], exponential law [15] and power law [16–24].

There was widespread interest in functionally graded materials, which has developed a lot of analytical methods for analysis of elasticity [25–32] and thermoelasticity [33–53] problems, some of which have become dominant in scientific literature. For the numerical methods, the isogeometric finite element method (FEM) has been used by Valizadeh et al. [54] for static characteristics of FGM and by Bhardwaj et al. [55] for solving crack problem of FGM. Nowadays, the boundary element method is a simple, efficient and powerful numerical tool which provides an excellent alternative to the finite element method for the solution of FGM problems, Sladek et al. [56–58] have been developed BEM formulation for transient thermal problems in FGMs. Gao et al. [59] developed fracture analysis of functionally graded materials by a BEM. Fahmy [60–72] developed BEM to solve elastic, thermoelastic and biomechanic problems in anisotropic functionally graded structures. Further details on the BEM are given in [73, 74] and the references therein.

In the present paper, we propose new FGA structures theory and new boundary element technique for modeling problems of nonlinear uncoupled magneto-thermoelasticity involving three temperatures. The boundary element method reduces the dimension of the problem, therefore, we obtain a reduction of numerical approximation, linear equations system and input data. Since there is strong nonlinearity in the proposed theory and its related problems. So, we develop new

boundary element technique for modeling such problems. The numerical results are presented graphically through the thickness of the homogeneous and functionally graded structures to show the effect of graded parameter on the temperatures and displacements. The numerical results demonstrate the validity and accuracy of our proposed model.

A brief summary of the chapter is as follows: Section 1 outlines the background and provides the readers with the necessary information to books and articles for a better understanding of mechanical behaviour of magneto-thermoelastic FGA structures and their applications. Section 2 describes the formulation of the new theory and its related problems. Section 3 discusses the implementation of the new BEM for solving the nonlinear radiative heat conduction equation, to obtain the three temperature fields. Section 4 studies the development of new BEM and its implementation for solving the move equation based on the known three temperature fields, to obtain the displacement field. Section 5 presents the new numerical results that describe the through-thickness mechanical behaviour of homogeneous and functionally graded structures.

## 2. Formulation of the problem

We consider a Cartesian coordinate system for 2D structure (see **Figure 1**) which is functionally graded along the 0x direction, and considering z-axis is the direction of the effect of the constant magnetic field  $H_0$ .

The fractional-order governing equations of three temperatures nonlinear uncoupled magneto-thermoelasticity in FGA structures can be written as follows [6].

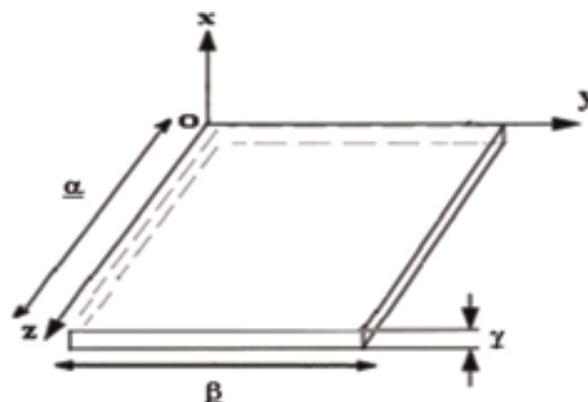
$$\sigma_{pj,j} + \tau_{pj,j} = \rho(x+1)^m \ddot{u}_k \quad (1)$$

$$\sigma_{pj} = (x+1)^m [C_{pjkl} u_{k,l} - \beta_{pj} T_\alpha(r, \tau)] \quad (2)$$

$$\tau_{pj} = \mu(x+1)^m (h_p H_j + H_j h_p - \delta_{pj} (h_k H_k)) \quad (3)$$

where  $\sigma_{pj}, \tau_{pj}, u_k, C_{pjkl}$  ( $C_{pjkl} = C_{klpj} = C_{kljp}$ ),  $\beta_{pj}$  ( $\beta_{pj} = \beta_{jp}$ ),  $\mu$  and  $h_p$  are respectively, mechanic stress tensor, Maxwell stress tensor, displacement, constant elastic moduli, stress-temperature coefficients, magnetic permeability and perturbed magnetic field.

The nonlinear time-dependent two dimensions three temperature (2D-3 T) radiation diffusion equations coupled by electron, ion and phonon temperatures may be written as follows



**Figure 1.**  
 Geometry of the FGA structure.

$$D_\tau^\alpha T_\alpha(r, \tau) = \xi \nabla [\mathbb{K}_\alpha \nabla T_\alpha(r, \tau)] + \xi \bar{\mathbb{W}}(r, \tau), \xi = \frac{1}{c_\alpha \rho \delta_1} \quad (4)$$

where

$$\bar{\mathbb{W}}(x, y, \tau) = \begin{cases} -\rho \mathbb{W}_{ei} (T_e - T_i) - \rho \mathbb{W}_{er} (T_e - T_r), & \alpha = e, \delta_1 = 1 \\ \rho \mathbb{W}_{ei} (T_e - T_i), & \alpha = i, \delta_1 = 1 \\ \rho \mathbb{W}_{ep} (T_e - T_p), & \alpha = p, \delta_1 = \frac{4}{\rho} T_p^3 \end{cases} \quad (5)$$

and

$$\mathbb{W}_{ei} = \rho \mathbb{A}_{ei} T_e^{-2/3}, \mathbb{W}_{ep} = \rho \mathbb{A}_{ep} T_e^{-1/2} \mathbb{K}_\alpha = \mathbb{A}_\alpha T_\alpha^{5/2}, \alpha = e, i, \mathbb{K}_p = \mathbb{A}_p T_p^{3+\mathbb{B}} \quad (6)$$

The total energy per unit mass can be expressed as follows

$$P = P_e + P_i + P_p, P_e = c_e T_e, P_i = c_i T_i, P_p = \frac{1}{\rho} c_p T_p^4 \quad (7)$$

where  $\mathbb{K}_\alpha$  are conductive coefficients,  $T_\alpha$  are temperature functions,  $c_\alpha$  ( $\alpha = e, i, p$ ) are isochore specific-heat coefficients,  $\rho$  is the density,  $\tau$  is the time. In which,  $c_\alpha$ ,  $\mathbb{A}_\alpha$  ( $\alpha = e, i, p$ ),  $\mathbb{B}$ ,  $\mathbb{A}_{ei}$ ,  $\mathbb{A}_{ep}$  are constant inside each subdomain,  $\mathbb{W}_{ei}$  and  $\mathbb{W}_{ep}$  are electron-ion coefficient and electron-phonon coefficient, respectively.

Initial and boundary conditions can be written as

$$T_\alpha(r, 0) = T_\alpha^0(r) = g_1(r, \tau), \quad (8)$$

$$u_k(r, 0) = \dot{u}_k(r, 0) = 0 \text{ for } r \in R \cup C(r, \tau), \quad (9)$$

$$\mathbb{K}_\alpha \frac{\partial T_\alpha}{\partial n} \Big|_{\Gamma_1} = 0, \alpha = e, i, T_r|_{\Gamma_1} = g_2(r, \tau) \quad (10)$$

$$u_k(r, \tau) = \Psi_k(r, \tau) \text{ for } r \in C_3 \quad (11)$$

$$t_k(r, \tau) = \delta_k(r, \tau) \text{ for } r \in C_4, C = C_3 \cup C_4, C_3 \cap C_4 = \emptyset \quad (12)$$

$$\mathbb{K}_\alpha \frac{\partial T_\alpha}{\partial n} \Big|_{\Gamma_2} = 0, \alpha = e, i, p \quad (13)$$

$$T(r, \tau) = H(r, \tau) \text{ for } r \in C_1, \quad \tau > 0 \quad (14)$$

$$q(r, \tau) = h(r, \tau) \text{ for } r \in C_2, \quad \tau > 0, C = C_1 \cup C_2, C_1 \cap C_2 = \emptyset \quad (15)$$

### 3. BEM numerical implementation for temperature field

This section outlines the solution of 2D nonlinear time-dependent three temperatures (electron, ion and phonon) radiation diffusion equations using a boundary element method.

Now, let us consider  $\xi = \frac{1}{c_\alpha \rho \delta_1} = 1$  and discretize the time interval  $[0, F]$  into  $F + 1$  equal time steps, where  $\tau^f = f \Delta \tau, f = 0, 1, 2, \dots, F$ . Let  $T^f(\mathbf{x}) = T(\mathbf{x}, \tau^f)$  be the solution at time  $\tau^f$ . Assuming that the time derivative of temperature within the time interval  $[\tau^f, \tau^{f+1}]$  can be approximated by.

$$\dot{T}_\alpha(r, \tau) = \frac{T_\alpha^{f+1}(r) - T_\alpha^f(r)}{\Delta \tau} + O(\Delta \tau) \quad (16)$$

$D_\tau^a$  denotes the Caputo fractional time derivative of order  $a$  defined by [75].

$$D_\tau^a T_\alpha(r, \tau) = \frac{1}{\Gamma(1-a)} \int_0^\tau \frac{\partial T_\alpha(r, s)}{\partial s} \frac{ds}{(\tau-s)^a}, 0 < a < 1 \quad (17)$$

By using a finite difference scheme of Caputo fractional time derivative of order  $a$  (17) at times  $(f + 1)\Delta \tau$  and  $f\Delta \tau$ , we obtain:

$$D_\tau^a T_\alpha^{f+1} + D_\tau^a T_\alpha^f \approx \sum_{j=0}^k W_{a,j} (T_\alpha^{f+1-j}(r) - T_\alpha^{f-j}(r)), (f = 1, 2, \dots, F) \quad (18)$$

Where

$$W_{a,0} = \frac{(\Delta \tau)^{-a}}{\Gamma(2-a)} \quad (19)$$

$$W_{a,j} = W_{a,0} ((j+1)^{1-a} - (j-1)^{1-a}), j = 1, 2, \dots, F \quad (20)$$

According to Eq. (18), the fractional order heat Eq. (4) can be replaced by the following system

$$\begin{aligned} & W_{a,0} T_\alpha^{f+1}(r) - \mathbb{K}_\alpha(x) T_{\alpha,II}^{f+1}(r) - \mathbb{K}_{\alpha,I}(x) T_{\alpha,I}^{f+1}(r) \\ & = W_{a,0} T_\alpha^f(r) - \mathbb{K}_\alpha(x) T_{\alpha,II}^f(r) - \mathbb{K}_{\alpha,I}(x) T_{\alpha,I}^f(r) \\ & - \sum_{j=1}^f W_{a,j} (T_\alpha^{f+1-j}(r) - T_\alpha^{f-j}(r)) + \overline{\mathbb{W}}_m^{f+1}(x, \tau) + \overline{\mathbb{W}}_m^f(x, \tau), f = 0, 1, 2, \dots, F \end{aligned} \quad (21)$$

According to Fahmy [60], and using the fundamental solution which satisfies the system (21), the boundary integral equations corresponding to nonlinear three temperature heat conduction-radiation equations can be written as

$$CT_\alpha = \frac{D}{\mathbb{K}_\alpha} \int_0^\tau \int_S [T_\alpha q^* - T_\alpha^* q] dS d\tau + \frac{D}{\mathbb{K}_\alpha} \int_0^\tau \int_R b T_\alpha^* dR d\tau + \int_R T_\alpha^i T_\alpha^*|_{\tau=0} dR \quad (22)$$

which can be written in the absence of internal heat sources as follows

$$CT_\alpha = \int_S [T_\alpha q^* - T_\alpha^* q] dS - \int_R \frac{\mathbb{K}_\alpha}{D} \frac{\partial T_\alpha^*}{\partial \tau} T_\alpha dR \quad (23)$$

Time temperature derivative can be written as

$$\frac{\partial T_\alpha}{\partial \tau} \cong \sum_{j=1}^N f^j(r)^j a^j(\tau) \quad (24)$$

where  $f^j(r)$  are known functions and  $a^j(\tau)$  are unknown coefficients.

We suppose that  $\hat{T}_\alpha^j$  is a solution of

$$\nabla^2 \hat{T}_\alpha^j = f^j \quad (25)$$

Then, Eq. (23) yields the following boundary integral equation

$$CT = \int_S [T_\alpha q^* - T_\alpha^* q] dS + \sum_{j=1}^N a^j(\tau) D^{-1} \left( C \hat{T}_\alpha^j - \int_S [T_\alpha^j q^* - \hat{q}^j T_\alpha^*] dS \right) \quad (26)$$

where

$$\hat{q}^j = -\mathbb{K}_\alpha \frac{\partial \hat{T}_\alpha^j}{\partial n} \quad (27)$$

and

$$a^j(\tau) = \sum_{i=1}^N f_{ji}^{-1} \frac{\partial T(r_i, \tau)}{\partial \tau} \quad (28)$$

In which the entries of  $f_{ji}^{-1}$  are the coefficients of  $F^{-1}$  with matrix  $F$  defined as [76].

$$\{F\}_{ii} = f^j(r_i) \quad (29)$$

Using the standard boundary element discretization scheme for Eq. (26) and using Eq. (28), we have

$$C \dot{T}_\alpha + H T_\alpha = G Q \quad (30)$$

The diffusion matrix can be defined as

$$C = -[H \hat{T}_\alpha - G \hat{Q}] F^{-1} D^{-1} \quad (31)$$

with

$$\{\hat{T}\}_{ij} = \hat{T}^j(\mathbf{x}_i) \quad (32)$$

$$\{\hat{Q}\}_{ij} = \hat{q}^j(\mathbf{x}_i) \quad (33)$$

In order to solve Eq. (30) numerically the functions  $T_\alpha$  and  $q$  are interpolated as

$$T_\alpha = (1 - \theta) T_\alpha^m + \theta T_\alpha^{m+1} \quad (34)$$

$$q = (1 - \theta) q^m + \theta q^{m+1} \quad (35)$$

where  $\theta = \frac{\tau - \tau^m}{\tau^{m+1} - \tau^m}$ ,  $0 \leq \theta \leq 1$  determines the practical time  $\tau$  in the current time step.

By differentiating Eq. (34) with respect to time we get

$$\dot{T}_\alpha = \frac{dT_\alpha}{d\theta} \frac{d\theta}{d\tau} = \frac{T_\alpha^{m+1} - T_\alpha^m}{\tau^{m+1} - \tau^m} = \frac{T_\alpha^{m+1} - T_\alpha^m}{\Delta\tau^m} \quad (36)$$

The substitution of Eqs. (34)–(36) into Eq. (30) leads to

$$\left(\frac{C}{\Delta\tau^m} + \theta H\right) T_\alpha^{m+1} - \theta G Q^{m+1} = \left(\frac{C}{\Delta\tau^m} - (1 - \theta)H\right) T_\alpha^m + (1 - \theta)G Q^m \quad (37)$$

By using initial and boundary conditions, we get

$$\mathbf{aX} = \mathbf{b} \quad (38)$$

This system yields the temperature, that can be used to solve (1) for the displacement.

#### 4. BEM numerical implementation for displacement field

Based on Eqs. (2) and (3), Eq. (1) can be rewritten as

$$L_{jl}u_k = \rho\ddot{u}_p - (D_{jl}u_k + D_{pj}T_\alpha) = \rho\ddot{u}_p - \rho b_p = \rho b'_p \quad (39)$$

where

$$L_{jl} = D_l \frac{\partial}{\partial x_j}, D_{jl} = \mu H_0^2 D_j \frac{\partial}{\partial x_l}, D_{pj} = -\beta_{pj} D_j, D_l = C_{pjkl} \frac{\partial}{\partial x_l}, D_j = \left(\frac{\partial}{\partial x_j} + \Lambda\right), \Lambda = \frac{m}{x+1} \quad (40)$$

when the temperatures are known, the displacement can be computed by solving (39) using BEM. By choosing  $u_p^*$  as the weight function and applying the weighted residual method, Eq. (39) can be reexpressed as

$$\int_{\mathbf{R}} (L_{jl}u_k - \rho b_p')u_p^* d\mathbf{R} = 0 \quad (41)$$

The first term in (41) can be integrated partially using Gau  $\beta$  theory yields

$$\int_{\mathbf{R}} C_{pjkl}u_{k,lj}u_p^* d\mathbf{R} = \int_{\mathbf{C}} C_{pjkl}u_{k,l}u_p^* n_j d\mathbf{C} - \int_{\mathbf{R}} C_{pjkl}u_{k,l}u_{p,j}^* d\mathbf{R} \quad (42)$$

The last term in (42) can be integrated partially twice using Gau  $\beta$  theory yields

$$\int_{\mathbf{R}} C_{pjkl}u_{k,l}u_{p,j}^* d\mathbf{R} = \int_{\mathbf{C}} C_{pjkl}u_{k,l}u_{p,j}^* n_l d\mathbf{C} - \int_{\mathbf{R}} C_{pjkl}u_{k,l}u_{p,jl}^* d\mathbf{R} \quad (43)$$

Based on Eq. (43), Eq. (42) can be rewritten as

$$\int_{\mathbf{R}} C_{pjkl}u_{k,lj}u_p^* d\mathbf{R} - \int_{\mathbf{R}} C_{pjkl}u_{k,l}u_{p,jl}^* d\mathbf{R} = \int_{\mathbf{C}} C_{pjkl}u_{k,l}u_p^* n_j d\mathbf{C} - \int_{\mathbf{C}} C_{pjkl}u_{k,l}u_{p,j}^* n_l d\mathbf{C} \quad (44)$$

which can be written as

$$\int_{\mathbf{R}} (L_{jl}u_k \cdot u_p^* - L_{jl}^*u_k^* \cdot u_p) d\mathbf{R} = \int_{\mathbf{C}} (G_{jl}u_k \cdot u_p^* - G_{jl}^*u_k^* \cdot u_p) d\mathbf{C} \quad (45)$$

The boundary tractions are

$$t_p = C_{pjkl}u_{k,l}n_j = G_{jl}u_k \text{ and } t_p^* = C_{pjkl}u_{k,j}^*n_l = G_{jl}^*u_k^* \quad (46)$$

By using the symmetry relation of elasticity tensor, we obtain

$$L_{jl} = C_{pjkl} \mathfrak{N} \frac{\partial^2}{\partial x_l \partial x_j} = C_{kjpil} \mathfrak{N} \frac{\partial^2}{\partial x_j \partial x_l} = L_{jl}^* \quad (47)$$

$$G_{jl} = C_{pjkl} \mathfrak{N} n_j \frac{\partial}{\partial x_l} = C_{kjpil} \mathfrak{N} n_l \frac{\partial}{\partial x_j} = G_{jl}^* \quad (48)$$

Using Eqs. (46)–(48), the Eq. (45) can be reexpressed as

$$\int_{\mathbf{R}} (C_{pjkl}u_{k,lj} \cdot u_p^* - C_{pjkl}u_{k,lj}^* \cdot u_p) d\mathbf{R} = \int_{\mathbf{C}} (t_p u_p^* - t_p^* u_p) d\mathbf{C} \quad (49)$$

We define the fundamental solution  $u_{mk}^*$  by the relation

$$L_{jl}u_{mk}^* = -\delta(x, \xi)\delta_{pm} \quad (50)$$

By modifying the weighting functions, Eq. (49) can be written as

$$\int_{\mathbf{R}} (C_{pjkl}u_{k,lj} \cdot u_{mp}^* - C_{pjkl}u_{mk,lj}^* \cdot u_p) d\mathbf{R} = \int_{\mathbf{C}} (t_p u_{mp}^* - t_{mp}^* u_p) d\mathbf{C} \quad (51)$$

From (39), (50) and (51), the representation formula may be written as

$$u_m(\xi) = \int_C (u_{mp}^*(x, \xi) t_p(x) - t_{mp}^*(x, \xi) u_p(x)) dC - \int_R u_{mp}^*(x, \xi) \rho b_p'(x) dR \quad (52)$$

Let

$$\begin{aligned} \rho b_p' &= -((D_{jl} + D_{pk} + \Lambda D_l) u_k + D_{pj} T_\alpha) \\ &\approx \sum_{q=1}^N f_{pn}^q \alpha_n^q = \sum_{q=1}^N (L_{jl} u_{kn}^q) \alpha_n^q, D_{pk} = -\rho \delta_{pk} \frac{\partial^2}{\partial \tau^2} \end{aligned} \quad (53)$$

The displacement particular solution may be defined as

$$u_{kn}^q = \delta_{kn} (r^2 + r^3) \quad (54)$$

Differentiation of (54) leads to.

$$u_{kn,l}^q = \delta_{kn} (2r + 3r^2) r_{,l}, \quad u_{kn,li}^q = \delta_{kn} ((2 + 3r) \delta_{li} + 3rr_{,j} r_{,l}) \quad (55)$$

Now, we obtain the traction particular solution  $t_{pn}^q$  and source function  $f_{pn}^q$  as

$$t_{pn}^q = C_{pjkl} u_{kn,l}^q n_j, \quad L_{jl} u_{kn}^q = f_{pn}^q \quad (56)$$

The domain integral may be approximated as follows

$$\int_R u_{mp}^* \rho b_p' dR \approx \sum_{q=1}^N \left( \int_R (L_{jl} u_{kn}^q u_{mp}^*) dR \right) \alpha_n^q \quad (57)$$

The use of (57) together with the dual reciprocity

$$\int_R (L_{jl} u_{kn}^q u_{mp}^* - L_{jl} u_{mk}^* u_{pn}^q) dR = \int_C (u_{mp}^* t_{pn}^q - t_{mp}^* u_{pn}^q) dC \quad (58)$$

Leads to

$$\begin{aligned} \int_R u_{mp}^* \rho b_p' dR &= \sum_{q=1}^N \int_R (L_{jl} u_{mk}^* u_{pn}^q) dR \\ &+ \int_C (u_{mp}^* t_{pn}^q - t_{mp}^* u_{pn}^q) dC \quad (59) \end{aligned}$$

From (50), we can write

$$\int_{\mathbf{R}} L_{jl} u_{mk}^* u_{pn}^q d\mathbf{R} = \int_{\mathbf{R}} -\delta(x, \xi) \delta_{pm} u_{pn}^q d\mathbf{R} = -u_{mn}^q(\xi) \quad (60)$$

By using (52), (59) and (60), we obtain

$$u_m(\xi) = \int_{\mathbf{C}} (u_{mp}^* t_p - t_{mp}^* u_p) d\mathbf{C} + \sum_{q=1}^N \left( u_{mn}^q(\xi) - \int_{\mathbf{C}} (u_{mp}^* t_{pn}^q - t_{mp}^* u_{pn}^q) d\mathbf{C} \right) \alpha_n^q \quad (61)$$

According to Fahmy [9–11], the right-hand side integrals of (61) can be reexpressed as

$$\begin{aligned} & \int_{\mathbf{C}} (u_{mp}^*(x, \xi) t_p(x) - t_{mp}^*(x, \xi) u_p(\xi)) d\mathbf{C} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{C}-\mathbf{C}_\varepsilon} (u_{mp}^*(x, \xi) t_p(x) - t_{mp}^*(x, \xi) u_p(\xi)) d\mathbf{C} \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{C}_\varepsilon} (u_{mp}^*(x, \xi) t_p(x) - t_{mp}^*(x, \xi) u_p(\xi)) d\mathbf{C}_\varepsilon \end{aligned} \quad (62)$$

and

$$\begin{aligned} & \int_{\mathbf{C}} (u_{mp}^*(x, \xi) t_{pn}^q(x) - t_{mp}^*(x, \xi) u_{pn}^q(\xi)) d\mathbf{C} = \\ & \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{C}-\mathbf{C}_\varepsilon} (u_{mp}^*(x, \xi) t_{pn}^q(x) - t_{mp}^*(x, \xi) u_{pn}^q(\xi)) d\mathbf{C} \\ & + \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{C}_\varepsilon} (u_{mp}^*(x, \xi) t_{pn}^q(x) - t_{mp}^*(x, \xi) u_{pn}^q(\xi)) d\mathbf{C}_\varepsilon \end{aligned} \quad (63)$$

According to Fahmy [12], Guiggiani and Gigante [77] and Mantič [78] Eqs. (62) and (63) can respectively be expressed as

$$\begin{aligned} & \int_{\mathbf{C}} (u_{mp}^*(x, \xi) t_p(x) - t_{mp}^*(x, \xi) u_p(\xi)) d\mathbf{C} \\ &= \int_{\mathbf{C}} u_{mp}^* t_p d\Gamma - \oint_{\mathbf{C}} u_p t_{mp}^* d\Gamma - c_{pj} u_p(\xi) \end{aligned} \quad (64)$$

$$\begin{aligned} & \int_{\mathbf{C}} (u_{mp}^*(x, \xi) t_{pn}^q(x) - t_{mp}^*(x, \xi) u_{pn}^q(\xi)) d\mathbf{C} \\ &= \int_{\mathbf{C}} u_{mp}^* t_{pn}^q d\Gamma - \oint_{\mathbf{C}} u_{pn}^q t_{mp}^* d\Gamma - c_{pj} u_{pn}^q(\varepsilon) \end{aligned} \quad (65)$$

By using (64) and (65), the dual reciprocity boundary integral equation becomes

$$c_{pj}u_p(\varepsilon) + \oint_C u_p t_{mp}^* d\Gamma - \int_C u_{mp}^* t_p d\Gamma = \sum_{q=1}^N \left( c_{pj}u_{pn}^q(\varepsilon) + \oint_C u_{pn}^q t_{mp}^* d\Gamma - \int_C u_{mp}^* t_{pn}^q d\Gamma \right) \alpha_n^q \quad (66)$$

On the basis of isoparametric concept, we can write

$$\{u, t\} \approx \sum_{k=1}^N \varphi_k \{\check{u}_k, \check{t}_k\} = \Phi^T \{\check{u}, \check{t}\} \quad (67)$$

$$\{u^q, t^q\} \approx \sum_{k=1}^N \varphi_k \{\check{u}_k^q, \check{t}_k^q\} = \Phi^T \{\check{u}^q, \check{t}^q\} \quad (68)$$

By implementing the point collocation procedure and using (67) and (68), Eq. (66) may be reexpressed as

$$\zeta \check{u} - \eta \check{t} = \sum_{q=1}^N (\zeta \check{u}^q - \eta \check{t}^q) \alpha^q(\tau) \quad (69)$$

Let us suppose that

$$\check{U} = [\check{u}^1 \ \check{u}^2 \ \dots \ \check{u}^N] \quad (70)$$

$$\check{\rho} = [\check{t}^1 \ \check{t}^2 \ \dots \ \check{t}^N] \quad (71)$$

$$\alpha = [\alpha^1 \ \alpha^2 \ \dots \ \alpha^N]^T \quad (72)$$

We can write (69) as follows

$$\zeta \check{u}(\tau) - \eta \check{t}(\tau) = (\zeta \check{U} - \eta \check{\rho}) \alpha(\tau) \quad (73)$$

By using the point collocation procedure,  $\alpha(\tau)$  can be calculated from (53) as

$$\rho \check{u}(\tau) - \rho \check{b}(\tau) = F \alpha(\tau) \quad (74)$$

Now, from (74), we may derive

$$\alpha(\tau) = F^{-1}(\rho \check{u}(\tau) - \rho \check{b}(\tau)) \quad (75)$$

From (73) using (75) we have

$$\mathcal{M} \check{u} + \zeta \check{u} = \eta \check{t}(\tau) + \check{B}(\tau) \quad (76)$$

where

$$\mathbf{U} = (\eta\check{\phi} - \zeta\check{U})F^{-1}, \quad \mathcal{M} = \rho\mathbf{U}, \quad \check{\mathcal{B}}(\tau) = \rho\mathbf{U}\check{b}(\tau). \quad (77)$$

By considering the following known  $k$  and unknown  $u$  superscripts nodal vectors

$$\{\check{u}^k, \check{t}^u\} \in C_3, \{\check{u}^u, \check{t}^k\} \in C_4 \quad (78)$$

Hence (76) may be written as

$$\begin{bmatrix} \mathcal{M}^{11} & \mathcal{M}^{12} \\ \mathcal{M}^{21} & \mathcal{M}^{22} \end{bmatrix} \begin{bmatrix} \check{u}^k(\tau) \\ \check{u}^u(\tau) \end{bmatrix} + \begin{bmatrix} \zeta^{11} & \zeta^{12} \\ \zeta^{21} & \zeta^{22} \end{bmatrix} \begin{bmatrix} \check{u}^k(\tau) \\ \check{u}^u(\tau) \end{bmatrix} = \begin{bmatrix} \eta^{11} & \eta^{12} \\ \eta^{21} & \eta^{22} \end{bmatrix} \begin{bmatrix} \check{t}^k(\tau) \\ \check{t}^u(\tau) \end{bmatrix} + \begin{bmatrix} \check{\mathcal{B}}^1(\tau) \\ \check{\mathcal{B}}^2(\tau) \end{bmatrix} \quad (79)$$

From the first row of (79), we can calculate the unknown fluxes  $\check{t}^u(\tau)$  as follows

$$\begin{aligned} \check{t}^u(\tau) &= (\eta^{12})^{-1} [\mathcal{M}^{11}\check{u}^k(\tau) + \mathcal{M}^{12}\check{u}^u(\tau) + \zeta^{11}\check{u}^k(\tau) \\ &+ \zeta^{12}\check{u}^u(\tau) - \eta^{11}\check{t}^k(\tau) - \check{\mathcal{B}}^1(\tau)] \end{aligned} \quad (80)$$

From the second row of (79) and using (80) we get

$$\mathcal{M}^u\check{u}^u(\tau) + \zeta^u\check{u}^u(\tau) = Q^k(\tau) \quad (81)$$

where

$$Q^k(\tau) = \check{\mathcal{B}}^k(\tau) + \eta^k\check{t}^k(\tau) - \mathcal{M}^k\check{u}^k(\tau) - \zeta^k\check{u}^k(\tau)$$

$$\mathcal{M}^u = \mathcal{M}^{22} - \eta^{22}(\eta^{12})^{-1}\mathcal{M}^{12}$$

$$\zeta^u = \zeta^{22} - \eta^{22}(\eta^{12})^{-1}\zeta^{12} \quad (82)$$

$$\mathcal{M}^k = \mathcal{M}^{21} - \eta^{22}(\eta^{12})^{-1}\mathcal{M}^{11}$$

$$\zeta^k = \zeta^{21} - \eta^{22}(\eta^{12})^{-1}\zeta^{11}$$

$$\check{\mathcal{B}}^k(\tau) = \mathcal{B}^2(\tau) - \eta^{22}(\eta^{12})^{-1}\mathcal{B}^1(\tau)$$

Eq. (81) can be written at  $(n + 1)$  time step as

$$\mathcal{M}^u\check{u}_{n+1}^u(\tau) + \zeta^u\check{u}_{n+1}^u(\tau) = Q_{n+1}^k(\tau) \quad (83)$$

where

$$Q_{n+1}^k(\tau) = \check{\mathcal{B}}_{n+1}^k(\tau) + \eta^k\check{t}_{n+1}^k(\tau) - \mathcal{M}^k\check{u}_{n+1}^k(\tau) - \zeta^k\check{u}_{n+1}^k(\tau) \quad (84)$$

In order to solve (83), The implicit backward finite difference scheme has been applied based on the Houbolt's algorithm and the following approximations

$$\check{u}_{n+1} \approx \frac{1}{6\Delta t} (11\check{u}_{n+1} - 18\check{u}_n + 9\check{u}_{n-1} - 2\check{u}_{n-2}) \quad (85)$$

$$\check{u}_{n+1} \approx \frac{1}{\Delta t^2} (2\check{u}_{n+1} - 5\check{u}_n + 4\check{u}_{n-1} - \check{u}_{n-2}) \quad (86)$$

By using (85) and (86), we have from (83)

$$\zeta^u \check{u}_{n+1}^u(\tau) = Q_{n+1}^k(\tau) \quad (87)$$

In which,

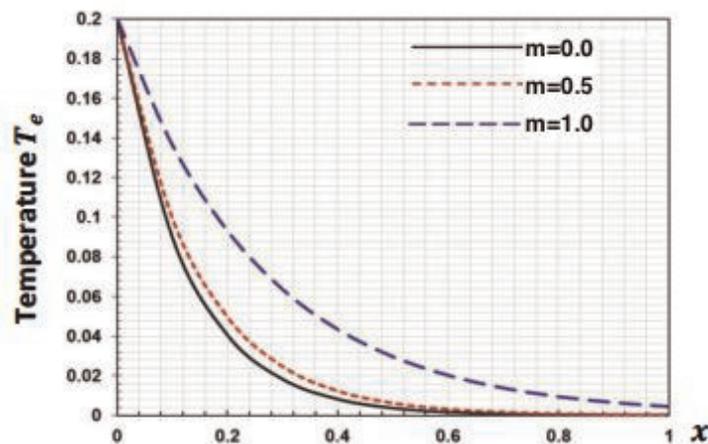
$$\zeta^u = \frac{2\mathcal{M}^u}{\Delta t^2} + \zeta^u \quad (88)$$

$$Q_{n+1}^k = Q_{n+1}^k + \frac{\mathcal{M}^u}{\Delta t^2} (5\check{u}_n - 4\check{u}_{n-1} + \check{u}_{n-2}) \quad (89)$$

We implement the successive over-relaxation (SOR) of Golub and Van Loan [79] for solving (87) to obtain  $\check{u}_{n+1}^u$ . Then, the unknown  $\check{u}_{n+1}^u$  and  $\check{u}_{n+1}^u$  can be obtained from (76) and (77), respectively. By using the procedure of Bathe [80], we obtain the traction vector  $t_{n+1}^u$  from (73).

## 5. Numerical results and discussion

The BEM that has been used in the current paper can be applicable to a wide variety of FGA structures problems associated with the proposed theory of three temperatures nonlinear uncoupled magneto-thermoelasticity. In order to evaluate the influence of graded parameter on the three temperatures and displacements, the numerical results are carried out and depicted graphically for homogeneous ( $m = 0$ ) and functionally graded ( $m = 0.5$  and  $1.0$ ) structures.



**Figure 2.**  
 Variation of the electron temperature  $T_e$  through the thickness coordinate  $x$ .

Figures 2–4 show the distributions of the three temperatures  $T_e$ ,  $T_i$  and  $T_p$  through the thickness coordinate  $Ox$ . It was shown from these figures that the three temperatures increase with increasing value of graded parameter  $m$ .

Figures 5 and 6 show the distributions of the displacements  $u_1$  and  $u_2$  through the thickness coordinate  $Ox$ . It was noticed from these figures that the displacement components increase with increasing value of graded parameter  $m$ .

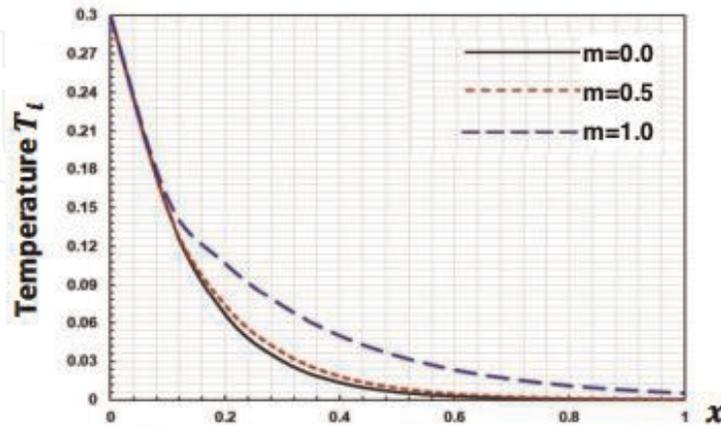


Figure 3.  
Variation of the ion temperature  $T_i$  through the thickness coordinate  $x$ .

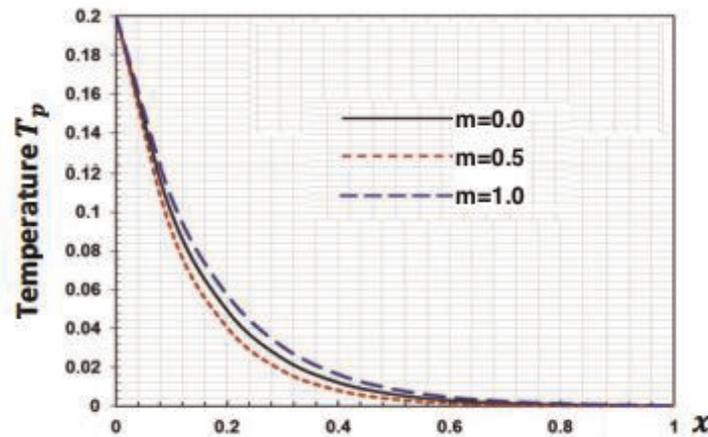


Figure 4.  
Variation of the photon temperature  $T_p$  through the thickness coordinate  $x$ .

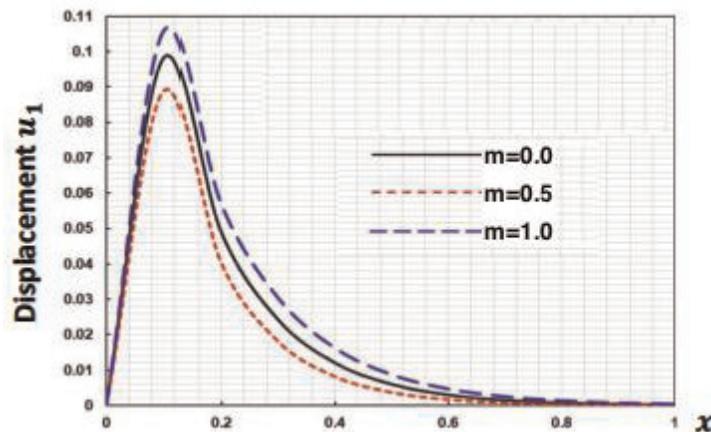


Figure 5.  
Variation of the displacement  $u_1$  through the thickness coordinate  $x$ .

Figures 7 and 8 show the distributions of the displacements  $u_1$  and  $u_2$  with the time for boundary element method (BEM), finite difference method (FDM) and finite element method (FEM) to demonstrate the validity and accuracy of our proposed technique. It is noted from numerical results that the BEM obtained results are agree quite well with those obtained using the FDM of Pazera and Jędrzyiak [81] and FEM of Xiong and Tian [82] results based on replacing heat conduction with three-temperature heat conduction.

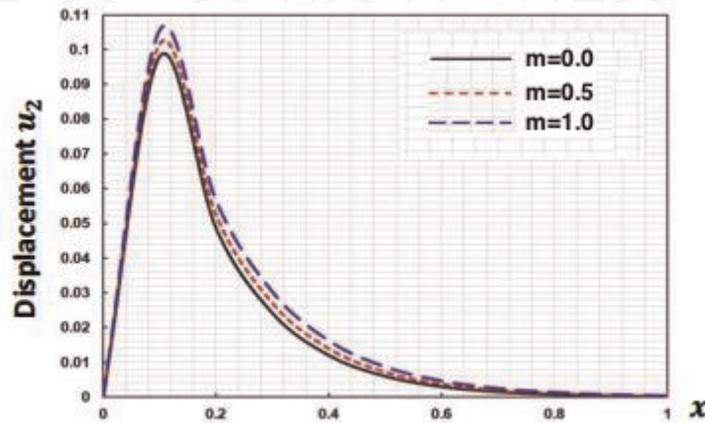


Figure 6.  
Variation of the displacement  $u_2$  through the thickness coordinate  $x$ .

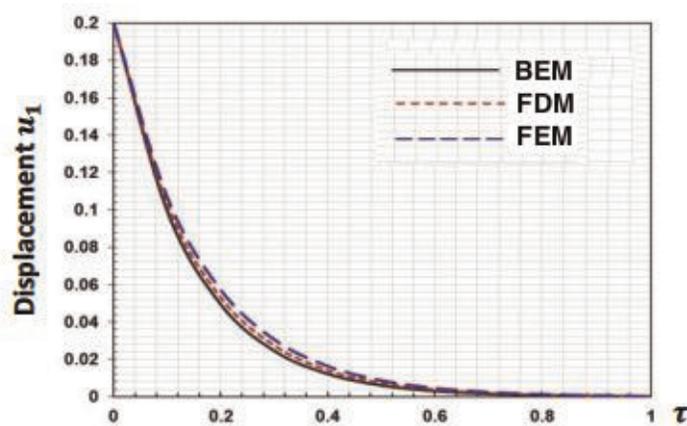


Figure 7.  
Variation of the displacement  $u_1$  with time  $\tau$ .

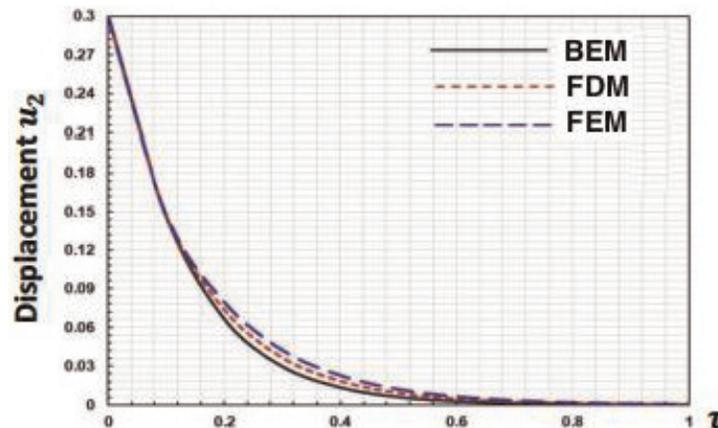


Figure 8.  
Variation of the displacement  $u_2$  with time  $\tau$ .

## 6. Conclusion

The main aim of this article is to introduce a new fractional-order theory called nonlinear uncoupled magneto-thermoelasticity theory involving three temperatures for FGA structures and new boundary element technique for solving problems related to the proposed theory. Since the nonlinear three temperatures radiative heat conduction equation is independent of the displacement field, we first determine the temperature field using the BEM, then based on the known temperature field, the displacement field is obtained by solving the move equation using the BEM. It can be seen from the numerical results that the graded parameter had a significant effect on the temperatures and displacements through the thickness of the functionally graded structures. Since there are no available results for the considered problem. So, some literatures may be considered as special cases from the considered problem based on replacing the heat conduction by three temperatures radiative heat conduction. The numerical results demonstrate the validity and accuracy of our proposed model. From the proposed BEM technique that has been performed in the present paper, it is possible to conclude that the proposed BEM should be applicable to any FGM uncoupled magneto-thermoelastic problem with three-temperature. BEM is more efficient, accurate and easy to use than FDM or FEM, because it only needs to solve the unknowns on the boundaries and BEM users need only to deal with real geometry boundaries. Also, BEM is reducing the computational cost of its solver. The present numerical results for our complex problem may provide interesting information for computer scientists, designers of new FGM materials and researchers in FGM science as well as for those working on the development of new functionally graded structures.

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