Chapter

A Study of Bounded Variation Sequence Spaces

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Abstract

In the theory of classes of sequence, a wonderful application of Hahn-Banach extension theorem gave rise to the concept of Banach limit, i.e., the limit functional defined on c can be extended to the whole space l_∞ and this extended functional is called as the Banach limit. After that, in 1948 Lorentz used this concept of a week limit to introduce a new type of convergence, named as the almost convergence. Later on, Raimi generalized the concept of almost convergence known as $\sigma-$ convergence and the sequence space BV_σ was introduced and studied by Mursaleen. The main aim of this chapter is to study some new double sequence spaces of invariant means defined by ideal, modulus function and Orlicz function. Furthermore, we also study several properties relevant to topological structures and inclusion relations between these spaces.

Keywords: invariant mean, bounded variation, ideal, filter, I-convergence, Orlicz function, modulus function, paranorm

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1] and Schoenberg [2]. There has been an effort to introduce several generalizations and variants of statistical convergence in different spaces. One such very important generalization of this notion was introduced by Kostyrko et al. [3] by using an ideal I of subsets of the set of natural numbers, which they called I-convergence. After that the idea of I-convergence for double sequence was introduced by Das et al. [4] in 2008.

Throughout a double sequence is defined by $x = (x_{ij})$ and we denote ${}_2\omega$ showing the space of all real or complex double sequences.

Let X be a nonempty set then a family $I \subset 2^X$ is said to be an **ideal** in X if $\emptyset \in I$, I is additive, i.e., for all $A, B \in I \Rightarrow A \cup B \in I$ and I is hereditary, i.e., for all $A \in I, B \subseteq A \Rightarrow B \in I$. A nonempty family of sets $\mathcal{F} \subset 2^X$ is said to be a **filter** on X if for all $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and for all $A \in \mathcal{F}$ with $A \subseteq B$ implies $B \in \mathcal{F}$. An ideal $I \subset 2^X$ is said to be **nontrivial** if $I \neq 2^X$, this non trivial ideal is said to be admissible if $I \supseteq \{\{x\} : x \in X\}$ and is said to be **maximal** if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset. For each ideal I there is a filter $\mathcal{F}(I)$ called as filter associate with ideal I, that is

$$\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}, \quad \text{where} \quad K^c = X \backslash K. \tag{1}$$

A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I-**convergent** [5–8] to a number L if for every $\epsilon > 0$, we have $\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I$. In this case, we write $I - \lim x_{ij} = L$. A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I-**Cauchy** if for every $\epsilon > 0$ there exists numbers $m = m(\epsilon), n = n(\epsilon)$ such that $\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I$.

A continuous linear functional ϕ on l_∞ is said to be an **invariant mean** [9, 10] or σ-mean if and only if:

1. $\phi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
2. $\phi(e) = 1$ where $e = \{1, 1, 1, 1, \ldots\}$,
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$,

where σ be an injective mapping of the set of the positive integers into itself having no finite orbits.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$, so we have

$$V_\sigma = \left\{x = (x_k) : \lim_{m \to \infty} t_{m,k}(x) = L \;\; \text{uniformly in} \;\; k, L = \sigma - \lim x\right\} \tag{2}$$

where m $\geq 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + \ldots + x_{\sigma^m(k)}}{m+1} \;\; \text{and} \;\; t_{-1,k} = 0, \tag{3}$$

where $\sigma^m(k)$ denote the mth-iterate of $\sigma(k)$ at k. In this case σ is the translation mapping, that is, $\sigma(k) = k + 1$, $\sigma-$ mean is called a Banach limit [11] and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (3) in which $\sigma(k) = k + 1$ was given by Lorentz [12] and the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c in the sense that

$$\phi(x) = \lim x, \quad \text{for all} \;\; x \in c. \tag{4}$$

Definition 1.1 A sequence $x \in l_\infty$ is of σ-**bounded variation** if and only if:

(i) $\sum |\phi_{m,k}(x)|$ converges uniformly in k,

(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

We denote by BV_σ, the space of all sequences of σ-bounded variation:

$$BV_\sigma = \left\{x \in l_\infty : \sum_m |\phi_{m,k}(x)| < \infty, \;\; \text{uniformly in} \;\; k\right\}.$$

is a Banach space normed by

$$\|x\| = \sup_k \sum_{m=0}^{\infty} |\phi_{m,k}(x)|. \tag{5}$$

A function $M : [0, \infty) \to [0, \infty)$ is said to be an **Orlicz function** [13, 14] if it satisfies the following conditions:

(i) M is continuous, convex and non-decreasing,

(ii) $M(0) = 0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark 1.1 If the convexity of an Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called **Modulus function** [15–17]. If M is an Orlicz function, then $M(\lambda X) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$. An Orlicz function M is said to satisfy Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \le KLM(u)$ for all values of $L > 1$ [18].

Definition 1.2 A double sequence space X is said to be:

[i] **solid** or **normal** if $(x_{ij}) \in X$ implies that $(\alpha_{ij}x_{ij}) \in X$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.

[ii] **symmetric** if $(x_{\pi(i,j)}) \in X$ whenever $(x_{ij}) \in X$, where $\pi(i,j)$ is a permutation on $\mathbb{N} \times \mathbb{N}$.

[iii] **sequence algebra** if $\left(x_{ij}y_{ij}\right) \in E$ whenever (x_{ij}), $\left(y_{ij}\right) \in X$.

[iv] **convergence free** if $\left(y_{ij}\right) \in X$ whenever $(x_{ij}) \in X$ and $x_{ij} = 0$ implies $y_{ij} = 0$, for all $(i,j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.3 Let $K = \{(n_i, k_j) : (i,j) : n_1 < n_2 < n_3 < \ldots. \text{ and } k_1 < k_2 < k_3 < \ldots.\} \subseteq \mathbb{N} \times \mathbb{N}$ and X be a double sequence space. A K-step space of X is a sequence space

$$\lambda_k^E = \{(\alpha_{ij}x_{ij}) : (x_{ij}) \in X\}.$$

A canonical preimage of a sequence $\left(x_{n_ik_j}\right) \in X$ is a sequence $(b_{nk}) \in X$ defined as follows:

$$b_{nk} = \begin{cases} a_{nk}, & \text{for n, k} \in K \\ 0, & \text{otherwise.} \end{cases}$$

A sequence space X is said to be **monotone** if it contains the canonical preimages of all its step spaces.

The following subspaces $l(p), l_\infty(p), c(p)$ and $c_0(p)$ where $p = (p_k)$ is a sequence of positive real numbers. These subspaces were first introduced and discussed by Maddox [16]. The following inequalities will be used throughout the section. Let $p = \left(p_{ij}\right)$ be a double sequence of positive real numbers [19]. For any complex λ with $0 < p_{ij} \le \sup_{ij} p_{ij} = G < \infty$, we have

$$|\lambda|^{p_{ij}} \le \max\left(1, |\lambda|^G\right).$$

Let $D = \max(1, 2^{G-1})$ *and* $H = \max\left\{1, \sup_{ij} p_{ij}\right\}$, then for the sequences (a_{ij}) and (b_{ij}) in the complex plane, we have

$$|a_{ij} + b_{ij}|^{p_{ij}} \le C\left(|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}}\right).$$

2. Bounded variation sequence spaces defined by Orlicz function

In this section, we define and study the concepts of I-convergence for double sequences defined by Orlicz function and present some basic results on the above definitions [8, 20].

$$_2BV_\sigma^I(M) = \left\{(x_{ij}) \in {}_2w : I - \lim M\left(\frac{|\phi_{mnij}(x) - L|}{\rho}\right) = 0, \ \textit{for some} \ \ L \in \mathbb{C}, \rho > 0\right\} \tag{6}$$

$$_2\left({}_0BV_\sigma^I(M)\right) = \left\{ \left(x_{ij}\right) \in {}_2w : I - \lim M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = 0, \rho > 0 \right\}, \tag{7}$$

$$_2\left({}_\infty BV_\sigma^I(M)\right) = \left\{ \left(x_{ij}\right) \in {}_2w : \left\{ (i,j) : \exists\, k > 0 \; s.t\; M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) \geq k \right\} \in I, \rho > 0 \right\} \tag{8}$$

$$_2({}_\infty BV_\sigma(M)) = \left\{ \left(x_{ij}\right) \in {}_2w : \sup M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) < \infty, \rho > 0 \right\}. \tag{9}$$

Now, we read some theorems based on these sequence spaces. These theorems are of general importance as indispensable tools in various theoretical and practical problems.

Theorem 2.1 Let M_1, M_2 be two Orlicz functions with Δ_2 condition, then

(a) $\chi(M_2) \subseteq \chi(M_1M_2)$

(b) $\chi(M_1) \cap \chi(M_2) \subseteq \chi(M_1 + M_2)$ for $\chi = {}_2BV_\sigma^I, {}_2\left({}_0BV_\sigma^I\right)$.

Proof. (a) Let $x = \left(x_{ij}\right) \in {}_2\left({}_0BV_\sigma^I(M_2)\right)$ be an arbitrary element, so there exists $\rho > 0$ such that

$$I - \lim M_2\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = 0. \tag{10}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \varepsilon$ for $0 < t \leq \delta$.

Write $y_{ij} = M_2\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)$. Consider,

$$\lim_{ij} M_1\left(y_{ij}\right) = \lim_{y_{ij} \leq \delta,\, i,j \in \mathbb{N}} M_1\left(y_{ij}\right) + \lim_{y_{ij} > \delta,\, i,j \in \mathbb{N}} M_1\left(y_{ij}\right). \tag{11}$$

Now, since M_1 is an Orlicz function so we have $M_1(\lambda x) \leq \lambda M_1(x)$, $0 < \lambda < 1$. Therefore, we have

$$\lim_{y_{ij} \leq \delta,\, i,j \in \mathbb{N}} M_1\left(y_{ij}\right) \leq M_1(2) \lim_{y_{ij} \leq \delta,\, i,j \in \mathbb{N}} \left(y_{ij}\right). \tag{12}$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Now, since M_1 is non-decreasing and convex, it follows that,

$$M_1\left(y_{ij}\right) < M_1\left(1 + \frac{y_{ij}}{\delta}\right) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1\left(\frac{2y_{ij}}{\delta}\right). \tag{13}$$

Since M_1 satisfies the Δ_2-condition, so we have

$$\begin{aligned} M_1\left(y_{ij}\right) &< \frac{1}{2}K\frac{y_{ij}}{\delta}M_1(2) + \frac{1}{2}KM_1\left(\frac{2y_{ij}}{\delta}\right) \\ &< \frac{1}{2}K\frac{y_{ij}}{\delta}M_1(2) + \frac{1}{2}K\frac{y_{ij}}{\delta}M_1(2) \\ &= K\frac{y_{ij}}{\delta}M_1(2). \end{aligned} \tag{14}$$

This implies that,

$$M_1\left(y_{ij}\right) < K\frac{y_{ij}}{\delta}M_1(2). \tag{15}$$

Hence, we have

$$\lim_{y_{ij}>\delta,\, i,j\in\mathbb{N}} M_1\left(y_{ij}\right) \le \max\left\{1, K\delta^{-1}M_1(2) \lim_{y_{ij}>\delta,\, i,j\in\mathbb{N}}\left(y_{ij}\right)\right\}. \tag{16}$$

Therefore from (12) and (16), we have

$$I-\lim_{ij} M_1\left(y_{ij}\right) = 0.$$
$$\Rightarrow I-\lim_{ij} M_1M_2\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = 0.$$

This implies that $x = \left(x_{ij}\right) \in {}_2\left({}_0BV_\sigma^I(M_1M_2)\right)$. Hence $\chi(M_2) \subseteq \chi(M_1M_2)$ for $\chi = {}_2\left({}_0BV_\sigma^I\right)$. The other cases can be proved in similar way.

(b) Let $x = \left(x_{ij}\right) \in {}_2\left({}_0BV_\sigma^I(M_1)\right) \cap {}_2\left({}_0BV_\sigma^I(M_2)\right)$. Let $\epsilon>0$ be given. Then there exist $\rho>0$, such that

$$I-\lim_{ij} M_1\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = 0, \tag{17}$$

and

$$I-\lim_{ij} M_2\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = 0. \tag{18}$$

Therefore

$$I-\lim_{ij}(M_1+M_2)\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = I-\lim_{ij} M_1\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) + I-\lim_{ij} M_2\left(\frac{|\phi_{mnij}(x)|}{\rho}\right),$$

from Eqs. (17) and (18), we get

$$I-\lim_{ij}(M_1+M_2)\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = 0.$$

so we have $x = \left(x_{ij}\right) \in {}_2\left({}_0BV_\sigma^I(M_1+M_2)\right)$.

Hence, ${}_2\left({}_0BV_\sigma^I(M_1)\right) \cap {}_2\left({}_0BV_\sigma^I(M_2)\right) \subseteq {}_2\left({}_0BV_\sigma^I(M_1+M_2)\right)$. For $\chi = {}_2BV_\sigma^I$ the inclusion are similar.

Corollary *$\chi \subseteq \chi(M)$ for $\chi = {}_2\left(BV_\sigma^I\right)$ and ${}_2BV_\sigma^I$.*

Proof. For this let $M(x) = x$, for all $x = \left(x_{ij}\right) \in X$. Let us suppose that $x = \left(x_{ij}\right) \in 2\left({}_0BV_\sigma^I\right)$. Then for any given $\epsilon>0$, we have

$$\left\{(i,j) : |\phi_{mnij}(x)| \ge \epsilon\right\} \in I.$$

Now let $A_1 = \left\{(i,j) : |\phi_{mnij}(x)| < \epsilon\right\} \in I$, be such that $A_1^c \in I$. Consider for $\rho > 0$,

$$M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = \frac{|\phi_{mnij}(x)|}{\rho} < \frac{\epsilon}{\rho} < \epsilon.$$

This implies that $I-\lim M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) = 0$, which shows that $x = \left(x_{ij}\right) \in {}_2\left({}_0BV_\sigma^I(M)\right)$.

Hence, we have

$$_2\left({}_0BV_\sigma^I\right) \subseteq {}_2\left({}_0BV_\sigma^I(M)\right).$$
$$\Rightarrow \chi \subseteq \chi(M).$$

Using the definition of convergence free sequence space, let us give another theorem which will be of particular importance in our future work:

Theorem 2.2 The spaces $_2\left({}_0BV_\sigma^I(M)\right)$ and $_2BV_\sigma^I(M)$ are not convergence free.

Example 2.1 To show this let $I = I_f$ and $M(x) = x$, for all $x = [0, \infty)$. Now consider the double sequence (x_{ij}), $\left(y_{ij}\right)$ which defined as follows:

$$x_{ij} = \frac{1}{i+j} \text{ and } y_{ij} = i + j, \forall i, j \in \mathbb{N}.$$

Then we have (x_{ij}) belong to both $_2\left({}_0BV_\sigma^I(M)\right)$ and $_2BV_\sigma^I(M)$, but $\left(y_{ij}\right)$ does not belong to $_2\left({}_0BV_\sigma^I(M)\right)$ and $_2BV_\sigma^I(M)$. Hence, the spaces $_2\left({}_0BV_\sigma^I(M)\right)$ and $_2BV_\sigma^I(M)$ are not convergence free.

To gain a good understanding of these double sequence spaces and related concepts, let us finally look at this theorem on inclusions:

Theorem 2.3 Let M be an Orlicz function. Then

$$_2\left({}_0BV_\sigma^I(M)\right) \subseteq {}_2BV_\sigma^I(M) \subseteq {}_2\left({}_\infty BV_\sigma^I(M)\right).$$

Proof. For this let us consider $x = (x_{ij}) \in {}_2\left({}_0BV_\sigma^I(M)\right)$. It is obvious that it must belong to $_2BV_\sigma^I(M)$. Now consider

$$M\left(\frac{|\phi_{mnij}(x) - L|}{\rho}\right) \leq M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) + M\left(\frac{|L|}{\rho}\right).$$

Now taking the limit on both sides we get

$$I - \lim_{ij} M\left(\frac{|\phi_{mnij}(x) - L|}{\rho}\right) = 0.$$

Hence $x = (x_{ij}) \in {}_2BV_\sigma^I(M)$. Now it remains to show that

$$_2\left(BV_\sigma^I(M)\right) \subseteq {}_2\left({}_\infty BV_\sigma^I(M)\right).$$

For this let us consider $x = (x_{ij}) \in {}_2BV_\sigma^I(M)$ this implies that there exist $\rho > 0$ s.t

$$I - \lim_{ij} M\left(\frac{|\phi_{mnij}(x) - L|}{\rho}\right) = 0.$$

Now consider,

$$M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) \leq M\left(\frac{|\phi_{mnij}(x) - L|}{\rho}\right) + M\left(\frac{|L|}{\rho}\right).$$

Now taking the supremum on both sides, we get

$$\sup_{ij} M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right) < \infty.$$

Hence, $x = (x_{ij}) \in {}_2\left({}_\infty BV_\sigma^I(M)\right)$. ∎

3. Paranorm bounded variation sequence spaces

In this section we study double sequence spaces by using the double sequences of strictly positive real numbers $p = \left(p_{ij}\right)$ with the help of BV_σ space and an Orlicz function M. We study some of its properties and prove some inclusion relations related to these new spaces. For m, n ≥ 0, we have

$$ {}_2BV_\sigma^I(M,p) = \{(x_{ij}) \in {}_2\omega : \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x)-L|}{\rho}\right)^{p_{ij}} \geq \epsilon\right\} \in I; \text{ for some } L \in \mathbb{C}, \rho>0 \tag{19}$$

$$ {}_2\left({}_0BV_\sigma^I(M,p)\right) = \left\{ (x_{ij}) \in {}_2\omega : \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} \geq \epsilon \right\} \in I, \rho>0 \right\}, \tag{20}$$

$$ {}_2\left({}_\infty BV_\sigma^I(M,p)\right) = \left\{ (x_{ij}) \in {}_2\omega : \left\{(i,j) : \exists K>0 : M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} \geq K \right\} \in I, \rho > 0 \right\} \tag{21}$$

$$ {}_2l_\infty(M,p) = \left\{ (x_{ij}) \in {}_2\omega : \sup M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} < \infty, \rho > 0 \right\}. \tag{22}$$

We also denote

$$ {}_2M_{BV_\sigma}^I(M,p) = {}_2BV_\sigma^I(M,p) \cap {}_2l_\infty(M,p)$$

and

$$ {}_2\left({}_0M_{BV_\sigma}^I(M,p)\right) = {}_2\left({}_0BV_\sigma^I(M,p)\right) \cap {}_2l_\infty(M,p).$$

We can now state and proof the theorems based on these double sequence spaces which are as follows:

Theorem 3.1 Let $p = \left(p_{ij}\right) \in {}_2l_\infty$ then the classes of double sequence ${}_2M_{BV_\sigma}^I(M,p)$ and ${}_2\left({}_0M_{BV_\sigma}^I(M,p)\right)$ are paranormed spaces, paranormed by

$$ g(x_{ij}) = \inf_{i,j\geq 1} \left\{ \rho^{\frac{p_{ij}}{H}} : \sup_{ij} M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} \leq 1, \text{ for some} \rho > 0 \right\}$$

where $H = \max\left\{1, \sup_{ij} p_{ij}\right\}$.

Proof. $(P1)$ It is clear that $g(x) = 0$ if and only if $x = 0$.

$(P2)$ $g(-x) = g(x)$ is obvious.

$(P3)$ Let $x = (x_{ij}), y = \left(y_{ij}\right) \in {}_2M_{BV_\sigma}^I(M,p)$. Now for $\rho_1, \rho_2 > 0$, we denote

$$ A_1 = \left\{ \rho_1 : \sup_{ij} M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} \leq 1 \right\} \tag{23}$$

$$ A_2 = \left\{ \rho_2 : \sup_{ij} M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} \leq 1 \right\} \tag{24}$$

Let us take $\rho_3 = \rho_1 + \rho_2$. Then by using the convexity of M, we have

$$M\left(\frac{|\phi_{mnij}(x+y)|}{\rho}\right) \leq \frac{\rho_1}{\rho_1+\rho_2} M\left(\frac{|\phi_{mnij}(x)|}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2} M\left(\frac{|\phi_{mnij}(y)|}{\rho_2}\right)$$

which in terms give us

$$\sup_{ij} M\left(\frac{|\phi_{mnij}(x+y)|}{\rho}\right)^{p_{ij}} \leq 1$$

and

$$\begin{aligned} g\left(x_{ij}+y_{ij}\right) &= \inf\left\{(\rho_1+\rho_2)^{\frac{p_{ij}}{H}} : \rho_1 \in A_1, \rho_2 \in A_2\right\} \\ &\leq \inf\left\{(\rho_1)^{\frac{p_{ij}}{H}} : \rho_1 \in A_1\right\} + \inf\left\{(\rho_2)^{\frac{p_{ij}}{H}} : \rho_2 \in A_2\right\} \\ &= g(x_{ij}) + g\left(y_{ij}\right). \end{aligned}$$

Therefore $g(x+y) \leq g(x) + g(y)$.

$(P4)$ Let (λ_{ij}) be a double sequence of scalars with $(\lambda_{ij}) \to \lambda\ (i,j \to \infty)$ and $(x_{ij}), L \in {}_2M^I_{BV_\sigma}(M,p)$ such that

$$x_{ij} \to L\ \ (i,j \to \infty),$$

in the sense that

$$g(x_{ij} - L) \to 0\ \ (i,j \to \infty).$$

Then, since the inequality

$$g(x_{ij}) \leq g(x_{ij} - L) + g(L)$$

holds by subadditivity of g, the sequence $g(x_{ij})$ is bounded.
Therefore,

$$\begin{aligned} g\left[(\lambda_{ij}x_{ij} - \lambda L)\right] &= g\left[(\lambda_{ij}x_{ij} - \lambda x_{ij} + \lambda x_{ij} - \lambda L)\right] \\ &= g\left[(\lambda_{ij} - \lambda)x_{ij} + \lambda(x_{ij} - L)\right] \\ &\leq g\left[(\lambda_{ij} - \lambda)x_{ij}\right] + g\left[\lambda(x_{ij} - L)\right] \\ &\leq \left|(\lambda_{ij} - \lambda)\right|^{\frac{p_{ij}}{M}} g(x_{ij}) + |\lambda|^{\frac{p_{ij}}{M}} g(x_{ij} - L) \to 0 \end{aligned}$$

as $(i,j \to \infty)$. That implies that the scalar multiplication is continuous. Hence ${}_2M^I_{BV_\sigma}(M,p)$ is a paranormed space. For another space ${}_2\left({}_0M^I_{BV_\sigma}(M,p)\right)$, the result is similar.

We shall see about the separability of these new defined double sequence spaces in the next theorem.

Theorem 3.2 The spaces ${}_2M^I_{BV_\sigma}(M,p)$ and ${}_2\left({}_0M^I_{BV_\sigma}(M,p)\right)$ are not separable.

Example 3.1 By counter example, we prove the above result for the space ${}_2M^I_{BV_\sigma}(M,p)$.

Let A be an infinite subset of increasing natural numbers, i.e., $A \subseteq \mathbb{N} \times \mathbb{N}$ such that $A \in I$.

Let

$$p_{ij} = \begin{cases} 1, & \text{if } (i,j) \in A \\ 2, & \text{otherwise.} \end{cases}$$

Let $P_0 = \{(x_{ij}) : x_{ij} = 0 \; or \; 1, \text{for } i, j \in M \text{ and } x_{ij} = 0, \text{otherwise}\}$.

Since A is infinite, so P_0 is uncountable. Consider the class of open balls

$$B_1 = \left\{B\left(z, \frac{1}{2}\right) : z \in P_0\right\}.$$

Let C_1 be an open cover of ${}_2M^I_{BV_\sigma}(M,p)$ containing B_1.

Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for ${}_2M^I_{BV_\sigma}(M,p)$. Thus ${}_2M^I_{BV_\sigma}(M,p)$ is not separable.

We shall now introduce a theorem which improves our work.

Theorem 3.3 Let (p_{ij}) and (q_{ij}) be two double sequences of positive real numbers. Then ${}_2\left({}_0M^I_{BV_\sigma}(M,p)\right) \supseteq {}_2\left({}_0M^I_{BV_\sigma}(M,q)\right)$ if and only if $\lim_{i,j \in K} \inf \frac{p_{ij}}{q_{ij}} > 0$, where $K^c \subseteq \mathbb{N} \times \mathbb{N}$ such that $K \in I$.

Proof. Let $\lim_{i,j \in K} \inf \frac{p_{ij}}{q_{ij}} > 0$ and $(x_{ij}) \in {}_2\left({}_0M^I_{BV_\sigma}(M,q)\right)$. Then, there exists $\beta > 0$ such that $p_{ij} > \beta \, q_{ij}$ for sufficiently large $(i,j) \in K$.

Since $(x_{ij}) \in {}_2\left({}_0M^I_{BV_\sigma}(M,q)\right)$. For a given $\epsilon > 0$, there exist $\rho > 0$ such that

$$B_0 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{q_{ij}} \geq \epsilon\right\} \in I.$$

Let $G_0 = K^c \cup B_0$. Then for all sufficiently large $(i,j) \in G_0$.

$$\left\{(i,j) : M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} \geq \epsilon\right\} \subseteq \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x)|}{\rho}\right)^{\beta q_{ij}} \geq \epsilon\right\} \in I.$$

Therefore, $(x_{ij}) \in {}_2\left({}_0M^I_{BV_\sigma}(M,p)\right)$. The converse part of the result follows obviously.

Remark 3.1 Let (p_{ij}) and (q_{ij}) be two double sequences of positive real numbers. Then ${}_2\left({}_0M^I_{BV_\sigma}(M,q)\right) \supseteq {}_2\left({}_0M^I_{BV_\sigma}(M,p)\right)$ if and only if $\lim_{i,j \in K} \inf \frac{q_{ij}}{p_{ij}} > 0$ and ${}_2\left({}_0M^I_{BV_\sigma}(M,q)\right) = {}_2\left({}_0M^I_{BV_\sigma}(M,p)\right)$ if and only if $\lim_{i,j \in K} \inf \frac{p_{ij}}{q_{ij}} > 0$ and $\lim_{i,j \in K} \inf \frac{q_{ij}}{p_{ij}} > 0$, where $K^c \subseteq \mathbb{N} \times \mathbb{N}$ such that $K \in I$.

Theorem 3.4 The set ${}_2M^I_{BV_\sigma}(M,p)$ is closed subspace of ${}_2l_\infty(M,p)$.

Proof. Let $\left(x^{(pq)}_{ij}\right)$ be a Cauchy double sequence in ${}_2M^I_{BV_\sigma}(M,p)$ such that $x^{(pq)} \to x$. We show that $x \in {}_2M^I_{BV_\sigma}(M,p)$. Since, $\left(x^{(pq)}_{ij}\right) \in {}_2M^I_{BV_\sigma}(M,p)$, then there exists a_{pq}, and $\rho > 0$ such that

$$\left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{pq}) - a_{pq}|}{\rho}\right)^{p_{ij}} \geq \epsilon\right\} \in I.$$

We need to show that

(1) (a_{pq}) converges to a.

(2) If $U = \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{pq}) - a|}{\rho}\right)^{p_{ij}} < \epsilon\right\}$, then $U^c \in I$.

Since $\left(x_{ij}^{(pq)}\right)$ be a Cauchy double sequence in ${}_2M_{BV_\sigma}^I(M,p)$ then for a given $\epsilon>0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sup_{ij} M\left(\frac{|\phi_{mnij}(x^{pq}) - \phi_{mnij}(x^{rs})|}{\rho}\right)^{p_{ij}} < \frac{\epsilon}{3}, \text{ for all } p,q,r,s \geq k_0.$$

For a given $\epsilon>0$, we have

$$B_{pqrs} = \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{pq}) - \phi_{mnij}(x^{rs})|}{\rho}\right)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\right\},$$
$$B_{pq} = \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{pq}) - a_{pq}|}{\rho}\right)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\right\},$$
$$B_{rs} = \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{rs}) - a_{rs}|}{\rho}\right)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\right\}.$$

Then $B_{pqrs}^c, B_{pq}^c, B_{rs}^c \in I$. Let $B^c = B_{pqrs}^c \cap B_{pq}^c \cap B_{rs}^c$,

where $B = \left\{(i,j) : M\left(\frac{|a_{pq} - a_{rs}|}{\rho}\right)^{p_{ij}} < \epsilon\right\}$, then $B^c \in I$. We choose $k_0 \in B^c$, then for each $p,q,r,s \geq k_0$, we have

$$\left\{(i,j) : M\left(\frac{|a_{pq} - a_{rs}|}{\rho}\right)^{p_{ij}} < \epsilon\right\} \supseteq \left[\left\{i,j \in \mathbb{N} : M\left(\frac{|\phi_{mnij}(x^{pq}) - \phi_{mnij}(x^{rs})|}{\rho}\right)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\right\}\right.$$
$$\cap \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{pq}) - a_{pq}|}{\rho}\right)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\right\}$$
$$\left.\cap \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{rs}) - a_{rs}|}{\rho}\right)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\right\}\right].$$

Then (a_{pq}) is a Cauchy double sequence in $\mathbb{C}$. So, there exists a scalar $a \in \mathbb{C}$ such that $(a_{pq}) \to a$, *as* $p,q \to \infty$.

(2) For the next step, let $0 < \delta < 1$ be given. Then, we show that if

$$U = \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{pq}) - a|}{\rho}\right)^{p_{ij}} \leq \delta\right\}$$

then $U^c \in I$. Since $x^{(pq)} \to x$, then there exists $p_0, q_0 \in \mathbb{N}$ such that,

$$P = \left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{p_0q_0}) - \phi_{mnij}(x)|}{\rho}\right)^{p_{ij}} < \left(\frac{\delta}{3D}\right)^H\right\} \tag{25}$$

where $D = \max\{1, 2^{G-1}\}$, $G = \sup_{ij} p_{ij} \geq 0$ and $H = \max\left\{1, \sup_{ij} p_{ij}\right\}$ implies $P^c \in I$. The number (p_0, q_0) can be so chosen that together with (25), we have

$$Q = \left\{(i,j) : M\left(\frac{|a_{p_0q_0} - a|}{\rho}\right)^{p_{ij}} < \left(\frac{\delta}{3D}\right)^H\right\}$$

such that $Q^c \in I$. Since $\left(x_{ij}^{(pq)}\right) \in {}_2M_{BV_\sigma}^I(M,p)$.

We have

$$\left\{(i,j) : M\left(\frac{|\phi_{mnij}(x^{p_0q_0}) - a_{p_0q_0}|}{\rho}\right)^{p_{ij}} \geq \delta\right\} \in I.$$

Then we have a subset $S \subseteq \mathbb{N} \times \mathbb{N}$ such that $S^c \in I$, where

$$S = \left\{ (i,j) : M\left(\frac{|\phi_{mnij}(x^{p_0 q_0}) - a_{p_0 q_0}|}{\rho} \right)^{p_{ij}} < \left(\frac{\delta}{3D} \right)^H \right\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where

$$U = \left\{ (i,j) : M\left(\frac{|\phi_{mnij}(x) - a|}{\rho} \right)^{p_{ij}} < \delta \right\}$$

Therefore, for $(i,j) \in U^c$, we have

$$\begin{aligned} &\left\{ (i,j) : M\left(\tfrac{|\phi_{mnij}(x) - a|}{\rho} \right)^{p_{ij}} < \delta \right\} \\ \supseteq &\left[\left\{ (i,j) : M\left(\tfrac{|\phi_{mnij}(x^{p_0 q_0}) - \phi_{mnij}(x)|}{\rho} \right)^{p_{ij}} < \left(\tfrac{\delta}{3D} \right)^H \right\} \right. \\ \cap &\left\{ (i,j) : M\left(\frac{|a_{p_0 q_0} - a|}{\rho} \right)^{p_{ij}} < \left(\frac{\delta}{3D} \right)^M \right\} \\ \cap &\left. \left\{ (i,j) : M\left(\frac{|\phi_{mnij}(x^{p_0 q_0}) - a_{p_0 q_0}|}{\rho} \right)^{p_{ij}} < \left(\frac{\delta}{3D} \right)^H \right\} \right]. \end{aligned}$$

Hence the result ${}_2M^I_{BV_\sigma}(M,p) \subset {}_2l_\infty(M,p)$ follows.

Since the inclusions ${}_2M^I_{BV_\sigma}(M,p) \subset {}_2l_\infty(M,p)$ and ${}_2\left({}_0M^I_{BV_\sigma}(M,p)\right) \subset {}_2l_\infty(M,p)$ are strict so in view of Theorem (3.3), we have the following result.

The above theorem is interesting and itself will have various applications in our future work.

4. Bounded variation sequence spaces defined by modulus function

In this section, we study some new double sequence spaces of invariant means defined by ideal and modulus function. Furthermore, we also study several properties relevant to topological structures and inclusion relations between these spaces. The following classes of double sequence spaces are as follows:

$${}_2BV^I_\sigma(f) = \left\{ (x_{ij}) \in {}_2\omega : \left\{ (i,j) : \sum_{m,n=0}^{\infty} f\left(|\phi_{mnij}(x) - L| \right) \geq \epsilon \right\} \in I; \text{for some } L \in \mathbb{C} \right\}; \tag{26}$$

$${}_2\left({}_0BV^I_\sigma(f)\right) = \left\{ (x_{ij}) \in {}_2\omega : \left\{ (i,j) : \sum_{m,n=0}^{\infty} f\left(|\phi_{mnij}(x)| \right) \geq \epsilon \right\} \in I \right\}; \tag{27}$$

$${}_2\left({}_\infty BV^I_\sigma(f)\right) = \left\{ (x_{ij}) \in {}_2\omega : \left\{ (i,j) : \exists K>0 : \sum_{m,n=0}^{\infty} f\left(|\phi_{mnij}(x)| \right) \geq K \right\} \in I \right\}; \tag{28}$$

$${}_2\left({}_\infty BV_\sigma(f)\right) = \left\{ (x_{ij}) \in {}_2\omega : \sup_{i,j} \sum_{m,n=0}^{\infty} f\left(|\phi_{mnij}(x)| \right) < \infty \right\}. \tag{29}$$

We also denote

$$_2M^I_{BV_\sigma}(f) = {_2}BV^I_\sigma(f) \cap {_2}({_\infty}BV_\sigma(f))$$

and

$$_2\left({_0}M^I_{BV_\sigma}(f)\right) = {_2}\left({_0}BV^I_\sigma(f)\right) \cap {_2}({_\infty}BV_\sigma(f)).$$

We shall now consider important theorems of these double sequence spaces by using modulus function.

Theorem 4.1 *For any modulus function f, the classes of double sequence $_2\left({_0}BV^I_\sigma(f)\right)$, $_2BV^I_\sigma(f)$, $_2\left({_0}M^I_{BV_\sigma}(f)\right)$ and $_2M^I_{BV_\sigma}(f)$ are linear spaces.*

Proof. Suppose $x = (x_{ij})$ and $y = \left(y_{ij}\right) \in {_2}BV^I_\sigma(f)$ be any two arbitrary elements. Let α, β are scalars. Now, since $(x_{ij}), \left(y_{ij}\right) \in {_2}BV^I_\sigma(f)$. Then this implies that there exists some positive numbers $L_1, L_2 \in \mathbb{C}$ and such that the sets

$$A_1 = \left\{(i,j) : \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(x) - L_1|\right) \geq \frac{\epsilon}{2}\right\} \in I, \tag{30}$$

$$A_2 = \left\{(i,j) : \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(y) - L_2|\right) \geq \frac{\epsilon}{2}\right\} \in I. \tag{31}$$

Now, assume

$$B_1 = \left\{(i,j) : \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(x) - L_1|\right) < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I), \tag{32}$$

$$B_2 = \left\{(i,j) : \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(y) - L_2|\right) < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I) \tag{33}$$

be such that $B_1^c, B_2^c \in I$. Since f is a modulus function, we have

$$\begin{aligned}
&\sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|\right)\\
&= \sum_{m,\,n=0}^{\infty} f\left(|\left(\alpha\phi_{mnij}(x) + \beta\phi_{mnij}(y)\right) - (\alpha L_1 + \beta L_2)|\right)\\
&= \sum_{m,\,n=0}^{\infty} f\left(|\alpha\left(\phi_{mnij}(x) - L_1\right) + \beta\left(\phi_{mnij}(y) - L_2\right)|\right)\\
&\leq \sum_{m,\,n=0}^{\infty} f\left(|\alpha||\phi_{mnij}(x) - L_1|\right) + \sum_{m,\,n=0}^{\infty} f\left(|\beta||\phi_{mnij}(y) - L_2|\right)\\
&\leq \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(x) - L_1|\right) + \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(y) - L_2|\right)\\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

This implies that $\left\{(i,j) : \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|\right) \geq \epsilon\right\} \in I$. Thus $\alpha(x_{ij}) + \beta\left(y_{ij}\right) \in {_2}BV^I_\sigma(f)$. As (x_{ij}) and $\left(y_{ij}\right)$ are two arbitrary element then

$\alpha(x_{ij}) + \beta(y_{ij}) \in {}_2BV_\sigma^I(f)$ for all (x_{ij}), $(y_{ij}) \in {}_2BV_\sigma^I(f)$ and for all scalars α, β. Hence ${}_2BV_\sigma^I(f)$ is linear space. The proof for other spaces will follow similarly. ■

We may go a step further and define another theorem on ideal convergence which basically depends upon the set in the filter associated with the same ideal.

Theorem 4.2 A sequence $x = (x_{ij}) \in {}_2M_{BV_\sigma}^I(f)$ I-convergent if and only if for every $\epsilon>0$, there exists $M_\varepsilon, N_\varepsilon \in \mathbb{N}$ such that

$$\left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij}) - \phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) < \epsilon\right\} \in \mathcal{F}(I).$$

Proof. Let $x = (x_{ij}) \in {}_2M_{BV_\sigma}^I(f)$. Suppose $I - \lim x = L$. Then, the set

$$B_\epsilon = \left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij}) - L|\right) < \frac{\epsilon}{2}\right\} \in F(I), \quad \text{for all} \quad \epsilon>0.$$

Fix $M_\varepsilon, N_\varepsilon \in B_\varepsilon$. Then we have

$$\begin{aligned}\sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij}) - \phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) &\leq \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{M_\epsilon, N_\epsilon}) - L|\right) \\ &+ \sum_{m,\, n=0}^{\infty} f\left(|L - \phi_{mnij}(x_{ij})|\right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

which holds for all $(i,j) \in B_\epsilon$.
Hence

$$\left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij}) - \phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) < \epsilon\right\} \in \mathcal{F}(I).$$

Conversely, suppose that

$$\left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij}) - \phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) < \epsilon\right\} \in \mathcal{F}(I).$$

Then, being f a modulus function and by using basic triangular inequality, we have

$$\left\{(i,j) : \left|\sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij})|\right) - \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right)\right| < \epsilon\right\} \in F(I), \text{for all} \quad \epsilon>0.$$

Then, the set

$$\begin{aligned}C_\varepsilon = \Bigg\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij})|\right) \in \\ \left[\sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) - \epsilon, \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) + \epsilon\right]\Bigg\} \in \mathcal{F}(I).\end{aligned}$$

Let $J_\epsilon = \left[\sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) - \epsilon, \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{M_\epsilon, N_\epsilon})|\right) + \epsilon\right]$.

If we fix $\epsilon > 0$ then, we have $C_\epsilon \in F(I)$ as well as $C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$.
Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi.$$

That is

$$\left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij})|\right) \in J\right\} \in \mathcal{F}(I).$$

This shows that

$$diam\ J \leq diam\ J_\epsilon$$

where the *diam J* denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I_k \supseteq \ldots$$

with the property that $diam\ I_k \leq \frac{1}{2} diam\ I_{k-1}$ for $(k = 2, 3, 4, \ldots)$ and $\left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij})|\right) \in I_k\right\} \in F(I)$ for $(k = 1, 2, 3, 4, \ldots)$.
Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that

$$\xi = I - \lim_{i,j} \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x_{ij})|\right),$$

showing that $x = (x_{ij}) \in {}_2M^I_{BV_\sigma}(f)$ is I-convergent. Hence the result holds.
As the reader knows about solid and monotone sequence space now turn to theorem on solid and monotone double sequence spaces of invariant mean defined by ideal and modulus function.

Theorem 4.3 For any modulus function f, the spaces ${}_2\left({}_0BV^I_\sigma(f)\right)$ and ${}_2\left({}_0M^I_{BV_\sigma}(f)\right)$ are solid and monotone.

Proof. We consider ${}_2\left({}_0BV^I_\sigma(f)\right)$ and for ${}_2\left({}_0M^I_{BV_\sigma}(f)\right)$ the proof shall be similar.
Let $x = (x_{ij}) \in {}_2\left({}_0BV^I_\sigma(f)\right)$ be an arbitrary element, then the set

$$\left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x)|\right) \geq \epsilon\right\} \in I. \tag{34}$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$.
Now, since f is a modulus function. Then the result follows from (2.18) and the inequality

$$f\left(|\alpha_{ij}\phi_{mnij}(x)|\right) \leq |\alpha_{ij}| f\left(|\phi_{mnij}(x)|\right) \leq f\left(|\phi_{mnij}(x)|\right).$$

Therefore,

$$\left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\alpha_{ij}\phi_{mnij}(x)|\right) \geq \epsilon\right\} \subseteq \left\{(i,j) : \sum_{m,\, n=0}^{\infty} f\left(|\phi_{mnij}(x)|\right) \geq \epsilon\right\} \in I$$

implies that

$$\left\{(i,j): \sum_{m,\,n=0}^{\infty} f\left(|\alpha_{ij}\phi_{mnij}(x)|\right) \geq \epsilon\right\} \in I.$$

Thus we have $(\alpha_{ij}x_{ij}) \in {}_2({}_0BV_\sigma^I(f))$. Hence ${}_2({}_0BV_\sigma^I(f))$ is solid. Therefore ${}_2({}_0BV_\sigma^I(f))$ is monotone. Since every solid sequence space is monotone.

Remark 4.1 The space ${}_2BV_\sigma^I(f)$ and ${}_2\left(M_{BV_\sigma^I}(f)\right)$ are neither solid nor monotone in general.

Example 4.1 Here we give counter example for establishment of this result. Let $X={}_2BV_\sigma^I$ and ${}_2\left(M_{BV_\sigma^I}\right)$. Let us consider $I = I_f$ and $f(x) = x$, for all $x = (x_{ij})$ and $x_{ij} \in [0, \infty)$. Consider, the K-step space $X_K(f)$ of $X(f)$ defined as follows:

Let $x = (x_{ij}) \in X(f)$ and $y = (y_{ij}) \in X_K(f)$ be such that

$$y_{ij} = \begin{cases} x_{ij}, & \text{if } i,\ j \text{ are even} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_{ij}) defined by $(x_{ij}) = 1$ for all $i, j \in \mathbb{N}$.

Then, $x = (x_{ij}) \in {}_2BV_\sigma^I(f)$ and ${}_2M_{BV_\sigma^I}(f)$, but K-step space preimage does not belong to $BV_\sigma^I(f)$ and ${}_2M_{BV_\sigma}^I(f)$. Thus, ${}_2BV_\sigma^I(f)$ and ${}_2M_{BV_\sigma}^I(f)$ are not monotone and hence they are not solid.

After discussing about solid and monotone sequence space now we come to the concept of sequence algebra which will help to understand our further work.

Theorem 4.4 For any modulus function f, the spaces ${}_2({}_0BV_\sigma^I(f))$ and ${}_2BV_\sigma^I(f)$ are sequence algebra.

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2({}_0BV_\sigma^I(f))$ be any two arbitrary elements.

Then, the sets

$$\left\{(i,j): \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(x)|\right) \geq \epsilon\right\} \in I$$

and

$$\left\{(i,j): \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(y)|\right) \geq \epsilon\right\} \in I.$$

Therefore,

$$\left\{(i,j): \sum_{m,\,n=0}^{\infty} f\left(|\phi_{mnij}(x).\phi_{mnij}(y)|\right) \geq \epsilon\right\} \in I.$$

Thus, we have $(x_{ij}).(y_{ij}) \in {}_2({}_0BV_\sigma^I(f))$. Hence ${}_2({}_0BV_\sigma^I(f))$ is sequence algebra. And for ${}_2BV_\sigma^I(f)$ the result can be proved similarly.

Remark 4.2 If I is not maximal and $I \neq I_f$ then the spaces ${}_2BV_\sigma^I(f)$ and ${}_2({}_0BV_\sigma^I(f))$ are not symmetric.

Example 4.2 Let $A \in I$ be an infinite set and $f(x) = x$ for all $x = (x_{ij})$ and $x_{ij} \in [0, \infty)$. If

$$x_{ij} = \begin{cases} 1, & \text{if } (i,j) \in A \\ 0, & \text{otherwise} \end{cases}$$

Then, it is clearly seen that $(x_{ij}) \in {}_2({}_0BV_\sigma^I(f)) \subset {}_2BV_\sigma^I(f)$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be such that $K \notin I$ and $K^c \notin I$. Let $\phi : K \to A$ and $\psi : K^c \to A^c$ be a bijective maps (as all four sets are infinite). Then, the mapping $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by

$$\pi(i,j) = \begin{cases} \phi(i,j), & \text{if } (i,j) \in K \\ \psi(i,j), & \text{otherwise.} \end{cases}$$

is a permutation on $\mathbb{N} \times \mathbb{N}$.

But $(x_{\pi(i,j)}) \notin {}_2BV_\sigma^I(f)$ and hence $(x_{\pi(i,j)}) \notin {}_2({}_0BV_\sigma^I(f))$ showing that ${}_2BV_\sigma^I(f)$ and ${}_2({}_0BV_\sigma^I(f))$ are not symmetric double sequence spaces.

5. Conclusion

In this chapter, we study different forms of BV_σ double sequence spaces of invariant means with the help of ideal, operators and some functions such as Orlicz function and modulus function. The chapter shows the potential of the new theoretical tools to deal with the convergence problems of sequences in sigma bounded variation, occurring in many branches of science, engineering and applied mathematics.

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Conflict of interest

The authors declare that they have no competing interests.

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