We are IntechOpen, the world’s leading publisher of Open Access books
Built by scientists, for scientists

6,600
Open access books available

177,000
International authors and editors

195M
Downloads

154
Countries delivered to

TOP 1%
Our authors are among the most cited scientists

12.2%
Contributors from top 500 universities

WEB OF SCIENCE™
Selection of our books indexed in the Book Citation Index in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com
Abstract

In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. Finally, we investigate the zeros of the 3-variable Hermite polynomials by using computer.

Keywords: differential equations, heat equation, Hermite polynomials, the 3-variable Hermite polynomials, generating functions, complex zeros

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers see [1–15]. The special polynomials of two variables provided new means of analysis for the solution of a wide class of differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

In [1], the Hermite polynomials are given by the exponential generating function

\[ \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{2xt - t^2}. \]

We can also have the generating function by using Cauchy’s integral formula to write the Hermite polynomials as
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \frac{n!}{2^n} \int e^{2te^2} \frac{e^{-t}}{t^{n+1}} dt$$

with the contour encircling the origin. It follows that the Hermite polynomials also satisfy the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

Further, the two variables Hermite Kampé de Fériet polynomials $H_n(x, y)$ defined by the generating function (see [3])

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{\mu^n}{n!} = e^{x^2 + y^2 t}$$

are the solution of heat equation

$$\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y), \quad H_n(x, 0) = x^n.$$ We note that

$$H_n(2x, -1) = H_n(x).$$

The 3-variable Hermite polynomials $H_n(x, y, z)$ are introduced [4].

$$H_n(x, y, z) = n! \sum_{k=0}^{[n/3]} \frac{\phi^k H_n-3k(x, y) k!(n-3k)!}{k!}.$$ The differential equation and the generating function for $H_n(x, y, z)$ are given by

$$\left(3z \frac{\partial^3}{\partial x^3} + 2y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n\right)H_n(x, y, z) = 0$$

and

$$e^{x^2 + y^2 t + z^3} = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{\mu^n}{n!}$$

respectively.

By (2), we get

$$\sum_{n=0}^{\infty} H_n(x_1 + x_2, y, z) \frac{\mu^n}{n!} = e^{(x_1 + x_2)^2 + y^2 + z^3}$$

$$= \sum_{n=0}^{\infty} x_2^{n} \frac{\mu^n}{n!} \sum_{n=0}^{\infty} H_n(x_1, y, z) \frac{\mu^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y, z) x_2^{n-l} \right) \frac{\mu^n}{n!}.$$
By comparing the coefficients on both sides of (3), we have the following theorem.

**Theorem 1.** For any positive integer \( n \), we have

\[
H_n(x_1 + x_2, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y, z)x_2^{n-l}.
\]

Applying Eq. (2), we obtain

\[
\sum_{n=0}^{\infty} H_n(x, y, z_1 + z_2) \frac{t^n}{n!} = e^{x^2 + y^2 + (z_1 + z_2)t^2}
\]

\[
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} H_l(x, y, z_1) \frac{t^l}{l!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{H_{n-3k}(x, y, z_1) z_1^k}{k!(n-3k)!} \right) \frac{t^n}{n!}.
\]

On equating the coefficients of the like power of \( t \) in the above, we obtain the following theorem.

**Theorem 2.** For any positive integer \( n \), we have

\[
H_n(x, y, z_1 + z_2) = n! \sum_{k=0}^{n} \frac{H_{n-3k}(x, y, z_1) z_1^k}{k!(n-3k)!}.
\]

Also, the 3-variable Hermite polynomials \( H_n(x, y, z) \) satisfy the following relations

\[
\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z),
\]

and

\[
\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z).
\]

The following elementary properties of the 3-variable Hermite polynomials \( H_n(x, y, z) \) are readily derived form (2). We, therefore, choose to omit the details involved.

**Theorem 3.** For any positive integer \( n \), we have

1. \( H_n(2x, -1, 0) = H_n(x) \).
2. \( H_n(x, y_1 + y_2, z) = n! \sum_{k=0}^{n} \frac{H_{n-3k}(x, y_1, z) y_1^k}{k!(n-2k)!} \).
3. \( H_n(x, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_l(x) H_{n-l}(-x, y + 1, z) \).
Theorem 4. For any positive integer \( n \), we have

\[ H_n(x_1 + x_2, y_1 + y_2, z) = \sum_{i=0}^{n} \binom{n}{i} H_i(x_1, y_1, z) H_{n-i}(x_2, y_2). \]

The 3-variable Hermite polynomials can be determined explicitly. A few of them are

\[
\begin{align*}
H_0(x, y, z) &= 1, \\
H_1(x, y, z) &= x, \\
H_2(x, y, z) &= x^2 + 2y, \\
H_3(x, y, z) &= x^3 + 6xy + 6z, \\
H_4(x, y, z) &= x^4 + 12x^2y + 12y^2 + 24xz, \\
H_5(x, y, z) &= x^5 + 20x^3y + 60x^2y^2 + 60x^2z + 120yz, \\
H_6(x, y, z) &= x^6 + 30x^4y + 180x^3y^2 + 120x^3z + 720xyz + 360z^2, \\
H_7(x, y, z) &= x^7 + 42x^5y + 420x^4y^2 + 840x^4z + 520x^2yz + 2520x^2z^2 + 2520xz^2, \\
H_8(x, y, z) &= x^8 + 56x^6y + 840x^5y^2 + 3360x^5z + 504x^3y^3 + 504x^3z^3 + 720x^2yz + 6720x^2z^2 + 20160xy^2z + 10080x^2y^2z + 20160yz^2.
\end{align*}
\]

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials (see [7, 8, 12, 16-19]). In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. In addition, we investigate the zeros of the 3-variable Hermite polynomials using numerical methods. Using computer, a realistic study for the zeros of the 3-variable Hermite polynomials is very interesting. Finally, we observe an interesting phenomenon of ‘scattering’ of the zeros of the 3-variable Hermite polynomials.

2. Differential equations associated with the 3-variable Hermite polynomials

In this section, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials.

Let

\[ F(t, x, y, z) = e^{tx^2 + ey^2 + 2tz} = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}, \quad x, y, z, t \in \mathbb{C}. \]
Then, by (4), we have
\[
F^{(1)} = \frac{\partial}{\partial t} F(t, x, y, z) = \frac{\partial}{\partial t} \left( e^{xt+yt+zt} \right) = e^{xt+yt+zt} (x + 2yt + 3zt^2) \tag{5}
\]

Continuing this process, we can guess that
\[
F^{(2)} = \frac{\partial}{\partial t} F^{(1)} (t, x, y, z) = (2yt + 3zt^2) F(t, x, y, z) + (x + 2yt + 3zt^2) F^{(1)} (t, x, y, z) \tag{6}
\]

Continuing this process, we can guess that
\[
F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \sum_{i=0}^{2N} a_i(N, x, y, z) t^i F(t, x, y, z), \quad (N = 0, 1, 2, \ldots) \tag{7}
\]

Differentiating (7) with respect to $t$, we have
\[
F^{(N+1)} = \frac{\partial F^{(N)}}{\partial t} = \sum_{i=0}^{2N} a_i(N, x, y, z) t^{i-1} F(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z) t^i F^{(1)} (t, x, y, z) \]
\[
= \sum_{i=0}^{2N} a_i(N, x, y, z) t^{i-1} F(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z) t^i \left( x + 2yt + 3zt^2 \right) F(t, x, y, z) \tag{8}
\]
\[
= \sum_{i=0}^{2N} a_i(N, x, y, z) t^{i-1} F(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z) t^i F(t, x, y, z) \]
\[
+ \sum_{i=0}^{2N} 2a_i(N, x, y, z) t^{i+1} F(t, x, y, z) + \sum_{i=0}^{2N} 3a_i(N, x, y, z) t^{i+2} F(t, x, y, z) \]
\[
+ \sum_{i=0}^{2N-1} (i+1) a_{i+1}(N, x, y, z) t^i F(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z) t F(t, x, y, z) \]
\[
+ \sum_{i=0}^{2N+2} 3a_{i+2}(N, x, y, z) t F(t, x, y, z)
\]

Hence we have
\[
F^{(N+1)} = \sum_{i=0}^{2N-1} (i+1) a_{i+1}(N, x, y, z) t^i F(t, x, y, z) + \sum_{i=0}^{2N} x a_i(N, x, y, z) t^i F(t, x, y, z) \]
\[
+ \sum_{i=1}^{2N+2} 2ya_{i+2}(N, x, y, z) t^i F(t, x, y, z) \tag{8}
\]

\[
F^{(N+1)} = \sum_{i=0}^{2N} x a_i(N, x, y, z) t^i F(t, x, y, z) + \sum_{i=1}^{2N+1} 2ya_{i+1}(N, x, y, z) t^i F(t, x, y, z) + \sum_{i=2}^{2N+2} 3za_{i+2}(N, x, y, z) t^i F(t, x, y, z).
\]
Now replacing $N$ by $N + 1$ in (7), we find

$$F^{(N+1)} = \sum_{i=0}^{2N+2} a_i (N+1, x, y, z) t^i F(t, x, y, z).$$  

(9)

Comparing the coefficients on both sides of (8) and (9), we obtain

$$a_0(N + 1, x, y, z) = a_0(N, x, y, z) + x a_0(N, x, y, z),$$
$$a_1(N + 1, x, y, z) = 2a_2(N, x, y, z) + x a_1(N, x, y, z) + 2y a_0(N, x, y, z),$$
$$a_{2N}(N + 1, x, y, z) = x a_{2N}(N, x, y, z) + 2y a_{2N-1}(N, x, y, z) + 3z a_{2N-2}(N, x, y, z),$$
$$a_{2N+1}(N + 1, x, y, z) = 2y a_{2N}(N, x, y, z) + 3z a_{2N-1}(N, x, y, z),$$
$$a_{2N+2}(N + 1, x, y, z) = 3z a_{2N}(N, x, y, z),$$

and

$$a_i(N + 1, x, y, z) = (i + 1) a_{i+1}(N, x, y, z) + x a_i(N, x, y, z)$$
$$+ 2y a_{i-1}(N, x, y, z) + 3z a_{i-2}(N, x, y, z), \quad (2 \leq i \leq 2N - 1).$$  

(10)

In addition, by (7), we have

$$F(t, x, y, z) = F^{(0)}(t, x, y, z) = a_0(0, x, y, z) F(t, x, y, z),$$

(12)

which gives

$$a_0(0, x, y, z) = 1.$$  

(13)

It is not difficult to show that

$$xF(t, x, y) + 2ytF(t, x, y, z) + 3z t^2 F(t, x, y, z)$$
$$= F^{(1)}(t, x, y, z)$$
$$= \sum_{i=0}^{2} a_i(1, x, y, z) F(t, x, y, z)$$

(14)

Thus, by (14), we also find

$$a_0(1, x, y, z) = x, \quad a_1(1, x, y, z) = 2y, \quad a_2(1, x, y, z) = 3z.$$  

(15)

From (10), we note that

$$a_0(N + 1, x, y, z) = a_1(N, x, y, z) + x a_0(N, x, y, z),$$
$$a_0(N, x, y, z) = a_1(N - 1, x, y, z) + x a_0(N - 1, x, y, z),$$
$$a_0(N + 1, x, y, z) = \sum_{i=0}^{N} x^i a_i(N - i, x, y, z) + x^{N+1},$$

(16)

and
Theorem 5. Therefore, we obtain the following theorem.

\[ a_{2N+2}(N+1, x, y, z) = 3za_{2N}(N, x, y, z), \]
\[ a_{2N}(N, x, y, z) = 3za_{2N-2}(N - 1, x, y, z), \] ...
\[ a_{2N+2}(N+1, x, y, z) = (3z)^{N+1}. \] (17)

Note that, here the matrix \( a_i(j, x, y) \) is given by

\[
\begin{pmatrix}
1 & 2y + x^2 & \cdots & \cdots \\
0 & 2y & 4xy + 6z & \cdots & \cdots \\
0 & 3z & 6xz + 4y^2 & \cdots & \cdots \\
0 & 0 & 12yz & \cdots & \cdots \\
0 & 0 & (3z)^2 & \cdots & \cdots \\
0 & 0 & (3z)^3 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & (3z)^{N+1}
\end{pmatrix}
\]

Therefore, we obtain the following theorem.

**Theorem 5.** For \( N = 0, 1, 2, \ldots \), the differential equation

\[ F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \left( \sum_{i=0}^{N} a_i(N, x, y, z)t^i \right) F(t, x, y, z) \]

has a solution

\[ F = F(t, x, y, z) = e^{2yt^2 + z^2}, \]

where

\[
\begin{align*}
a_0(N+1, x, y, z) &= \sum_{i=0}^{N} x^i a_1(N - i, x, y, z) + x^{N+1}, \\
a_1(N+1, x, y, z) &= 2a_2(N, x, y, z) + xa_1(N, x, y, z) + 2ya_0(N, x, y, z), \\
a_{2N}(N+1, x, y, z) &= x_0 a_{2N}(N, x, y, z) + 2ya_{2N-1}(N, x, y, z) + 3za_{2N-2}(N, x, y, z), \\
a_{2N+1}(N+1, x, y, z) &= 2ya_{2N}(N, x, y, z) + 3za_{2N-1}(N, x, y, z), \\
a_{2N+2}(N+1, x, y, z) &= (3z)^{N+1},
\end{align*}
\]
\[
a_i(N + 1, x, y, z) = (i + 1)a_{i+1}(N, x, y, z) + xa_i(N, x, y, z) \\
+ 2ya_{i-1}(N, x, y, z) + 3za_i(N, x, y, z), \quad (2 \leq i \leq 2N - 1).
\]

From (4), we note that
\[
F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \sum_{k=0}^{\infty} H_{k+N}(x, y, z) \frac{t^k}{k!}.
\tag{18}
\]

By (4) and (18), we get
\[
e^{-mt} \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} \frac{t^k}{k!} \left( \sum_{m=0}^{\infty} H_{m+N}(x, y, z) \frac{t^m}{m!} \right)
\tag{19}
\]

By the Leibniz rule and the inverse relation, we have
\[
e^{-mt} \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} \frac{t^k}{k!} \left( \sum_{m=0}^{\infty} H_{m+N}(x, y, z) \frac{t^m}{m!} \right)
\tag{20}
\]

Hence, by (19) and (20), and comparing the coefficients of \( \frac{t^m}{m!} \) gives the following theorem.

**Theorem 6.** Let \( m, n, N \) be nonnegative integers. Then
\[
\sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} H_{N+k}(x, y, z) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} H_{m+k}(x - n, y, z).
\tag{21}
\]

If we take \( m = 0 \) in (21), then we have the following corollary.

**Corollary 7.** For \( N = 0, 1, 2, \ldots \), we have
\[
H_N(x, y, z) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} H_k(x - n, y, z).
\]

For \( N = 0, 1, 2, \ldots \), the differential equation
\[
F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \left( \sum_{i=0}^{N} a_i(N, x, y, z) t^i \right) F(t, x, y, z)
\]

has a solution
\[ F = F(t, x, y, z) = e^{xt} + y^2 + z^2. \]

Here is a plot of the surface for this solution. In Figure 1(left), we choose \(-2 \leq z \leq -2, -1 \leq t \leq 1, x = 2,\) and \(y = -4.\) In Figure 1(right), we choose \(-5 \leq x \leq 5, -1 \leq t \leq 1, y = -3,\) and \(z = -1.\)

3. Distribution of zeros of the 3-variable Hermite polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the 3-variable Hermite polynomials \(H_n(x, y, z).\) By using computer, the 3-variable Hermite polynomials \(H_n(x, y, z)\) can be determined explicitly. We display the shapes of the 3-variable Hermite polynomials \(H_n(x, y, z)\) and investigate the zeros of the 3-variable Hermite polynomials \(H_n(x, y, z)\). We investigate the beautiful zeros of the 3-variable Hermite polynomials \(H_n(x, y, z)\) by using a computer. We plot the zeros of the \(H_n(x, y, z)\) for \(n = 20, y = 1, -1 + i, -1 - i, z = 3, -3, 3 + i, -3 - i\) and \(x \in \mathbb{C}\) (Figure 2). In Figure 2(top-left), we choose \(n = 20, y = 1,\) and \(z = 3.\) In Figure 2(top-right), we choose \(n = 20, y = -1,\) and \(z = -3.\) In Figure 2(bottom-left), we choose \(n = 20, y = 1 + i,\) and \(z = 3 + i.\) In Figure 2(bottom-right), we choose \(n = 20, y = -1 - i,\) and \(z = -3 - i.\)

In Figure 3(top-left), we choose \(n = 20, x = 1,\) and \(y = 1.\) In Figure 3(top-right), we choose \(n = 20, x = -1,\) and \(y = -1.\) In Figure 3(bottom-left), we choose \(n = 20, x = 1 + i,\) and \(y = 1 + i.\) In Figure 3(bottom-right), we choose \(n = 20, x = -1 - i,\) and \(y = -1 - i.\)

Stacks of zeros of the 3-variable Hermite polynomials \(H_n(x, y, z)\) for \(1 \leq n \leq 20\) from a 3-D structure are presented (Figure 3). In Figure 4(top-left), we choose \(n = 20, y = 1,\) and \(z = 3.\) In Figure 4(top-right), we choose \(n = 20, y = -1,\) and \(z = -3.\) In Figure 4(bottom-left), we choose \(n = 20, y = 1 + i,\) and \(z = 3 + i.\) In Figure 4(bottom-right), we choose \(n = 20, y = -1 - i,\) and \(z = -3 - i.\)

Figure 1. The surface for the solution \(F(t, x, y, z).\)
Our numerical results for approximate solutions of real zeros of the 3-variable Hermite polynomials \( H_n(x, y, z) \) are displayed (Tables 1–3).

The plot of real zeros of the 3-variable Hermite polynomials \( H_n(x, y, z) \) for \( 1 \leq n \leq 20 \) structure are presented (Figure 5).

In Figure 5(left), we choose \( y = 1 \) and \( z = 3 \). In Figure 5(right), we choose \( y = -1 \) and \( z = -3 \).

Stacks of zeros of \( H_n(x, -2, 4) \) for \( 1 \leq n \leq 40 \), forming a 3D structure are presented (Figure 6). In Figure 6(top-left), we plot stacks of zeros of \( H_n(x, -2, 4) \) for \( 1 \leq n \leq 20 \). In Figure 6(top-right), we draw \( x \) and \( y \) axes but no \( z \) axis in three dimensions. In Figure 6(bottom-left), we draw \( y \)
and $z$ axes but no $x$ axis in three dimensions. In Figure 6 (bottom-right), we draw $x$ and $z$ axes but no $y$ axis in three dimensions.

It is expected that $H_n(x, y, z), x \in \mathbb{C}, y, z \in \mathbb{R}$, has $\text{Im}(x) = 0$ reflection symmetry analytic complex functions (see Figures 2–7). We observe a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $y, z \in \mathbb{R}$. We also hope to verify a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $y, z \in \mathbb{R}$ (Tables 1 and 2). Next, we calculated an approximate solution satisfying $H_n(x, y, z) = 0, x \in \mathbb{C}$. The results are given in Tables 3 and 4.
The plot of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \leq n \leq 20$ structure are presented (Figure 7).

In Figure 7(left), we choose $x = 1$ and $y = 2$. In Figure 7(right), we choose $x = -1$ and $y = -2$.

Finally, we consider the more general problems. How many zeros does $H_n(x, y, z)$ have? We are not able to decide if $H_n(x, y, z) = 0$ has $n$ distinct solutions. We would also like to know the number of complex zeros $C_{H_n(x, y, z)}$ of $H_n(x, y, z), \text{Im}(x) \neq 0$. Since $n$ is the degree of the polynomial $H_n(x, y, z)$, the number of real zeros $R_{H_n(x, y, z)}$ lying on the real line $\text{Im}(x) = 0$ is then $R_{H_n(x, y, z)} = n - C_{H_n(x, y, z)}$, where $C_{H_n(x, y, z)}$ denotes complex zeros. See Tables 1 and 2 for

Figure 4. Stacks of zeros of $H_n(x, y, z)$, $1 \leq n \leq 20$. 

Differential Equations - Theory and Current Research
<table>
<thead>
<tr>
<th>Degree $n$</th>
<th>Real zeros</th>
<th>Complex zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 1. Numbers of real and complex zeros of $H_n(x, 1, 3)$.

<table>
<thead>
<tr>
<th>Degree $n$</th>
<th>Real zeros</th>
<th>Complex zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 2. Numbers of real and complex zeros of $H_n(x, -1, -3)$. 

Differential Equations Arising from the 3-Variable Hermite Polynomials and Computation of Their Zeros

http://dx.doi.org/10.5772/intechopen.74355
tabulated values of $R_{H_n(x,y,z)}$ and $C_{H_n(x,y,z)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the 3-variable Hermite polynomials $H_n(x,y,z)$ which appear in mathematics and physics. The reader may refer to [2, 11, 13, 20] for the details.
Figure 6. Stacks of zeros of $H_n(x, -2, 4)$ for $1 \leq n \leq 20$.

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$-1.4142, 1.4142$</td>
</tr>
<tr>
<td>3</td>
<td>3.3681</td>
</tr>
<tr>
<td>4</td>
<td>0.16229, 5.0723</td>
</tr>
<tr>
<td>5</td>
<td>$-1.3404, 1.4745, 6.6661$</td>
</tr>
<tr>
<td>6</td>
<td>2.9754, 8.1678</td>
</tr>
<tr>
<td>7</td>
<td>0.31213, 4.3783, 9.5946</td>
</tr>
</tbody>
</table>
Table 4. Approximate solutions of $H_n(x, -1, -3) = 0, x \in \mathbb{R}$.

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$-1.2604, 1.5304, 5.7274, 10.959$</td>
</tr>
<tr>
<td>9</td>
<td>$2.8224, 7.0271, 12.270$</td>
</tr>
<tr>
<td>10</td>
<td>$0.44594, 4.0615, 8.2834, 13.535$</td>
</tr>
<tr>
<td>11</td>
<td>$-1.1740, 1.5825, 5.2667, 9.5013, 14.760$</td>
</tr>
<tr>
<td>12</td>
<td>$-1.4659, -0.87728, 2.7469, 6.4398, 10.685, 15.949$</td>
</tr>
</tbody>
</table>

Figure 7. Real zeros of $H_n(x, y, z), 1 \leq n \leq 20$.

Acknowledgements

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1A2B4006092).

Author details

Cheon Seoung Ryoo
Address all correspondence to: ryoocs@hnu.kr
Department of Mathematics, Hannam University, Daejeon, Korea
References


