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Chapter 5

On Six DOF Relative Orbital Motion of Satellites

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Abstract

In this chapter, we reveal a dual-tensor-based procedure to obtain exact expressions for the six degree of freedom (6-DOF) relative orbital law of motion in the specific case of two Keplerian confocal orbits. The result is achieved by pure analytical methods in the general case of any leader and deputy motion, without singularities or implying any secular terms. Orthogonal dual tensors play a very important role, with the representation of the solution being, to the authors' knowledge, the shortest approach for describing the complete onboard solution of the 6-DOF orbital motion problem. The solution does not depend on the local-vertical–local-horizontal (LVLH) properties involves that is true in any reference frame of the leader with the origin in its mass center. A representation theorem is provided for the full-body initial value problem. Furthermore, the representation theorems for rotation part and translation part of the relative motion are obtained.

Keywords: relative orbital motion, full body problem, dual algebra, Lie group, Lie algebra, closed form solution

1. Introduction

The relative motion between the leader and the deputy in the relative motion is a six-degrees-of-freedom (6-DOF) motion engendered by the joining of the relative translational motion with the rotational one. Recently, the modeling of the 6-DOF motion of spacecraft gained a special attention [1–5], similar to the controlling the relative pose of satellite formation that became a very important research subject [6–10]. The approach implies to consider the relative translational and rotational dynamics in the case of chief-deputy spacecraft formation to be modeled using vector and tensor formalism.

In this chapter we reveal a dual algebra tensor based procedure to obtain exact expressions for the six D.O.F relative orbital law of motion for the case of two Keplerian confocal orbits.

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Orthogonal dual tensors play a very important role, the representation of the solution being, to the authors' knowledge, the shortest approach for describing the complete onboard solution of the six D.O.F relative orbital motion problem. Because the solution does not depend on the LVLH properties involves that is true in any reference frame of the Leader with the origin in its mass center. To obtain this solution, one has to know only the inertial motion of the Leader spacecraft and the initial conditions of the deputy satellite in the local-vertical-local-horizontal (LVLH) frame. For the full body initial value problem, a general representation theorem is given. More, the real and imaginary parts are split and representation theorems for the rotation and translation parts of the relative orbital motion are obtained. Regarding translation, we will prove that this problem is super-integrable by reducing it to the classic Kepler problem.

The chapter is structured as following. The second section is dedicated to the rigid body motion parameterization using orthogonal dual tensors, dual quaternions and other different vector parameterization. The Poisson-Darboux problem is extended in dual Lie algebra. In the third section, the state equations for a rigid body motion relative to an arbitrary non-inertial reference frame are determined. Using the obtained result, in the fourth section, the representation theorem and the complete solution for the case of onboard full-body relative orbital motion problem is given. The last section is designated to the conclusions and to the future works.

2. Rigid body motion parameterization using dual Lie algebra

The key notions that will be presented in this section are tensorial, vectorial and non-vectorial parameterizations that can be used to properly describe the rigid-body motion. We discuss the properties of proper orthogonal dual tensorial maps. The proper orthogonal tensorial maps are related with the skew-symmetric tensorial maps via the Darboux–Poisson equation. Orthogonal dual tensorial maps are a powerful instrument in the study of the rigid motion with respect to an inertial and noninertial reference frames. More on dual numbers, dual vectors and dual tensors can be found in [2, 16–23].

2.1. Isomorphism between Lie group of the rigid displacements $S\mathbb{E}_3$ and Lie group of the orthogonal dual tensors $\underline{S\mathbb{O}}_3$

Let the orthogonal dual tensor set be denoted by.

$$\underline{S\mathbb{O}}_3 = \left\{\underline{\boldsymbol{R}} \in \mathrm{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) \middle| \underline{\boldsymbol{R}}\underline{\boldsymbol{R}}^T = \underline{\boldsymbol{I}}, \det\underline{\boldsymbol{R}} = 1\right\} \tag{1}$$

where $\underline{S\mathbb{O}_3}$ is the set of special orthogonal dual tensors and $\underline{\boldsymbol{I}}$ is the unit orthogonal dual tensor.

The internal structure of any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$ is illustrated in a series of results which were detailed in our previous work [17, 18, 23].

Theorem 1. (Structure Theorem). *For any* $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$ *a unique decomposition is viable*

$$\underline{\boldsymbol{R}} = (\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}})\boldsymbol{Q} \tag{2}$$

where $\boldsymbol{Q} \in S\mathbb{O}_3$ and $\boldsymbol{\rho} \in \boldsymbol{V}_3$ are called **structural invariants**, $\varepsilon^2 = 0$, $\varepsilon \neq 0$.

Taking into account the Lie group structure of $\underline{S\mathbb{O}}_3$ and the result presented in previous theorem, it can be concluded that any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$ can be used globally parameterize displacements of rigid bodies.

Theorem 2 (Representation Theorem). *For any orthogonal dual tensor* $\underline{\boldsymbol{R}}$ *defined as in Eq. (2), a dual number* $\underline{\alpha} = \alpha + \varepsilon d$ *and a dual unit vector* $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{u}_0$ *can be computed to have the following Eq.* [17, 18]:

$$\underline{\boldsymbol{R}}(\underline{\alpha}, \underline{\mathbf{u}}) = \boldsymbol{I} + \sin\underline{\alpha}\underline{\widetilde{\boldsymbol{u}}} + (1 - \cos\underline{\alpha})\underline{\widetilde{\boldsymbol{u}}}^2 = \exp\left(\underline{\alpha}\underline{\widetilde{\boldsymbol{u}}}\right) \tag{3}$$

The parameters $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are called the **natural invariants** of $\underline{\boldsymbol{R}}$. The unit dual vector $\underline{\mathbf{u}}$ gives the Plücker representation of the Mozzi-Chalses axis [16, 24] while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translated distance d.

The Lie algebra of the Lie group $\underline{S\mathbb{O}}_3$ is the skew-symmetric dual tensor set denoted by $\underline{s\mathbb{o}}_3 = \left\{\underline{\widetilde{\boldsymbol{\alpha}}} \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) \middle| \underline{\widetilde{\boldsymbol{\alpha}}} = -\underline{\widetilde{\boldsymbol{\alpha}}}^{\mathrm{T}}\right\}$, where the internal mapping is $\langle\underline{\widetilde{\boldsymbol{\alpha}}}_1, \underline{\widetilde{\boldsymbol{\alpha}}}_2\rangle = \widetilde{\underline{\widetilde{\boldsymbol{\alpha}}}_1\,\underline{\boldsymbol{\alpha}}_2}$.

The link between the Lie algebra $\underline{s\mathbb{o}}_3$, the Lie group $\underline{S\mathbb{O}}_3$, and the exponential map is given by the following.

Theorem 3. *The mapping is well defined and surjective.*

$$\begin{aligned} &\exp : \underline{s\mathbb{o}}_3 \to \underline{S\mathbb{O}}_3, \\ &\exp\left(\underline{\widetilde{\boldsymbol{\alpha}}}\right) = e^{\underline{\widetilde{\boldsymbol{\alpha}}}} = \sum_{\mathrm{k}=0}^{\infty}\frac{\underline{\widetilde{\boldsymbol{\alpha}}}^{\mathrm{k}}}{\mathrm{k}!} \end{aligned} \tag{4}$$

Any screw axis that embeds a rigid displacement is parameterized by a unit dual vector, whereas the screw parameters (angle of rotation around the screw and the translation along the screw axis) is structured as a dual angle. The computation of the screw axis is bound to the problem of finding the logarithm of an orthogonal dual tensor $\underline{\boldsymbol{R}}$, that is a multifunction defined by the following equation:

$$\begin{aligned} &\log : \underline{S\mathbb{O}}_3 \to \underline{s\mathbb{o}}_3, \\ &\log\underline{\boldsymbol{R}} = \left\{\underline{\widetilde{\boldsymbol{\psi}}} \in \underline{s\mathbb{o}}_3 \middle| \exp\left(\underline{\widetilde{\boldsymbol{\psi}}}\right) = \underline{\boldsymbol{R}}\right\} \end{aligned} \tag{5}$$

and is the inverse of Eq. (4).

From **Theorem 2** and **Theorem 3**, for any orthogonal dual tensor $\underline{\boldsymbol{R}}$, a dual vector $\underline{\boldsymbol{\psi}} = \underline{\alpha}\underline{\mathbf{u}} = \boldsymbol{\psi} + \varepsilon\boldsymbol{\psi}_0$ is computed, represents the **screw dual vector** or **Euler dual vector** (that includes the screw axis and screw parameters) and the form of $\underline{\boldsymbol{\psi}}$ implies that $\underline{\widetilde{\boldsymbol{\psi}}} \in \log\underline{\mathbf{R}}$. The types of rigid displacements that is parameterized by the Euler dual vector $\underline{\boldsymbol{\psi}}$ as below:

i. roto-translation if $\boldsymbol{\psi} \neq \mathbf{0}$, $\boldsymbol{\psi}_0 \neq \mathbf{0}$ and $\boldsymbol{\psi}\cdot\boldsymbol{\psi}_0 \neq 0 \Leftrightarrow \left|\underline{\boldsymbol{\psi}}\right| \in \mathbb{R}$ and $|\boldsymbol{\psi}| \notin \varepsilon\mathbb{R}$;

ii. pure translation if if $\boldsymbol{\psi} = \mathbf{0}$ and $\boldsymbol{\psi}_0 \neq \mathbf{0} \Leftrightarrow \left|\underline{\boldsymbol{\psi}}\right| \in \underline{\varepsilon\mathbb{R}}$;

iii. pure rotation if $\boldsymbol{\psi}\neq\mathbf{0}$ and $\boldsymbol{\psi}\cdot\boldsymbol{\psi}_0=0 \Leftrightarrow \left|\underline{\boldsymbol{\psi}}\right| \in \mathbb{R}$.

Also, $\|\boldsymbol{\psi}\| < 2\pi$, **Theorem 2** and **Theorem 3** can be used to uniquely recover the screw dual vector $\underline{\boldsymbol{\psi}}$, which is equivalent with computing $\log \underline{\boldsymbol{R}}$.

Theorem 4. *The natural invariants* $\underline{\alpha} = \alpha + \varepsilon d$, $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ *can be used to directly recover the structural invariants* $\boldsymbol{Q}$ *and* $\boldsymbol{\rho}$ *from Eq. (2):*

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{I} + \sin\alpha\widetilde{\boldsymbol{u}} + (1 - \cos\alpha)\widetilde{\boldsymbol{u}}^2 \\ \boldsymbol{\rho} &= d\mathbf{u} + \sin\alpha\mathbf{u}_0 + (1 - \cos\alpha)\mathbf{u}\times\mathbf{u}_0 \end{aligned} \tag{6}$$

To prove Eq. (6), we need to use Eqs. (2) and (3). If these equations are equal, then the structure of their dual parts leads to the result presented in Eq. (6).

Theorem 5. (Isomorphism Theorem): *The special Euclidean group* $(S\mathbb{E}_3, \cdot)$ *and* $\left(\underline{S\mathbb{O}}_3, \cdot\right)$ *are connected via the isomorphism of the Lie groups*

$$\begin{aligned} &\Phi : S\mathbb{E}_3 \to \underline{S\mathbb{O}}_3, \\ &\Phi(g) = (\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}})\boldsymbol{Q} \end{aligned} \tag{7}$$

where $g = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$, $\Phi \in S\mathbb{O}_3$, $\boldsymbol{\rho} \in \boldsymbol{V}_3$.

Proof. For any $g_1, g_2 \in S\mathbb{E}_3$, the map defined in Eq. (7) yields

$$\Phi(g_1 \cdot g_2) = \Phi(g_1) \cdot \Phi(g_2) \tag{8}$$

Let $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$. Based on **Theorem 1**, which ensures a unique decomposition, we can conclude that the only choice for g, such that $\Phi(g) = \underline{\boldsymbol{R}}$ is $g = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$. This underlines that Φ is a bijection and keeps all the internal operations.

Remark 1: The inverse of Φ is

$$\Phi^{-1} : \underline{S\mathbb{O}}_3 \leftrightarrow S\mathbb{E}_3; \Phi^{-1}(\underline{\boldsymbol{R}}) = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix} \tag{9}$$

where $\boldsymbol{Q} = \mathrm{Re}(\underline{\boldsymbol{R}})$, $\boldsymbol{\rho} = vect\left(Du(\underline{\boldsymbol{R}}) \cdot \boldsymbol{Q}^{\mathrm{T}}\right)$.

2.2. Dual tensor-based parameterizations of rigid-body motion

The Lie group $\underline{S\mathbb{O}}_3$ admits multiple parameterization and few of them will be discussed in this section.

2.2.1. The exponential parameterization (the Euler dual vector parameterization)

If $\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}(\underline{\alpha}, \underline{\mathbf{u}})$, then we can construct the **Euler dual vector** (screw dual vector) $\underline{\boldsymbol{\psi}} = \underline{\alpha}\underline{\mathbf{u}}$, $\underline{\boldsymbol{\psi}} \in \underline{\mathbf{V}}_3$ which combined with **Theorem 2** and **Theorem 3** lead to

$$\underline{\boldsymbol{R}} = \exp\left(\underline{\tilde{\boldsymbol{\psi}}}\right) = \underline{\boldsymbol{I}} + \mathrm{sinc}\left|\underline{\boldsymbol{\psi}}\right|\underline{\tilde{\boldsymbol{\psi}}} + \frac{1}{2}\mathrm{sinc}^2\frac{\left|\underline{\boldsymbol{\psi}}\right|}{2}\underline{\tilde{\boldsymbol{\psi}}}^2 \tag{10}$$

where

$$\mathrm{sinc}(|\underline{x}|) = \begin{cases} \dfrac{\sin|\underline{x}|}{|\underline{x}|}, & |\underline{x}| \notin \varepsilon\mathbb{R} \\ 1, & |\underline{x}| \in \varepsilon\mathbb{R} \end{cases} \tag{11}$$

2.2.2. Dual quaternion parameterization

One of the most important non-vectorial parameterizations for the orthogonal dual tensor $\underline{S\mathbb{O}}_3$ is given by the dual quaternions [20, 21]. A dual quaternion can be defined as an associated pair of a dual scalar quantity and a free dual vector:

$$\underline{\hat{\mathbf{q}}} = \left(\underline{q}, \underline{\mathbf{q}}\right), \underline{q} \in \underline{\mathbb{R}}, \underline{\mathbf{q}} \in \underline{\mathbf{V}}_3 \tag{12}$$

The set of dual quaternions will be denoted $\underline{\mathbf{Q}}$ and is organized as a $\underline{\mathbb{R}}$-module of rank 4, if dual quaternion addition and multiplication with dual numbers are considered.

The product of two dual quaternions $\underline{\hat{\mathbf{q}}}_1 = \left(\underline{q}_1, \underline{\mathbf{q}}_1\right)$ and $\underline{\hat{\mathbf{q}}}_2 = \left(\underline{q}_2, \underline{\mathbf{q}}_2\right)$ is defined by

$$\underline{\hat{\mathbf{q}}}_1\underline{\hat{\mathbf{q}}}_2 = \left(\underline{q}_1 \cdot \underline{q}_2 - \underline{\mathbf{q}}_1 \cdot \underline{\mathbf{q}}_2, \underline{q}_1\underline{\mathbf{q}}_2 + \underline{q}_2\underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_1 \times \underline{\mathbf{q}}_2\right) \tag{13}$$

From the above properties, results that the $\underline{\mathbb{R}}$-module $\underline{\mathbf{Q}}$ becomes an associative, non-commutative linear dual algebra of rank 4 over the ring of dual numbers. For any dual quaternion defined by Eq. (12), the conjugate denoted by $\underline{\hat{\mathbf{q}}}^* = \left(\underline{q}, -\underline{\mathbf{q}}\right)$ and the norm denoted by $\left|\underline{\hat{\mathbf{q}}}\right|^2 = \underline{\hat{\mathbf{q}}}\underline{\hat{\mathbf{q}}}^*$ can be computed. For $\left|\underline{\hat{\mathbf{q}}}\right| = 1$, any dual quaternion is called unit dual quaternion. Regarded solely as a free $\underline{\mathbb{R}}$-module, $\underline{\mathbf{Q}}$ contains two remarkable sub-modules: $\underline{\mathbf{Q}}_{\mathbb{R}}$ and $\underline{\mathbf{Q}}_{\underline{V}_3}$. The first one composed from pairs $\left(\underline{q}, \underline{\mathbf{0}}\right)$, $\underline{q} \in \underline{\mathbb{R}}$, isomorphic with $\underline{\mathbb{R}}$, and the second one, containing the pairs $\left(\underline{0}, \underline{\mathbf{q}}\right)$, $\underline{\mathbf{q}} \in \underline{V}_3$, isomorphic with $\underline{V}_3$. Also, any dual quaternion can be written as $\underline{\hat{\mathbf{q}}} = \underline{q} + \underline{\mathbf{q}}$, where $\underline{q} = \left(\underline{q}, \underline{\mathbf{0}}\right)$ and $\underline{\mathbf{q}} = \left(\underline{0}, \underline{\mathbf{q}}\right)$, or $\underline{\hat{\mathbf{q}}} = \hat{\mathbf{q}} + \varepsilon\hat{\mathbf{q}}_0$, where $\hat{\mathbf{q}}, \hat{\mathbf{q}}_0$ are real quaternions. The scalar and the vector parts of a dual unit quaternion are also known as **dual Euler parameters** [19].

Let denote with $\mathbb{U}$ the set of unit quaternions and with $\underline{\mathbb{U}}$ the set of unit dual quaternions. For any $\underline{\hat{\mathbf{q}}} \in \underline{\mathbb{U}}$, the following equation is valid [17, 20]:

$$\underline{\widehat{\mathbf{q}}} = \left(1 + \varepsilon \frac{1}{2}\widehat{\boldsymbol{\rho}}\right)\widehat{\mathbf{q}} \tag{14}$$

where $\boldsymbol{\rho} \in \mathbf{V}_3$ and $\widehat{\mathbf{q}} \in \mathbb{U}$. This representation is the quaternionic counterpart to Eq. (2). Also a dual number $\underline{\alpha}$ and a unit dual vector $\underline{\mathbf{u}}$ exist so that:

$$\underline{\widehat{\mathbf{q}}} = \cos\frac{\underline{\alpha}}{2} + \underline{\mathbf{u}}\sin\frac{\underline{\alpha}}{2} = \exp\left(\frac{1}{2}\underline{\alpha}\,\underline{\mathbf{u}}\right). \tag{15}$$

Remark 2: The mapping $\exp\,:\underline{\boldsymbol{V}}_3 \to \underline{\mathbb{U}}, \underline{\widehat{\mathbf{q}}} = \exp\frac{1}{2}\underline{\boldsymbol{\Psi}}$, is well defined and surjective.

Remark 3: The dual unit quaternions set $\underline{\mathbb{U}}$, by the multiplication of dual quaternions, is a Lie group with $\underline{\mathbf{V}}_3$ being it's associated Lie algebra (with the cross product between dual vectors as the internal operation).

Using the internal structure of any element from $\underline{S\mathbb{O}_3}$ the following theorem is valid:

Theorem 6. *The Lie groups* $\underline{\mathbb{U}}$ *and* $\underline{S\mathbb{O}_3}$ *are linked by a surjective homomorphism*

$$\Delta : \underline{\mathbb{U}} \to \underline{S\mathbb{O}_3}, \Delta\left(\underline{q} + \underline{\mathbf{q}}\right) = \underline{\boldsymbol{I}} + 2\underline{q}\underline{\widetilde{q}} + 2\underline{\widetilde{q}}^2 \tag{16}$$

Proof. Taking into account that any $\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}$ can be decomposed as in Eq. (15), results that $\Delta\left(\underline{\widehat{\mathbf{q}}}\right) = \exp\left(\underline{\alpha}\underline{\widetilde{u}}\right) \in \underline{S\mathbb{O}_3}$. This shows that relation Eq. (16) is well defined and surjective. Using direct calculus, we can also acknowledge that $\Delta\left(\underline{\widehat{\mathbf{q}}}_2\underline{\widehat{\mathbf{q}}}_1\right) = \Delta\left(\underline{\widehat{\mathbf{q}}}_2\right)\Delta\left(\underline{\widehat{\mathbf{q}}}_1\right)$.

An important property of the previous homomorphism is that for $\underline{\widehat{\mathbf{q}}}$ and $-\underline{\widehat{\mathbf{q}}}$ we can associate the same orthogonal dual tensor, which shows that Eq. (16) is not injective and $\underline{\mathbb{U}}$ is a double cover of $\underline{S\mathbb{O}_3}$.

2.2.3. N-order modified fractional Cayley transform for dual vectors

Next, we present a series of results that are the core of our research. These results are obtained after using a set of Cayley transforms that are different than the ones already reported in literature [17, 25–27].

Theorem 7. *The fractional order Cayley map* $f\,:\underline{\boldsymbol{V}}_3 \to \underline{\mathbb{U}}$

$$cay_{\frac{n}{2}}(\underline{\mathbf{v}}) = f(\underline{\mathbf{v}}) = (1+\underline{\mathbf{v}})^{\frac{n}{2}}(1-\underline{\mathbf{v}})^{-\frac{n}{2}}, n \in \mathbb{N}^* \tag{17}$$

is well defined and surjective.

Proof. Using direct calculus results that $f(\underline{\mathbf{v}})f^*(\underline{\mathbf{v}}) = 1$ and $|f(\underline{\mathbf{v}})| = 1$. The surjectivity is proved by the following theorem.

Theorem 8. *The inverse of the previous fractional order Cayley map, is a multifunction with n branches* $f^{-1} : \underline{\mathbb{U}} \to \underline{\boldsymbol{V}}_3$ given by

$$\underline{\mathbf{v}} = \frac{\sqrt[n]{\widehat{\underline{\mathbf{q}}}^2} - 1}{\sqrt[n]{\widehat{\underline{\mathbf{q}}}^2} + 1}. \tag{18}$$

Remark 4: *If* $|\underline{\mathbf{v}}| \in \mathbb{R}$ *then* $cay_{\frac{n}{2}}(\underline{\mathbf{v}})$ *is the parameterization of a pure rotation about an axis which does not necessarily pass through the origin of reference system. Meanwhile, if* $|\underline{\mathbf{v}}| \in \boldsymbol{\varepsilon}\mathbb{R}$ *the mapping* $cay_{\frac{n}{2}}(\underline{\mathbf{v}})$ *is the parameterization of a pure translation. Otherwise,* $cay_{\frac{n}{2}}(\underline{\mathbf{v}})$ *is the parameterization of roto-translation.*

Taking into account that a dual number $\underline{\alpha}$ and a dual vector $\underline{\mathbf{u}}$ exist in order to have

$$\widehat{\underline{\mathbf{q}}} = \cos\frac{\underline{\alpha}}{2} + \underline{\mathbf{u}}\sin\frac{\underline{\alpha}}{2}, \tag{19}$$

from Eq. (18), results that:

$$\underline{\mathbf{v}} = \tan\frac{\underline{\alpha} + 2k\pi}{2n}\underline{\mathbf{u}}, k = \{0, 1, \ldots, n-1\}. \tag{20}$$

The previous equation contains both the principal parameterization $\underline{\mathbf{v}}_0 = \tan\frac{\underline{\alpha}}{2n}\underline{\mathbf{u}}$, which is the **higher order Rodrigues dual vector**, while for $k = \{1, \ldots, n-1\}$ the dual vectors $\underline{\mathbf{v}}_k = \tan\frac{\underline{\alpha}+2k\pi}{2n}\underline{\mathbf{u}}$ are the **shadow parameterization** [25] that can be used to describe the same pose. Based on $|\underline{\mathbf{v}}_0| = \tan\frac{\underline{\alpha}}{2n}$ and $|\underline{\mathbf{v}}_k| = \tan\frac{\underline{\alpha}+2k\pi}{2n}$, results that $|\underline{\mathbf{v}}_k| = \frac{|\underline{\mathbf{v}}_0| + \tan\frac{k\pi}{n}}{1 - |\underline{\mathbf{v}}_0|\tan\frac{k\pi}{n}}$.

If $\mathrm{Re}(|\underline{\mathbf{v}}_0|) \to \infty$ then $\mathrm{Re}(|\underline{\mathbf{v}}_k|) \to -\cot\frac{k\pi}{n}$, which allows the avoidance of any singularity of type $\mathrm{Re}\left(\frac{\underline{\alpha}}{2n}\right) = \frac{\pi}{2} + \pi\mathbb{Z}$.

Theorem 9. *If* $\underline{\mathbf{v}} \in \underline{\mathbf{V}}_3$ *is the parameterization of a displacement obtained from Eq. (20), then*

$$\pm\widehat{\underline{\mathbf{q}}} = \frac{1}{\sqrt{\left(1 + |\underline{\mathbf{v}}|^2\right)^n}}\left[p_n(|\underline{\mathbf{v}}|) + q_n(|\underline{\mathbf{v}}|)\underline{\mathbf{v}}\right] \tag{21}$$

where

$$p_n(X) = \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{n} X^{2k} \tag{22}$$

$$q_n(X) = \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{2k+1}{n} X^{2k} \tag{23}$$

In Eqs. (22) and (23), [.] represents the floor of a number and $\binom{k}{n}$ are binomial coefficients.

Remark 5. The structure of the polynomials $p_n(X)$ and $q_n(X)$, given by Eqs. (22) and (23), can be used to obtain the following iterative expressions:

$$\begin{aligned} p_{n+1}(X) &= p_n(X) - X^2 q_n(X) \\ q_{n+1}(X) &= q_n(X) + q_n(X) \\ p_1(X) &= 1, q_1(X) = 1. \end{aligned} \tag{24}$$

In order to evaluate the usefulness of the iterative expressions, we provide the second to third order polynomials and the resulting dual quaternions and dual orthogonal tensors:

$$\begin{aligned} p_1(X) = 1; q_1(X) = 1; \underline{\mathbf{v}} &= \tan\frac{\alpha}{2}\underline{\mathbf{u}}; \\ \pm\widehat{\underline{\mathbf{q}}} &= \frac{1}{\sqrt{\underline{1} + |\underline{\mathbf{v}}|^2}}[1 + \underline{\mathbf{v}}]; \\ \underline{\boldsymbol{R}} &= \underline{\boldsymbol{I}} + \frac{2}{1 + |\underline{\mathbf{v}}|^2}\left[\underline{\widetilde{v}} + \underline{\widetilde{v}}^{\mathbf{2}}\right]; \end{aligned} \tag{25}$$

$$\begin{aligned} &p_2(X) = 1 - X^2; q_2 = 2; \underline{\mathbf{v}} = \tan\frac{\alpha + 2k\pi}{4}\underline{\mathbf{u}}; \mathrm{k} = \overline{0, 1}; \\ &\pm\widehat{\underline{\mathbf{q}}} = \frac{1}{\underline{1} + |\underline{\mathbf{v}}|^2}\left[1 - |\underline{\mathbf{v}}|^2 + 2\underline{\mathbf{v}}\right]; \underline{\boldsymbol{R}} = \underline{\boldsymbol{I}} + \frac{4}{\left(1 + |\underline{\mathbf{v}}|^2\right)^2}\left[\left(1 - |\underline{\mathbf{v}}|^2\right)\underline{\widetilde{v}} + 2\underline{\widetilde{v}}^{\mathbf{2}}\right]; \end{aligned} \tag{26}$$

$$\begin{aligned} &p_3(X) = 1 - 3X^2; q_3 = 3 - X^2; \underline{\mathbf{v}} = \tan\frac{\alpha + 2k\pi}{6}\underline{\mathbf{u}}; \\ &\mathrm{k} = \overline{0, 2;}\ \pm\widehat{\underline{\mathbf{q}}} = \frac{1}{\sqrt{\left(\underline{1} + |\underline{\mathbf{v}}|^2\right)^3}}\left[1 - 3|\underline{\mathbf{v}}|^2 + \left(3 - |\underline{\mathbf{v}}|^2\right)\underline{\mathbf{v}}\right]; \\ &\underline{\boldsymbol{R}} = \underline{\boldsymbol{I}} + \frac{2\left(3 - |\underline{\mathbf{v}}|^2\right)}{\left(1 + |\underline{\mathbf{v}}|^2\right)^3}\left[\left(1 - 3|\underline{\mathbf{v}}|^2\right)\underline{\widetilde{v}} + \left(3 - |\underline{\mathbf{v}}|^2\right)\underline{\widetilde{v}}^{\mathbf{2}}\right]. \end{aligned} \tag{27}$$

2.3. Poisson-Darboux problems in dual Lie algebra and vector parameterization

Consider the functions $\boldsymbol{Q} = \boldsymbol{Q}(t) \in S\mathbb{O}_3^{\mathbb{R}}$ and $\boldsymbol{\rho} = \boldsymbol{\rho}(t) \in \boldsymbol{V}_3^{\mathbb{R}}$ to be the parametric equations of any rigid motion. Thus, any rigid motion can be parameterized by a curve in $\underline{S\mathbb{O}}_3$ where $\underline{\boldsymbol{R}}(t) = (I + \varepsilon\widetilde{\boldsymbol{\rho}}(t))\boldsymbol{Q}(t)$, where t is time variable. Let $\underline{\mathbf{h}}_0$ embed the Plücker coordinates of a line feature at $t = t_0$. At a time stamp t the line is transformed into:

$$\underline{\mathbf{h}}(t) = \underline{\boldsymbol{R}}(t)\underline{\mathbf{h}}_0 \tag{28}$$

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Theorem 10. *In a general rigid motion, described by an orthogonal dual tensor function* $\underline{\boldsymbol{R}}$*, the velocity dual tensor function* $\underline{\boldsymbol{\Phi}}$ *defined as*

$$\dot{\underline{\mathbf{h}}} = \underline{\boldsymbol{\Phi}}\underline{\mathbf{h}}, \forall \underline{\mathbf{h}} \in \underline{V}_3 \tag{29}$$

is expressed by

$$\underline{\boldsymbol{\Phi}} = \dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{R}}^{\mathrm{T}}. \tag{30}$$

Let $\underline{\boldsymbol{\Phi}} = \dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{R}}^{\mathrm{T}}$, then $\dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{R}}^{\mathrm{T}} + \underline{\boldsymbol{R}}\dot{\underline{\boldsymbol{R}}}^{\mathrm{T}} = \underline{\mathbf{0}}$, equivalent with $\underline{\boldsymbol{\Phi}} = -\underline{\boldsymbol{\Phi}}^{\mathbf{T}}$, which shows that $\underline{\boldsymbol{\Phi}} \in \underline{so}_3^{\mathbb{R}}$.

The dual vector $\underline{\boldsymbol{\omega}} = vect\dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{R}}^{\mathrm{T}}$ is called **dual angular velocity of the rigid body** and has the form:

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon \mathbf{v} \tag{31}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the rigid body and $\mathbf{v} = \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho}$ represents the linear velocity of the point of the body that coincides instantaneously with the origin of the reference frame. The pair $(\boldsymbol{\omega}, \mathbf{v})$ is usually refereed as the **twist of the rigid body**.

2.3.1. Poisson-Darboux equation in dual Lie algebra

The next Theorem permits the reconstruction of the rigid body motion knowing in any moment the twist of the rigid body that is equivalent with knowing the dual angular velocity [5, 18].

Theorem 11. *For any continuous function* $\underline{\boldsymbol{\omega}} \in \underline{\mathbf{V}}_3^{\mathbb{R}}$ *a unique dual tensor* $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$ *exists so that*

$$\begin{aligned} \dot{\underline{\boldsymbol{R}}} &= \widetilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) &= \underline{\boldsymbol{R}}_0, \underline{\boldsymbol{R}}_0 \in \underline{S\mathbb{O}}_3 \end{aligned} \tag{32}$$

Due to the fact that orthogonal dual tensor $\underline{\boldsymbol{R}}$ completely models the six degree of freedom motion, we can conclude that the **Theorem 11** is the dual form of the **Poisson-Darboux problem** [28] for the case when the rotation tensor is computed from the instantaneous angular velocity. So, in order to recover $\underline{\boldsymbol{R}}$, it is necessary to find out how the dual angular velocity vector $\underline{\boldsymbol{\omega}}$ behaves in time and also the value of $\underline{\boldsymbol{R}}$ at time $t = t_0$.

The dual tensor $\underline{\boldsymbol{R}}$ can be derived from $\underline{\boldsymbol{\omega}}$, when is positioned in space, or from $\underline{\boldsymbol{\omega}}^B$, which denotes the dual angular velocity vector to be positioned in the rigid body.

Remark 6. The dual angular velocity vector positioned in the rigid body can be recovered from $\underline{\boldsymbol{\omega}}^B = \underline{\boldsymbol{R}}^{\mathrm{T}}\underline{\boldsymbol{\omega}}$, thus transforming Eq. (32) into:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\boldsymbol{R}}\widetilde{\underline{\boldsymbol{\omega}}}^B \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0, \underline{\boldsymbol{R}}_0 \in \underline{S\mathbb{O}}_3 \end{cases} \tag{33}$$

Eqs. (32) and (33) represent the dual replica of the classical orientation Poisson-Darboux problem [17, 28, 29].

The tensorial Eqs. (32) and (33) are equivalent with 18 scalar differential equations. The previous parameterizations of the orthogonal dual tensors allow us to determine some solutions of smaller dimension in order to solve the dual Poisson- Darboux problem.

2.3.2. Kinematic equation for Euler dual vector parameterization

Consider $\underline{\boldsymbol{\Psi}} \in V_3^{\mathbb{R}}$ such that $\underline{\boldsymbol{R}} = \exp \widetilde{\underline{\boldsymbol{\Psi}}}$. According to the Eq. (10), the Poisson-Darboux problem (32) is equivalent to

$$\begin{cases} \dot{\underline{\boldsymbol{\Psi}}} = \underline{\boldsymbol{T}}\underline{\boldsymbol{\omega}} \\ \underline{\boldsymbol{\Psi}}(t_0) = \underline{\boldsymbol{\Psi}}_0 \end{cases} \tag{34}$$

where $\exp \widetilde{\underline{\boldsymbol{\Psi}}}_0 = \underline{\boldsymbol{R}}_0$, and $\underline{\boldsymbol{T}}$ is the following dual tensor:

$$\underline{\boldsymbol{T}} = \frac{|\underline{\boldsymbol{\Psi}}|}{2} \cot \frac{|\underline{\boldsymbol{\Psi}}|}{2} \underline{\boldsymbol{I}} - \frac{1}{2} \widetilde{\underline{\boldsymbol{\Psi}}} - \frac{1}{2|\underline{\boldsymbol{\Psi}}|} \cot \frac{|\underline{\boldsymbol{\Psi}}|}{2} \widetilde{\underline{\boldsymbol{\Psi}}}^2 \tag{35}$$

The representation of the Poisson-Darboux problem from Eq. (33) is equivalent to

$$\begin{cases} \dot{\underline{\boldsymbol{\Psi}}} = \underline{\boldsymbol{T}}^{\mathrm{T}}\underline{\boldsymbol{\omega}}^{\mathrm{B}} \\ \underline{\boldsymbol{\Psi}}(t_0) = \underline{\boldsymbol{\Psi}}_0 \end{cases} \tag{36}$$

2.3.3. Kinematic equation for high order Rodrigues dual vector parameterization

Let $\underline{\mathbf{v}} \in \underline{\mathbf{V}}_3^{\mathbb{R}}$ such that $\underline{\boldsymbol{R}} = \mathrm{cay}_n \underline{\mathbf{v}}$. The problems (32) and (33) are equivalent to:

$$\begin{cases} \dot{\underline{\mathbf{v}}} = \underline{\boldsymbol{S}}\underline{\boldsymbol{\omega}} \\ \underline{\mathbf{v}}(t_0) = \underline{\mathbf{v}}_{\mathbf{0}}, \end{cases} \tag{37}$$

$$\begin{cases} \dot{\underline{\mathbf{v}}} = \underline{\boldsymbol{S}}^{\mathrm{T}}\underline{\boldsymbol{\omega}}^{B} \\ \underline{\mathbf{v}}(t_0) = \underline{\mathbf{v}}_0 \end{cases} \tag{38}$$

where $\mathrm{cay}_n \underline{\mathbf{v}}_0 = \underline{\boldsymbol{R}}_0$, and $\underline{\boldsymbol{S}}$ is the following dual tensor [29]:

$$\underline{\boldsymbol{S}} = \frac{p_n |\underline{\mathbf{v}}|}{2q_n |\underline{\mathbf{v}}|} \underline{\boldsymbol{I}} - \frac{1}{2} \widetilde{\underline{v}} + \frac{\left(1 + |\mathbf{v}|^2\right) q_n \left(|\underline{\mathbf{v}}| - n p_n |\underline{\mathbf{v}}|\right)}{2n |\underline{\mathbf{v}}|^2 q_n |\underline{\mathbf{v}}|} \widetilde{\underline{v}}^2 \tag{39}$$

and the polynomials p_n, q_n are given by the Eqs. (22)–(24).

Eqs. (34), (36)–(38) are equivalent with six scalar differential equations. This is a minimal parameterization of the Poisson-Darboux problem in dual algebra.

2.3.4. Kinematic equation for dual quaternion parameterization

Let $\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}^{\mathbb{R}}$ such that $\Delta\left(\underline{\widehat{\mathbf{q}}}\right) = \underline{\mathbf{R}}$. According to Eq. (16), the Poisson–Darboux problems (32) and (33) are equivalent to:

$$\begin{cases} \dot{\underline{\widehat{\mathbf{q}}}} = \frac{1}{2}\underline{\boldsymbol{\omega}}\underline{\widehat{\mathbf{q}}} \\ \underline{\widehat{\mathbf{q}}}(t_0) = \underline{\widehat{\mathbf{q}}}_0 \end{cases} \tag{40}$$

and

$$\begin{cases} \dot{\underline{\widehat{\mathbf{q}}}} = \frac{1}{2}\underline{\widehat{\mathbf{q}}}\underline{\boldsymbol{\omega}}^{B} \\ \underline{\widehat{\mathbf{q}}}(t_0) = \underline{\widehat{\mathbf{q}}}_0 \end{cases} \tag{41}$$

where $\Delta\left(\underline{\widehat{\mathbf{q}}}_0\right) = \underline{\mathbf{R}}_0$

Eqs. (40) and (41) are equivalent to eight scalar differential equations.

3. Rigid body motion in arbitrary non-inertial frame revised

To the author's knowledge, in the field of astrodynamics there aren't many reports on how the motion of rigid body can be studied in arbitrary non-inertial frames. Next, we proposed a dual tensors based model for the motion of the rigid body in arbitrary non-inertial frame. The proposed method eludes the calculus of inertia forces that contributes to the rigid body relative state. So, the free of coordinate state equation of the rigid body motion in arbitrary non-inertial frame will be obtained.

Let $\underline{\boldsymbol{R}}_D$ and $\underline{\boldsymbol{R}}_C$ be the dual orthogonal tensors which describe the motion of two rigid bodies relative to the inertial frame.

If $\underline{\boldsymbol{R}}$ is the orthogonal dual tensor which embeds the six degree of freedom relative motion of rigid body C relative to rigid body D, then:

$$\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}_C^T \underline{\boldsymbol{R}}_D \tag{42}$$

Let $\underline{\boldsymbol{\omega}}_{\mathrm{C}}$ denote the dual angular velocity of the rigid body C and $\underline{\boldsymbol{\omega}}_{\mathrm{D}}$ the dual angular velocity of the rigid body D, both being related to inertial reference frame. In the followings, the inertial motion of the rigid body C is considered to be known. If $\underline{\boldsymbol{\omega}}$ is the dual angular velocity of the rigid body D relative to the rigid body C, then, conforming with Eq. (42):

$$\underline{\boldsymbol{\omega}} = \underline{\boldsymbol{\omega}}_D - \underline{\boldsymbol{\omega}}_C \tag{43}$$

Considering $\underline{\boldsymbol{\omega}}_D^B$ being the dual angular velocity vector of the rigid body D in the body frame, the dual form of the Euler equation given in [30] results that:

$$\underline{\boldsymbol{M}}\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_D^B = \underline{\boldsymbol{\tau}}^B \tag{44}$$

In Eq. (44) $\underline{\boldsymbol{\tau}}^B = \mathbf{F}^B + \varepsilon\boldsymbol{\tau}^B$, where $\mathbf{F}^B$ the force applied in the mass center and $\boldsymbol{\tau}^B$ is the torque. Also in Eq. (44), $\underline{\boldsymbol{M}}$ represents the inertia dual operator, which is given by $\underline{\boldsymbol{M}} = m_D \frac{d}{d\varepsilon}\boldsymbol{I} + \varepsilon\boldsymbol{J}$, where $\boldsymbol{J}$ is the inertia tensor of the rigid body D related to its mass center and m_D is the mass of the rigid body D. Combining $\underline{\boldsymbol{M}}^{-1} = \boldsymbol{J}^{-1}\frac{d}{d\varepsilon} + \varepsilon\frac{1}{m_D}\boldsymbol{I}$ with Eq. (44) results:

$$\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{M}}^{-1}\left(\underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_D^B\right) = \underline{\boldsymbol{M}}^{-1}\underline{\boldsymbol{\tau}}^B \tag{45}$$

Taking into account that $\underline{\boldsymbol{\omega}}_D = \underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B$, the dual angular velocity vector can be computed from

$$\underline{\boldsymbol{\omega}} = \underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B - \underline{\boldsymbol{\omega}}_C \tag{46}$$

this through differentiation gives:

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{\omega}}_D^B + \underline{\boldsymbol{R}}\dot{\underline{\boldsymbol{\omega}}}_D^B \tag{47}$$

If the previous equation is multiplied by $\underline{\boldsymbol{R}}^T$, then

$$\underline{\boldsymbol{R}}^T(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C) = \underline{\boldsymbol{R}}^T\dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{\omega}}_D^B + \dot{\underline{\boldsymbol{\omega}}}_D^B \tag{48}$$

which combined with $\dot{\underline{\boldsymbol{R}}} = \underline{\widetilde{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}$ generates:

$$\underline{\boldsymbol{R}}^T(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C) = \underline{\boldsymbol{R}}^T\underline{\widetilde{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B + \dot{\underline{\boldsymbol{\omega}}}_D^B \tag{49}$$

After a few steps, Eq. (49) is transformed into

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \tag{50}$$

which combined with Eq. (45) gives:

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\boldsymbol{M}}^{-1}\underline{\boldsymbol{\tau}}^B - \underline{\boldsymbol{R}}\underline{\boldsymbol{M}}^{-1}\left(\underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_D^B\right) + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \tag{51}$$

Because $\underline{\boldsymbol{\omega}}_D^B = \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C)$, the final equation is:

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\boldsymbol{M}}^{-1}\left[\underline{\boldsymbol{\tau}}^B - \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) \times \underline{\boldsymbol{M}}\underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C)\right] + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \tag{52}$$

The system:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\widetilde{\boldsymbol{\omega}}}\underline{\boldsymbol{R}} \\ \dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\boldsymbol{M}}^{-1}[\underline{\boldsymbol{R}^T\boldsymbol{\tau}} - \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) \times \\ \times\underline{\boldsymbol{M}}\underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C)] + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \\ \underline{\boldsymbol{\omega}}(\mathrm{t}_0) = \underline{\boldsymbol{\omega}}_0, \underline{\boldsymbol{\omega}}_0 \in \underline{V}_3 \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0, \underline{\boldsymbol{R}}_0 \in \underline{S\mathbb{O}}_3 \end{cases} \tag{53}$$

is a compact form which can be used to model the six D.O.F relative motion problem. In the previous equation the state of the rigid body D in relation with the rigid body C is modeled by the dual tensor $\underline{\boldsymbol{R}}$ and the dual angular velocities field $\underline{\boldsymbol{\omega}}$. This initial value problem can be used to study the behavior of the rigid body D in relation with the frame attached to the rigid body C. In Eq. (53), all the vectors are represented in the body frame of C, which shows that the proposed solution is onboard and has the property of being coupled in $\underline{\boldsymbol{R}}$ and $\underline{\boldsymbol{\omega}}$.

Next, we present a procedure that allows the decoupling of the proposed solution.

In order to describe the solution to Eq. (53), we consider the following change of variable:

$$\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) \tag{54}$$

This change of variable leads to $\dot{\underline{\boldsymbol{\omega}}}_* = \dot{\underline{\boldsymbol{R}}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) + \underline{\boldsymbol{R}}^T(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C) = -\underline{\boldsymbol{R}}^T\widetilde{\underline{\boldsymbol{\omega}}}(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) + \underline{\boldsymbol{R}}^T(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C)$. The result is equivalent with $\dot{\underline{\boldsymbol{\omega}}}_* = \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}}_C \times \underline{\boldsymbol{\omega}} + \dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C)$ or

$$\underline{\boldsymbol{\omega}}_C \times \underline{\boldsymbol{\omega}} + \dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\,\dot{\underline{\boldsymbol{\omega}}}_* \tag{55}$$

After some steps of algebraic calculus, from Eqs. (54), (55) and (52), results that:

$$\begin{cases} \underline{\boldsymbol{M}}\,\dot{\underline{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{\boldsymbol{M}}\,\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_* \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_*^0 \end{cases} \tag{56}$$

Where $\underline{\boldsymbol{\tau}}_* = \underline{\boldsymbol{R}}^T\underline{\boldsymbol{\tau}}$ is the dual torque related to the mass center in the body frame of the rigid body D and $\underline{\boldsymbol{\omega}}_*^0 = \underline{\boldsymbol{R}}_0^T(\underline{\boldsymbol{\omega}}_0 + \underline{\boldsymbol{\omega}}_C(t_0))$. Eq. (56) is a dual Euler fixed point classic problem.

For any $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$, the solution of Eq. (53) emerges from

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \widetilde{\underline{\boldsymbol{\omega}}}\,\underline{\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0 \end{cases} \tag{57}$$

Making use of Eq. (54), results that $\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C$. If $\widetilde{\ }$ operator used, the previous calculus is transformed into $\widetilde{\underline{\boldsymbol{R}}\,\underline{\boldsymbol{\omega}}_*} = \widetilde{\underline{\boldsymbol{\omega}}} + \widetilde{\underline{\boldsymbol{\omega}}}_C \Leftrightarrow \underline{\boldsymbol{R}}\,\widetilde{\underline{\boldsymbol{\omega}}}_*\underline{\boldsymbol{R}}^T = \dot{\underline{\boldsymbol{R}}}\,\underline{\boldsymbol{R}}^T + \widetilde{\underline{\boldsymbol{\omega}}}_C$. After multiplying the last expression by $\underline{\boldsymbol{R}}$, we obtain the initial value problem:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\boldsymbol{R}}\,\widetilde{\underline{\boldsymbol{\omega}}}_* - \widetilde{\underline{\boldsymbol{\omega}}}_C\,\underline{\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0 \end{cases} \tag{58}$$

Using the variable change Eq. (54), the initial value problem (53) has been decoupled into two distinct initial value problems (56) and (58).

Let $\underline{\boldsymbol{R}}_{-\underline{\omega}_C} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$ be the unique solution of the following Poisson-Darboux problem:

$$\begin{cases} \underline{\dot{\boldsymbol{R}}} + \underline{\widetilde{\omega}}_C \, \underline{\boldsymbol{R}} = 0 \\ \underline{\boldsymbol{R}}(t_0) = \boldsymbol{I} - \varepsilon \widetilde{\mathbf{r}}_C(t_0) \end{cases} \tag{59}$$

Considering $\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}_{-\underline{\omega}_C} \underline{\boldsymbol{R}}_*$, a representation theorem of the solution of Eq. (53) can be formulated.

Theorem 12. (Representation Theorem). *The solution of Eq. (53) results from the application of the tensor* $\underline{\boldsymbol{R}}_{-\underline{\omega}_C}$ *from Eq. (59) to the solution of the classical dual Euler fixed point problem:*

$$\begin{cases} \underline{\dot{\boldsymbol{R}}}_* = \underline{\boldsymbol{R}}_* \underline{\widetilde{\omega}}_* \\ \underline{\boldsymbol{M}} \, \underline{\dot{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{\boldsymbol{M}} \, \underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_* \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_{*0} \\ \underline{\boldsymbol{R}}_*(t_0) = \underline{\boldsymbol{R}}_{*0} \end{cases} \tag{60}$$

where $\underline{\boldsymbol{\omega}}_{*0} = \underline{\boldsymbol{R}}_0^T(\underline{\boldsymbol{\omega}}_0 + \underline{\boldsymbol{\omega}}_C(t_0)), \underline{\boldsymbol{R}}_{*0} = \left(\boldsymbol{I} + \varepsilon \widetilde{r}_C(t_0)\right)\underline{\boldsymbol{R}}_0, \underline{\boldsymbol{\tau}}_* = \underline{\boldsymbol{R}}^T \underline{\boldsymbol{\tau}}$.

Different representations can be considered for the problem (60).

Using dual quaternion representation $\underline{\boldsymbol{R}}_* = \Delta\left(\underline{\widehat{\mathbf{q}}}_*\right)$, Eq. (60) is equivalent with the following one:

$$\begin{cases} \dot{\underline{\widehat{\mathbf{q}}}}^* = \dfrac{1}{2} \underline{\widehat{\mathbf{q}}}_* \underline{\boldsymbol{\omega}}_* \\ \underline{\boldsymbol{M}} \, \underline{\dot{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{\boldsymbol{M}} \, \underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_* \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_{*0} \\ \underline{\widehat{\mathbf{q}}}_*(t_0) = \underline{\widehat{\mathbf{q}}}_{*\mathbf{0}} \end{cases} \tag{61}$$

For the n-th order of Cayley transform based representation $\underline{\boldsymbol{R}}_* = cay_n\left(\underline{\xi}\right), \underline{\xi} = \tan \frac{\alpha}{2n} \underline{\mathbf{u}}$, the Eq. (60) becomes:

$$\begin{cases} \underline{\dot{\xi}} = \underline{\boldsymbol{S}}\left(\underline{\xi}\right) \underline{\boldsymbol{\omega}}_* \\ \underline{\boldsymbol{M}} \, \underline{\dot{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{\boldsymbol{M}} \, \underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_* \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_{*0} \\ \underline{\xi}(t_0) = \underline{\xi}_0 \end{cases} \tag{62}$$

where the tensor $\underline{\boldsymbol{S}}$ is:

$$\underline{\boldsymbol{S}} == \frac{p_n\left(\left|\underline{\xi}\right|\right)}{2q_n\left(\left|\underline{\xi}\right|\right)} \underline{\boldsymbol{I}} + \frac{1}{2} \underline{\widetilde{\xi}} + \frac{\left(1 + \left|\underline{\xi}\right|^2\right) q_n\left(\underline{\xi}\right) - n p_n\left(\left|\underline{\xi}\right|\right)}{2n\left|\underline{\xi}\right|^2 q_n\left(\left|\underline{\xi}\right|\right)} \underline{\xi} \otimes \underline{\xi} \tag{63}$$

when $p_n(X)$ and $q_n(X)$ are defined by Eqs. (22) and (23).

Different particular cases can be analyzed for Eq. (62):

1. Let $\underline{\xi} = \tan\frac{\alpha}{2}\underline{\mathbf{u}}$ be the **Rodrigues dual vector** for n = 1:

$$\underline{S} = \frac{1}{2}\underline{I} + \frac{1}{2}\underline{\tilde{\xi}} + \frac{1}{2}\underline{\xi}\otimes\underline{\xi}$$

2. Let $\underline{\xi} = \tan\frac{\alpha}{4}\underline{\mathbf{u}}$ be the modified Rodrigues dual vector (Wiener-Milenkovic dual vector) for n = 2:

$$\underline{S} = \frac{1-|\underline{\xi}|^2}{4}\underline{I} + \frac{1}{2}\underline{\tilde{\xi}} + \frac{1}{2}\underline{\xi}\otimes\underline{\xi}.$$

The initial value problem (62) is a minimum parameterization of the six degrees of freedom motion problem. The singularity cases can be avoided using the **shadow parameters** of the **n-th order Modified Rodrigues Parameter dual vector.**

4. A dual tensor formulation of the six degree of freedom relative orbital motion problem

The results from the previous paragraphs will be used to study the six degrees of freedom relative orbital motion problem.

The relative orbital motion problem may now be considered classical one considering the many scientific papers written on this subject in the last decades. Also, the problem is quite important knowing its numerous applications: rendezvous operations, spacecraft formation flying, distributed spacecraft missions [3, 4, 6–10].

The model of the relative motion consists in two spacecraft flying in Keplerian orbits due to the influence of the same gravitational attraction center. The main problem is to determine the pose of the Deputy satellite relative to a reference frame originated in the Leader satellite center of mass. This non-inertial reference frame, known as “LVLH (Local-Vertical-Local- Horizontal)” is chosen as following: the C_x axis has the same orientation as the position vector of the Leader with respect to an inertial reference frame with the origin in the attraction center; the orientation of the C_z is the same as the Leader orbit angular momentum; the C_y axis completes a right-handed frame. The angular velocity of the LVLH is given by vector $\boldsymbol{\omega}_C$, which has the expression:

$$\boldsymbol{\omega}_C = \dot{f}_C\frac{\mathbf{h}_C}{h_C} = \frac{1}{r_C^2}\mathbf{h}_C = \left[\frac{1+e_C\cos f_C(t)}{p_C}\right]^2\mathbf{h}_C \tag{64}$$

where vector $\mathbf{r}_C$ is

$$\mathbf{r}_C = \frac{p_C}{1+e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{r_C^0} \tag{65}$$

where p_C is the conic parameter, $\mathbf{h}_C$ is the angular momentum of the Leader, $f_C(t)$ being the true anomaly and e_C is the eccentricity of the Leader.

We propose dual tensors based model for the motion and the pose for the mass center of the Deputy in relation with LVLH. Both, the Leader satellite and the Deputy satellite can be considered rigid bodies.

Furthermore, the time variation of $\mathbf{r}_C$ is:

$$\dot{\mathbf{r}}_C = \frac{e_C|\mathbf{h}_C| \sin f_C(t)}{p_C} \frac{\mathbf{r}_C^0}{r_C^0} \tag{66}$$

In order to a more easy to read list of notations, for $t = t_0$ there will be used the followings:

$$\boldsymbol{\omega}_C^0 = \left[\frac{1 + e_C \cos f_C(t_0)}{p_C}\right]^2 \mathbf{h}_C \tag{67}$$

$$\dot{\mathbf{r}}_C^0 = \frac{e_C|\mathbf{h}_C| \sin f_C(t_0)}{p_C} \frac{\mathbf{r}_C^0}{r_C^0} \tag{68}$$

where $\frac{\mathbf{r}_C^0}{r_C^0}$ is the unity vector of the X-axis from LVLH.

The full-body relative orbital motion is described by Eq. (53) where the dual angular velocity of the Chief satellite is:

$$\underline{\boldsymbol{\omega}}_C = \boldsymbol{\omega}_C + \varepsilon(\dot{\mathbf{r}}_C + \boldsymbol{\omega}_C \times \mathbf{r}_C) \tag{69}$$

and the dual torque related to the mass center of Deputy satellite is:

$$\underline{\boldsymbol{\tau}} = -\frac{\mu}{|\mathbf{r}_c + \mathbf{r}|^3}(\mathbf{r}_c + \mathbf{r}) + \varepsilon\boldsymbol{\tau}. \tag{70}$$

The representation theorem (**Theorem 12**) is applied in this case using the conditions (66)–(69), the solution of the Poisson-Darboux problem (59) is:

$$\underline{\boldsymbol{R}}_{-\underline{\omega}_C} = \left(\boldsymbol{I} - \varepsilon\widetilde{r}_C(t)\right)\left(\boldsymbol{I} - \sin f_c^0 \frac{\widetilde{\boldsymbol{h}}_C}{h_c} + (1 - \cos f_c^0)\frac{\widetilde{\boldsymbol{h}}_C^{\,2}}{h_c^{\,2}}\right). \tag{71}$$

In (71), we've noted $h_c = \|\mathbf{h_c}\|$ and $f_c^0 = f_c(t) - f_c(t_0)$.

Theorem 13. (Representation Theorem of the full body relative orbital motion). *The solution of Eq. (53) results from the application of the tensor* $\underline{\boldsymbol{R}}_{-\underline{\omega}_C}$ *from Eq. (71) to the solution of the classical dual Euler fixed point problem* (60).

4.1. The rotational and translational parts of the relative orbital motion

The complete solution of Eq. (53) can be recovered in two steps.

Consider first the real part of Eq. (53). This leads to an initial value problem:

$$\begin{cases} \dot{\boldsymbol{Q}} = \widetilde{\boldsymbol{\omega}}\boldsymbol{Q} \\ \dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}}_c = \boldsymbol{Q}\boldsymbol{J}^{-1}[\boldsymbol{Q}^T\boldsymbol{\tau} - \boldsymbol{Q}^T(\boldsymbol{\omega}+\boldsymbol{\omega}_c)\times \\ \times\boldsymbol{J}\boldsymbol{Q}^T(\boldsymbol{\omega}+\boldsymbol{\omega}_c)] + \boldsymbol{\omega}\times\boldsymbol{\omega}_c \\ \boldsymbol{\omega}(t_0) = \boldsymbol{\omega}_0, \boldsymbol{\omega}_0 \in V_3 \\ \boldsymbol{Q}(t_0) = \boldsymbol{Q}_0, \boldsymbol{Q}_0 \in S\mathbb{O}_3 \end{cases} \tag{72}$$

which has the solution $\boldsymbol{Q} = \boldsymbol{Q}(t)$, the real tensor $\boldsymbol{Q}$ being the attitude of Deputy in relation with LVLH. In Eq. (72), $\boldsymbol{\omega}$ is the angular velocity of the Deputy in relation with LVLH, $\boldsymbol{\omega}_c$ is the angular velocity of LVLH, $\boldsymbol{\tau}$ is the resulting torque of the forces applied on the Deputy in relation with is mass center, $\boldsymbol{J}$ is the inertia tensor of the Deputy in relation with its mass center. The angular velocity of Deputy in respect to LVLH at time t_0 is denoted with $\boldsymbol{\omega_0}$ and $\boldsymbol{Q_0}$ is the orientation of Deputy in respect to LVLH at time t_0.

Consider now the dual part of Eq. (53). Taking into account the internal structure of $\underline{\boldsymbol{R}}$, which is given by Eq. (2), after some basic algebraic calculus we obtain a second initial value problem that models the translation of the Deputy satellite mass center with respect to the LVLH reference frame:

$$\begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega}_c \times \dot{\mathbf{r}} + \boldsymbol{\omega}_c \times (\boldsymbol{\omega}_c \times \mathbf{r}) + \dot{\boldsymbol{\omega}}_c \times \mathbf{r} + \\ \qquad + \dfrac{\mu}{|\mathbf{r}_c+\mathbf{r}|^3}(\mathbf{r}_c+\mathbf{r}) - \dfrac{\mu}{r_c^3}\mathbf{r}_c = 0 \\ \mathbf{r}(t_0) = \mathbf{r}_0, \dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \end{cases} \tag{73}$$

where $\mu > 0$ is the gravitational parameter of the attraction center and $\mathbf{r_0}, \mathbf{v_0}$ represent the relative position and relative velocity vectors of the mass center of the Deputy spacecraft with respect to LVLH at the initial moment of time $t_0 \geq 0$.

Based on the **representation theorem 12**, the following theorem results.

Theorem 14. *The solutions of problems Eqs. (72) and (73) are given by*

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{R}_{-\omega_C}\boldsymbol{Q}_* \\ \mathbf{r} &= \boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \mathbf{r}_c \end{aligned} \tag{74}$$

where $\boldsymbol{Q}_*$ and $\mathbf{r}_*$ *are the solutions of the the classical Euler fixed point problem and, respectively, Kepler's problem*:

$$\begin{cases} \dot{\boldsymbol{Q}}_* = \boldsymbol{Q}_*\widetilde{\boldsymbol{\omega}}_* \\ \boldsymbol{J}\dot{\boldsymbol{\omega}}_* + \boldsymbol{\omega}_* \times \boldsymbol{J}\boldsymbol{\omega}_* = \boldsymbol{\tau}_* \\ \boldsymbol{\omega}_*(t_0) = \boldsymbol{Q}_0^T(\boldsymbol{\omega}_0 + \boldsymbol{\omega}_c(t_0)) \\ \boldsymbol{Q}_*(t_0) = \boldsymbol{Q}_0 \end{cases} \tag{75}$$

and

$$\begin{cases} \ddot{\mathbf{r}}_* + \dfrac{\mu}{r_*^3}\mathbf{r}_* = \mathbf{0}; \\ \mathbf{r}_*(t_0) = \mathbf{r}_c^0 + \mathbf{r}_0; \\ \dot{\mathbf{r}}_*(t_0) = \dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times \left(\mathbf{r}_C^0 + \mathbf{r}_0\right) \end{cases} \tag{76}$$

where

$$\boldsymbol{R}_{-\omega_C} = \boldsymbol{I} - \sin f_c^0 \frac{\widetilde{\boldsymbol{h}}_C}{|\mathbf{h}_c|} + (1 - \cos f_c^0) \frac{\widetilde{\boldsymbol{h}}_C^{\,2}}{|\mathbf{h}_c|^2}) \tag{77}$$

and $\mathbf{r}_c$ *is given by Eq. (65).*

Remark 7: The problems (72) and (73) are coupled because, in general case, the torque $\boldsymbol{\tau}$ depends of the position vector $\mathbf{r}$.

The relative velocity of the translation motion may be computed as:

$$\mathbf{v} = \boldsymbol{R}_{-\omega_C}\dot{\mathbf{r}}_* - \widetilde{\omega_c}\boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \frac{e_c|\mathbf{h}_c|\sin f_c(t)}{p_c}\frac{\mathbf{r}_c^0}{r_c^0} \tag{78}$$

This result shows a very interesting property of the translational part of the relative orbital motion problem (73). We have proven that this problem is super-integrable by reducing it to the classic Kepler problem [11, 12, 31, 32]. The solution of the translational part of the relative orbital motion problem is expressed thus:

$$\mathbf{r} = \mathbf{r}(t, t_0, \mathbf{r}_0, \mathbf{v}_0); \mathbf{v} = \mathbf{v}(t, t_0, \mathbf{r}_0, \mathbf{v}_0) \tag{79}$$

The exact closed form, free of coordinate, solution of the translational motion can be found in [11, 12, 31, 32, 34].

5. Conclusions

The chapter proposes a new method for the determination of the onboard complete solution to the full-body relative orbital motion problem.

Therefore, the isomorphism between the Lie group of the rigid displacements $S\mathbb{E}_3$ and the Lie group of the orthogonal dual tensors $\underline{S\mathbb{O}}_3$ is used. It is obtained a Poisson-Darboux like problem written in the Lie algebra of the group $\underline{S\mathbb{O}}_3$, an algebra that is isomorphic with the Lie algebra of the dual vectors. Different vectorial and non-vectorial parameterizations (obtained with n-th order Cayley-like transforms) permit the reduction of the Poisson-Darboux problem in dual Lie algebra to the simpler problems in the space of the dual vectors or dual quaternions.

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Using the above results, the free of coordinate state equation of the rigid body motion in arbitrary non-inertial frame is obtained.

The results are applied in order to offer a coupled (rotational and translational motion) state equation and a representation theorem for the onboard complete solution of full body relative orbital motion problem. The obtained results interest the domains of the spacecraft formation flying, rendezvous operation, autonomous mission and control theory.

Nomenclature

a	real number
$\underline{a}$	dual number
$\mathbf{a}$	real vector
$\underline{\mathbf{a}}$	dual vector
$\boldsymbol{A}$	real tensor
$\underline{\boldsymbol{A}}$	dual tensor
$\boldsymbol{V}_3$	real vectors set
$\underline{\mathbb{V}}_3$	dual vectors set
$\boldsymbol{V}_3^{\mathbb{R}}$	time depending real vectorial functions
$\underline{\mathbb{V}}_3^{\mathbb{R}}$	time depending dual vectorial functions
$\widetilde{\underline{\mathbf{a}}}$	skew-symmetric dual tensor corresponding to the dual vector $\underline{\mathbf{a}}$
f_c	true anomaly
p_c	conic parameter
$\boldsymbol{h}_c$	specific angular momentum of the leader satellite
$\boldsymbol{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3)$	dual tensor set
$\widehat{\mathbf{q}}$	real quaternion
$\underline{\widehat{\mathbf{q}}}$	dual quaternion
$\mathbb{R}$	real numbers set
$\underline{\mathbb{R}}$	dual numbers set
$S\mathbb{O}_3$	orthogonal real tensors set
$\underline{S\mathbb{O}}_3$	orthogonal dual tensor set

$\mathcal{SO}_3^{\mathbb{R}}$ time depending real tensorial functions

$\underline{\mathcal{SO}}_3^{\mathbb{R}}$ time depending dual tensorial functions

$\mathbb{U}$ unit quaternions set

$\underline{\mathbb{U}}$ unit dual quaternions set

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