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A Perturbation Theory for Nonintegrable Equations with Small Dispersion

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Abstract

We describe an approach called the “weak asymptotics method” to construct multisoliton asymptotic solutions for essentially nonintegrable equations with small dispersion. This paper contains a detailed review of the method and a perturbation theory to describe the interaction of distorted solitons for equations with small perturbations. All constructions have been realized for the gKdV equation with the nonlinearity $u^\mu$, $\mu \in (1, 5)$.

Keywords: generalized Korteweg-de Vries equation, soliton, interaction, perturbation, weak asymptotics method

2010 Mathematics Subject Classification: 35D30, 35Q53, 46F10

1. Introduction

We consider the problem of propagation and interaction of soliton-type solutions of nonlinear equations. Our basic example is the nonhomogeneous version of the generalized KdV equation

$$\frac{\partial u}{\partial t} + \frac{\partial u^\mu}{\partial x} + \varepsilon \frac{\partial f}{\partial x} u^{\mu-1} = f(u, \varepsilon \frac{\partial u}{\partial x}), \quad x \in \mathbb{R}, \quad t > 0,$$

where $\mu \in (1, 5)$, $\varepsilon \ll 1$, $f(u, z)$ is a known smooth function such that $f(0, 0) = 0$. Note that the restriction on $\mu$ implies both the soliton-type solution and the stability of the equation with respect to initial data (see, for example [1, 2]).

In the special case $f \equiv 0$ and $\mu = 2$ ($\mu = 3$), Eq. (1) is the famous KdV (modified KdV) equation. It is well known that KdV (mKdV) solitons are stable and interact in the elastic manner: after
the collision, they preserve the original amplitudes and velocities shifting the trajectories only
(see [3] and other bibliographies devoted to the inverse scattering transform (IST) method). In
the case of $\mu = 2$ ($\mu = 3$) but with $f \neq 0$, Eq. (1) is a nonintegrable one. However, using the small-
ness of $\epsilon$ (or of $f$ for other scaling), it is possible to create a perturbation theory that describes
the evolution of distorted solitons (see the approaches by Karpman and E. Maslov [4] and
Kaup and Newell [5] on the basis of the IST method, and the “direct” method by V. Maslov
and Omel’yanov [6]). Moreover, the approach by V. Maslov and Omel’yanov [6] can be easily
extended to essentially nonintegrable equations ($\mu \neq 2, 3$), but for a single soliton only. In fact,
it is impossible to use any direct method in the classical sense for the general problem of the
wave interaction. To explain this proposition, let us consider the homogeneous gKdV equation
$$\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u^\nu}{\partial x} + \epsilon^2 \frac{\partial^3 u}{\partial x^3} = 0, & \quad x \in \mathbb{R}, \ t > 0.
\end{align*}$$

It is easy to find the explicit soliton solution of (2),
$$u(x, t, \epsilon) = A\omega(\beta(x - Vt)/\epsilon), \quad \omega(\eta) = \cosh^{-1}(\eta/\gamma),
$$
$$\gamma = 2/(\mu - 1), \quad V = \beta^2, \quad A^{\nu-1} = V(\mu + 1)/2.
$$

Next let us consider two-soliton initial data
$$u|_{t=0} = \sum_{i=1}^2 A_i \omega(\beta_i(x - x_{(i,0)}/\epsilon),
$$
where $x_{(1,0)} > x_{(2,0)}$ and $A_2 > A_1$. Obviously, since $(x_{(2,0)} - x_{(1,0)})/\epsilon \to \infty$ as $\epsilon \to 0$, the sum of the waves (3)
$$u = \sum_{i=1}^2 \omega(\beta_i(x - V_i t - x_{(i,0)})/\epsilon)
$$
approximates the problem (2), (5) solution with the precision $O(\epsilon^n)$ but for $t \ll 1$ only.
Conversely, the sum (6) does not satisfy the gKdV equation for $t \sim O(1)$ in view of the trajec-
tories $x = Vt + x_{(i,0)}$ intersection at a point $(x', t')$.

Let us consider shortly how it is possible to analyze the problem (2), (5). There are some
different cases:

1. Let $A_1 \ll A_2$. Then, one can construct an asymptotic solution
$$u = W((x - \varphi_2(t))/\epsilon, t, x, \epsilon, \nu),
$$
where $\nu = A_2/A_1 \ll 1$ and $W((x - \varphi_2(t))/\epsilon, t, x, \epsilon, \nu) = A_2\omega(\beta_2(x - V_2 t - x_{(2,0)})/\epsilon) + O(\nu + \epsilon)$. Thus, to
find the leading term of the asymptotics, we obtain an equation with nonlinear ordinary dif-
fential operator; whereas to construct the corrections, it is enough to analyze the linearization
of this operator. This construction (with a little bit of other viewpoints) has been realized
by Ostrovsky et al. [7].
Let $A_2 - A_1 \ll 1$. We write again the ansatz in the form (7), where $\nu = A_2 - A_1 \ll 1$ now, and we assume $\nu/\varepsilon \ll 1$. In fact, this case coincides with the problem considered in [7].

The amplitudes $A_2 > A_1$ are arbitrary numbers. Then, we should write a two-phase ansatz

$$u(x, t, \varepsilon) = W((x - \varphi_1(t))/\varepsilon, (x - \varphi_2(t))/\varepsilon, t, x, \varepsilon)$$

without any additional parameter. Substituting (8) into equation (2), we obtain for the leading term $W_0(\tau_1, \tau_2, t)$:

$$A^2 W_0 + B^3 W_0 = 0$$

(9)

Since $\varphi_1 \neq \varphi_2$, we can pass to new variables,

$$\eta = (\tau_1 - \tau_2)/(\varphi_2 - \varphi_1), \xi = (\varphi_1 \tau_2 - \varphi_2 \tau_1)/(\varphi_1 - \varphi_2),$$

and transform equation (9) to the gKdV form (2) again

$$\frac{\partial W_0}{\partial \eta} + \frac{\partial W_0}{\partial \xi} + \frac{\partial^3 W_0}{\partial \xi^3} = 0.$$  

(10)

Therefore, to construct two-phase asymptotics, we should solve (10) explicitly what is impossible for any essentially nonintegrable case.

This difficulty can be overcome by using the weak asymptotics method. The main point here is that solitons tend to distributions as $\varepsilon \to 0$. Thus, it is possible to pass to the weak description of the problem, ignore the actual shape of the multiwave solutions, and find only the main solution characteristics, that is, the time dynamics of wave amplitudes and velocities. The weak asymptotics method has been proposed at first for shock wave type solutions [8] and for soliton-type solutions [9] many years ago. Further generalizations, modifications, and adaptations to other problems can be found in publications by M. Colombeau, Danilov, Mitrovic, Omel’yanov, Shelkovich, and others, see, for example, [10–20] and references therein.

The contents of the paper are the following: in Section 2, we present a detailed survey of the weak asymptotics method application to the problem of multisoliton asymptotics and Section 3 contains new results, namely a perturbation theory to describe the evolution and collision of distorted solitons for equation (1).

### 2. Weak asymptotics method

#### 2.1. Main definitions

Let us associate equation (2) with first two conservation laws written in the differential form:

$$\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} = \varepsilon^2 \frac{\partial^3 R_j}{\partial x^3}, \quad j = 1, 2,$$

(11)

$$Q_1 = u, \quad P_1 = u^\mu, \quad Q_2 = u^2, \quad P_2 = 2\mu u^\mu/(\mu + 1) - 3 (\varepsilon u)_x^3,$$

(12)

and $R_1 = u, R_2 = u^2$. Next, we define smallness in the weak sense:
At the same time for any left-hand sides of the relations (11). It is easy to check that \( \psi = \text{const} > 0 \) and a piecewise continuous function uniformly in \( \varepsilon \geq 0 \).

Following [9, 17, 18], we define two-soliton weak asymptotics:

**Definition 1.** A function \( v(t, x, \varepsilon) \) is said to be of the value \( O_{\varepsilon}(\cdot) \) if the relation \( \int_\varepsilon^\infty v(t, x, \varepsilon)\psi(x)dx = O(\varepsilon) \) holds uniformly in \( t \) for any test function \( \psi \in \mathcal{D}(\mathbb{R}) \). The right-hand side here is a \( c^* \)-function for \( \varepsilon = \text{const} > 0 \) and a piecewise continuous function uniformly in \( \varepsilon \geq 0 \).

Following [9, 17, 18], again, we write the asymptotic ansatz in the form:

\[
\begin{align*}
  u &= \sum_{i} G_i(\tau)\omega(\beta_i(x-\varphi_i(t, \tau, \varepsilon))/\varepsilon), \quad G_i(\tau) = A_i + S_i(\tau). 
\end{align*}
\]

Here \( \varphi_i = \varphi_i(t) + \varepsilon \psi_i(\tau) \), where \( \varphi_i = Vt + x_{\omega_i} \) are the trajectories of noninteracting solitary waves, \( \tau = \varphi_i(t)/\varepsilon \) denotes the “fast time”, \( \psi_i(t) = \beta_i(\varphi_i(t) - \varphi_i(t)) \), and the phase and amplitude corrections \( \varphi_i, \psi_i \) are smooth functions such that with exponential rates

\[
\begin{align*}
  \varphi_i(\tau) \to 0 & \quad \text{as} \quad \tau \to -\infty, \quad \varphi_i(\tau) \to \varphi_i^* = \text{const}, \quad \text{as} \quad \tau \to \infty, \\
  S_i(\tau) \to 0 & \quad \text{as} \quad \tau \to \pm\infty.
\end{align*}
\]

### 2.2. Two-wave asymptotic construction

To construct the asymptotics, we should calculate the weak expansions of the terms from the left-hand sides of the relations (11). It is easy to check that

\[
\begin{align*}
  u &= \varepsilon \sum_{i} a_i \frac{G_i}{\beta_i} \delta(x - \varphi_i) + O_{\varepsilon}(\varepsilon^3),
\end{align*}
\]

where \( \delta(x) \) is the Dirac delta-function. Here and in what follows, we use the notation

\[
\begin{align*}
  a_i \overset{\text{def}}{=} \int_{-\infty}^{\infty} (\omega(\eta))^k d\eta, \quad k > 0, \quad a_i \overset{\text{def}}{=} \int_{-\infty}^{\infty} (\omega(\eta))^2 d\eta.
\end{align*}
\]

At the same time for any \( F(u, \varepsilon \partial u/\partial x) \in C^1 \), we have

\[
\begin{align*}
  \int_{-\infty}^{\infty} F \left( \sum_i G_i \omega(\beta_i(x-\varphi_i)), \sum_i \beta_i G_i \omega(\beta_i(x-\varphi_i)) \right) \psi(x) dx \\
  = \varepsilon \sum_i \frac{1}{\beta_i} \int_{-\infty}^{\infty} F(A_i \omega(\eta), \beta_i A_i \omega(\eta)) \psi(\varphi_i + \varepsilon \eta) d\eta \\
  + \varepsilon \sum_i \frac{1}{\beta_i} \int_{-\infty}^{\infty} \left[ F \left( \sum_i G_i \omega(\eta), \sum_i \beta_i G_i \omega(\eta) \right) \right] \psi(\varphi_i + \varepsilon \eta) d\eta \\
  - \sum_i \int_{-\infty}^{\infty} (\beta_i A_i \omega(\eta)) \psi(\varphi_i + \varepsilon \eta) d\eta \\
  \overset{\text{def}}{=} \int_{-\infty}^{\infty} (\omega(\eta))^2 d\eta,
\end{align*}
\]
\[
\eta_{12} = \theta \eta - \sigma, \quad \eta_{22} = \eta, \quad \sigma = \beta_1 (\psi_1 - \phi_2)/\epsilon, \quad \theta = \beta_1 / \beta_2.
\]

We take into account that the second integrand in the right-hand side of (18) vanishes exponentially fast as \(|\psi_1 - \phi_2|\) grows; thus, its main contribution is at the point \(x^*\). We write

\[
\psi_\circ = x^* + V_0 (t - t^*) = x^* + \epsilon V_0 \tau / \psi_0 \quad \text{and} \quad \phi_2 = x^* + \epsilon \chi_1.
\]

where \(\psi_\circ = \beta_1 (V_0 - V_0) \tau / \psi_0 \phi_2 = V_0 / \psi_0 + \phi_2\). It remains to apply the formula

\[
f(\tau) \delta (x - \phi_2) = f(\tau) \delta (x - x^*) - \epsilon \chi_1 f(\tau) \delta (x - x^*) + \mathcal{O}_d (\epsilon^2),
\]

which holds for each \(\phi_2\) of the form (20) with slowly increasing \(\chi_1\) and for \(f(\tau)\) from the Schwartz space. Moreover, the second term in the right-hand side of (21) is \(\mathcal{O}_d (\epsilon)\). Thus, under the assumptions (14) and (15), we obtain the weak asymptotic expansion of \(F(u, \epsilon du/\partial x)\) in the final form:

\[
F(u, \epsilon u_x) = \epsilon \sum_{n=0}^{\infty} \left( \frac{a_n^{(0)}}{\beta_1} \delta (x - \phi_0) - \frac{a_n^{(1)}}{\beta_1} \delta (x - x^*) \right) + \frac{\epsilon}{\beta_1} \left( \mathcal{R}_f^{(0)} \delta (x - x^*) - \epsilon \mathcal{R}_f \delta (x - x^*) \right) + \mathcal{O}_d (\epsilon^2),
\]

where

\[
a_n^{(0)} = \int \eta' F(A_0, \omega (\eta), \beta_1, \omega' (\eta)) d\eta,
\]

\[
\mathcal{R}_f^{(0)} = \int \eta' \left( \mathcal{R}_f \sum \beta_i G_i \omega' (\eta_i) + \mathcal{R}_f \sum A_i, \omega (\eta_i) \right) d\eta.
\]

Here, we take into account that to define \(\delta u^2/\partial x \mod \mathcal{O}_d (\epsilon^2)\), it is necessary to calculate \(u^2\) with the precision \(\mathcal{O}_d (\epsilon)\). Thus, using (22) with \(F(u) = u^2\) and transforming (16) with the help of (21), we obtain modulo \(\mathcal{O}_d (\epsilon)\):

\[
u = \epsilon \sum a_i K_{i,1}^{(0)} \delta (x - \phi_0) + \epsilon \sum a_i K_{i,1}^{(1)} \left( \delta (x - x^*) - \epsilon \chi_1 \delta (x - x^*) \right),
\]

\[
u^2 = \epsilon \sum a_i K_{i,1}^{(2)} \delta (x - \phi_0) + \frac{\epsilon}{\beta_1} \left( \mathcal{R}_f^{(1)} \delta (x - x^*) - \epsilon \mathcal{R}_f \delta (x - x^*) \right),
\]

where

\[
K_{i,1}^{(0)} = G_i / \beta_1, \quad K_{i,1}^{(1)} = A_i / \beta_1, \quad K_{i,1}^{(2)} = \mathcal{R}_f.\]

Calculating weak expansions for other terms from Definition 2 and substituting them into (11), we obtain linear combinations of \(\epsilon \delta (x - \phi_0), i = 1, 2\), \(\delta (x - x^*)\), and \(\epsilon \delta (x - x^*)\). Therefore, we pass to the system:

\[
a_1 V_i K_{i,1}^{(0)} - a_i^{(0)} / \beta_1 = 0, \quad a_1 V_i K_{i,1}^{(1)} / \beta_1 = 0, \quad i = 1, 2,
\]

\[
\sum_{i=1}^{2} K_{i,1}^{(0)} = 0, \quad \mathcal{R}_f^{(0)} = 0, \quad i = 1, 2.
\]
\[
\psi_0 \frac{d}{dt} \sum_{i=1}^{2} [K^{(1)}_i q_{1i} + \chi_i K^{(2)}_i] = f, \quad \psi_0 \frac{d}{dt} \left( \sum_{i=1}^{2} a_i K^{(2)}_i q_{1i} + \mathcal{R}_i \right) = F,
\]

(30)

where

\[
f = \frac{1}{a_1 \beta_2} \mathcal{R}^{(2)}_i, \quad F = \frac{1}{\beta_2} \mathcal{R}^{(2)}_i - a_1 \psi_0 \sum_{i=1}^{2} q_{1i} \frac{dK^{(0)}_i}{dt}.
\]

(31)

The first four algebraic equations (28) imply again the relation (4) among \(A_i, \beta_i, \) and \(V_i.\) Furthermore, there exists a number \(\theta^* \in (0, 1)\) such that equations (29), (30) have the required solution \(S_i, q_{1i}\) with the properties (14) and (15) under the sufficient condition \(\theta \leq \theta^*\) (see [9, 17]). It is obvious that the existence of the weak asympotics (13) with the properties (14) and (15) implies that the solitary waves interact like the KdV solitons at least in the leading term.

**Theorem 1.** Let \(\theta \leq \theta^*.\) Then (13) describes mod \(O(\varepsilon^2)\) the elastic scenario of the solitary waves interaction for the \(\mu\)-gKdV equation (2).

Numerical simulations ([14, 15, 17]) confirm the traced analysis, see Figure 1. Note that a small oscillating tail appears after the soliton collision, see [15] for detail. Obviously, this effect is similar to the “radiation” appearance for the perturbed KdV [21].

### 2.3. Multisoliton interaction

\(N\)-wave solutions of the form similar to waves (13) contain \(2N\) free functions \(S_i, q_{1i}\). Thus, to describe an \(N\)-soliton collision, we should consider \(N\) conservation laws. However, nonintegrability implies the existence of a finite number of conservation laws only. For this reason, we need to involve into the consideration balance laws. For the gKdV-4 equation, the first conservation and balance laws have the form

\[
\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} + \varepsilon^{-1} K_j = O_\varepsilon(\varepsilon^2),
\]

(32)

Figure 1. Evolution of two solitary waves for \(\mu = 4\) and \(\varepsilon = 0.1.\)
where $Q_j, P_j, j = 1, 2$, coincide with (12) for $\mu = 4, K = 0, i = 1, 2, 3$,

$$Q_3 = (\varepsilon u_x)^2 - \frac{2}{3} u_x^3, \quad P_3 = 16 u^3 (\varepsilon u_x)^2 - u^8 - 3 (\varepsilon^2 u_{xx})^2, \quad (33)$$

$$Q_4 = \frac{1}{2} (\varepsilon^2 u_{xx})^2 + \frac{5}{21} u^8 - \frac{10}{3} u^3 (\varepsilon u_x)^2, \quad K_4 = - (\varepsilon u_x)^3, \quad (34)$$

$$P_4 = 12 u^3 (\varepsilon^2 u_{xx})^2 - 19 u (\varepsilon u_x)^3 - \frac{3}{2} (\varepsilon^3 u_{xxx})^2 + \frac{160}{231} u^4 - \frac{100}{3} u^8 (\varepsilon u_x)^2. \quad (35)$$

Note that the nondivergent "production" $\varepsilon^{-1}K_j$ has the same value $O(\varepsilon^{-1})$ (in the $C$-sense and for rapidly varying functions) as the first ones in (32).

The formal scheme of the asymptotic construction is similar to the one described above: we write the ansatz of the form (13) but with $N$ summands, found weak representations for all terms in (32), and pass to a system similar to (28)–(30). The main obstacle here is the proof that this system admits a solution with the properties of (14), (15). This idea has been realized in [18, 19] for the problem of three soliton collisions for the gKdV-4 equation.

**Theorem 2.** Let us denote $A_i$ the amplitudes of the original solitons and $x_{(i,0)}$ their initial positions such that $A_i+1 > A_i, x_{(i,0)} > x_{(i+1,0)}$, and $i = 1, 2$. Let all trajectories $x = \varphi_0(t)$ have an intersection point $(x^*, t^*)$. Then, under the assumption

$$\beta_3/\beta_3 = \nu^3, \quad \beta_i/\beta_3 = \nu^{3(3+\alpha)/2}, \quad \alpha \in [0, 1) \quad (36)$$

with sufficiently small $\nu < 1$, the three-phase asymptotic solution exists and describes $mod O(\varepsilon^{-1})$ the elastic scenario of the solitary waves interaction.

**Figure 2** depicts the evolution of a three-wave solution [14].

![Figure 2](http://dx.doi.org/10.5772/intechopen.71030)
2.4. Asymptotic equivalence

Let us come back to the case of two-phase asymptotics and transform the ansatz (13) to the following form:

\[ R = \sum_i G(\tau) \omega \left( \beta_1 \frac{x - \varphi(l, t, \epsilon)}{\epsilon} \right) + \mathcal{E}(\tau) W \left( \beta_1 \frac{x - \varphi(l, t, \epsilon)}{\epsilon} \right) \right), \]  

(37)

where \( \mathcal{E}(\tau), i = 1, 2 \) are arbitrary functions from the Schwartz space,

\[ W(\eta) = d^{2\alpha_1} \omega(\eta)/d\eta^{2\alpha_1}, \]

(38)

and \( l \geq 1 \) is an arbitrary integer. Calculating the weak representations for \( \bar{u} \) and \( \bar{u}^\prime \), we obtain

\[ \bar{u} = u + O_\eta(\epsilon^{2\alpha_1}), \]
\[ \bar{u}^2 = \epsilon \sum_i a_i K^{(i)}(x - \varphi_1) + \frac{\epsilon}{\beta_1} R^{(i)}(x - \varphi_1) + O_\eta(\epsilon^2), \]

(39)

where

\[ R^{(i)} = \int \left( \sum_i \left( G(\eta) + \mathcal{E}(\eta) \right) \right)^2 d\eta, \]

(40)

and \( \bar{u} \) in the right-hand side in (39) is the representation (25). Thus, the difference between \( u \) of the forms (13) and (37) is arbitrarily small in the sense \( \mathcal{D}(R) \). At the same time, instead of (29), (30), we obtain

\[ \sum_i K_n^{(i)}(x) = 0, \quad R^{(i)} = 0, \quad i = 1, 2, \]

(41)

\[ \psi_0 \frac{d}{dt} \sum_i \left( K_n^{(i)}(x) + x \right) K_n^{(i)}(x) = \tilde{f}, \quad \psi_0 \frac{d}{dt} \left( \sum_i a_i K_n^{(i)} q_n \right) = \tilde{f}, \]

(42)

where \( \tilde{f}, \tilde{F} \) differ from \( f, F \) in the same manner as \( R^{(i)}(x) \) differs from \( R^{(i)}(x) \). The system (41) and (42) have a solution with the properties (14) and (15) [9, 12]; however, it differs from the solution of Eqs. (29) and (30) with the value \( O(1) \) in the C-sense. Moreover, the asymptotic solutions (13) and (37) differ with the precision \( O_\eta(\epsilon) \) in the sense of Definition 1. This implies the principal impossibility to describe explicitly neither the real shape of the waves at the time instant of the collision nor the real \( \epsilon \)-size displacements of the trajectories after the interaction. However, the nonuniqueness of the value \( O(\epsilon) \) is concentrated within \( O(\epsilon^{1-\nu}) \)-neighborhood of the time instant \( t' \) of the interaction, \( \nu > 0 \). Thus, it is small in the \( \mathcal{D}(R) \) sense. We set

**Definition 3.** Functions \( u_1(x, t, \epsilon) \) and \( u_2(x, t, \epsilon) \) are said to be asymptotically equivalent if for any test function \( \psi \in \mathcal{D}(R) \)

\[ \int_{-\infty}^t \int_{-\infty}^t (u_1(x, t, \epsilon) - u_2(x, t, \epsilon)) \psi(x, t) dx \, dt = O(\epsilon^2). \]

(43)

In this sense, the solutions (13) and (37) are asymptotically equivalent.

We now focus attention on another question: how to choose, from the set of all possible conservation and balance laws, those that allow to construct a multiphase asymptotic solution? It seems
that there is not any rule and it is possible to use arbitrary combination of the laws. Thus, there appears the next question: what is the difference between such solutions? This problem has been discussed in [20] for two-phase asymptotic solutions of the gKdV-4 equation. Let us define two-phase asymptotics in the following manner:

**Definition 4.** Let $1 \leq k_0 < k_1 \leq 4$ and let a sequence $u_{i,k_0} = u_{i,k_1}(t,x,\epsilon)$ belong to the same functional space as $u(t,x,\epsilon)$ in Definition 2. Then, $u_{i,k_0}$ is called a weak asymptotic mod $O(\mathcal{D}'(\epsilon^2))$ solution of (2) if the relations (32) hold for $j = k_0$ and $j = k_1$ uniformly in $t$.

A detailed analysis implies the assertion [20].

**Theorem 3.** Let $\theta$ be sufficiently small. Then, the weak asymptotic solutions $u_{1,k_1}$ and $u_{1,k_1}'$ of the problem (2), (5) exist and they are asymptotically equivalent for all $k_1, k_1' \in \{2, 3, 4\}$.

### 3. Collision of distorted solitons

We consider now the nonhomogeneous version of the gKdV equation (1). It is easy to verify that, in the case of rapidly varying solutions, the right-hand side $f$ can be treated as a “small perturbation.”

An approach to construct one-phase self-similar asymptotic solutions for (1) had been created in [6] (see also [17]). Let us generalize this approach to the multiphase case. From the beginning, we state that equation (1) is associated with balance laws, the first two of which are

$$\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} + K_j = O(\epsilon^2), \quad j = 1, 2,$$

(44)

where $Q_j$ and $P_j$ coincide with ones described in (12),

$$K_1 = -f(u, \epsilon u), \quad K_2 = -uf(u, \epsilon u).$$

(45)

Note that, in contrast to $K_j$ in (32), productions here are regularly degenerating functions with the value $O(1)$ in the $C$-sense.

Let us first construct a two-phase version of self-similar asymptotics, which assumes a special initial data for (1) and discuss afterward how to treat it for more realistic initial data. By analogy with Definition 2, we write:

**Definition 5.** Let a sequence $u = u(t,x,\epsilon)$ belong to the same functional space as in Definition 2. Then, $u$ is called a weak asymptotic mod $O(\epsilon^2)$ solution of (1) if the relation (44) hold uniformly in $t \in (0, T), \tau = \min(\mu, 2)$.

Generalizing one-phase asymptotics, we write the ansatz as

$$u = \sum_{i=1}^{2} \left[ G_i(\tau, t) \omega(\eta_i) + \epsilon (z_0(\tau, t) \eta_i + \theta_i(\tau) \omega(\eta_i)) \right],$$

(46)

$$G_i(\tau, t) = A_i(\tau) + S(\tau), \quad \eta_i = \beta_i(\eta - \varphi_i)/\epsilon, \quad \varphi_i(\tau, t, \epsilon) = \varphi_{i,0} + \epsilon \varphi_{i,1}.$$

(47)
Here $A(t)$, $q_1(t) = \dot{q}_1(t)$, $\varepsilon(t) = \gamma A^r(t)$, $\omega(t)$, $S(t)$, $q_2(t)$ are the same as in (13); $\tau = \psi_1(t)/\varepsilon$ with $\psi_1(t) = q_2(t) - q_0(t)$ denotes the "fast time" again; $z(x,t) \in C^r$; and $\psi$, $S$ are smooth functions such that

$$\Theta_1(\tau) \to 0 \text{ as } \tau \to -\infty, \quad \Theta_2(\tau) \to \Theta_2^0 = \text{const} \text{ as } \tau \to +\infty,$$

with exponential rates. We assume also the intersection of the trajectories $q_1 = q_2, t = 1, 2$ at a point $x^* = q_2(t)$ namely,

$$\exists \tau^* > 0 \text{ such that } q_{20}(t^*) = q_{20}(t^*), \quad \sum_i \frac{\partial}{\partial t} (q_{20}(t) - q_{20}(t))_{|_{t^*}} \neq 0.$$

It is easy to verify the weak representations with the precision $O(\varepsilon)$:

$$u = \varepsilon \sum_i \left\{ \left. H(q_i) \delta(x-q_i) + a_i K_i^{(1)}(x) \delta(x-x^*) \right. \right\} \frac{\partial}{\partial t} + \varepsilon \sum_i \frac{\partial}{\partial t} H(q_i) - x^*)$$

$$= \varepsilon \sum_i \left\{ \psi_i \left. \frac{\partial}{\partial t} \left( q_i + \varepsilon \frac{\partial}{\partial t} \delta(x-q_i) \right) \right. \right\} \delta(x-x^*)$$

$$= \varepsilon \sum_i \left\{ \psi_i \left. \frac{\partial}{\partial t} \left( \delta(x-q_i) + \frac{\partial}{\partial t} \right) \right. \right\} \delta(x-x^*)$$

where $H(x)$ is the Heaviside function, $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$; $\varepsilon \frac{\partial}{\partial t} \delta(x-q_i)$; and $\psi_1(t,0)$ is the solution of the equation $q_{20}(t + \Phi) - q_{20}(t + \Phi) = \varepsilon \tau$, which exists in accordance with (50).

Next, the existence of nonsoliton summands in (51) implying a correction of formula (22), namely

$$F(u, h, \varepsilon) = \varepsilon \sum_i \left\{ \psi_i \left. \frac{\partial}{\partial t} \delta(x-q_i) + \frac{\partial}{\partial t} \right) \right\} \delta(x-x^*)$$

$$= \varepsilon \sum_i \left\{ \psi_i \left. \frac{\partial}{\partial t} \delta(x-q_i) + \varepsilon \frac{\partial}{\partial t} \right) \right\} \delta(x-x^*)$$

where $\varepsilon \frac{\partial}{\partial t} \delta(x-q_i)$ and $\psi_1(t,0)$ are defined in (23), (24), $F(0,0,0) = \Delta \frac{\partial (u,0)}{\partial u}$; and $\psi_1(t,0)$ is the solution of the equation $q_{20}(t + \Phi) - q_{20}(t + \Phi) = \varepsilon \tau$, which exists in accordance with (50).

Repeating the same calculations as above, we obtain linear combinations of $\varepsilon \delta(x-q_i)$, $\varepsilon \delta(x-q_i) - \varepsilon \delta(x-q_i)$, $i = 1, 2; \delta(x-x^*)$, $\varepsilon \delta(x-x^*)$, and $\varepsilon \delta(x-x^*)$. Equating zero, the coefficients of $\varepsilon \delta(x-q_i)$ and $\varepsilon \delta(x-q_i)$ yield

$$a_i A_i \frac{\partial q_i}{\partial t} = a_i^{(0)} q_i \frac{\partial^2}{\partial t^2} + a_i^{(0)} \frac{\partial}{\partial t},$$

$$a_i A_i \frac{\partial q_i}{\partial t} = a_i^{(0)} q_i \frac{\partial^2}{\partial t^2} + a_i^{(0)} \frac{\partial}{\partial t}.$$
Consequently, the condition Furthermore, equating zero the coefficients of \( \frac{d}{dt} \beta_i \) will assume equalities in equations (4) and rewrite the model equation for \( \omega(\eta) \) as follows:

\[
\frac{d}{d\eta} \left[ -\omega + \frac{\mu + 1}{2} \mu^\prime \omega + \frac{d^2}{d\eta^2} \right] = 0.
\]

Simple manipulations with (56) allow us to find relations between structural constants:

\[
a_i = (\mu + 1) a_i / 2, \quad a_i = (\mu + 3) a_i^{-1} / 2, \quad a_i = (\mu - 1) a_i^{-1} / 4.
\]

Next, we use (57), the equality \( \beta_i = \gamma A_i \), add the initial conditions, and obtain from (54) the Cauchy problem

\[
\begin{align*}
\frac{dA_i}{dt} &= -c_i a_i A_i^{-1}, & \frac{d\varrho_i}{dt} &= \frac{a_i}{a_i^{-1}} A_i^{-1}, & t > 0, \\
A_i \bigg|_{t=0} &= A_i^0, & \varrho_i \bigg|_{t=0} &= \varrho_i^0,
\end{align*}
\]

where \( c_i = 2/(a_i(5 - \mu)) \); \( A_i^0 \) > 0 and \( \varrho_i^0 \) are arbitrary numbers; and \( i = 1, 2 \). Note also that the first equalities in equations (54) and (55) are equivalent.

Next, equating zero the coefficients of the Heaviside functions, we obtain the equations

\[
\frac{\partial z_i}{\partial t} = f_i(0,0) z_i, \quad x < \varrho_i(t), \quad t > 0, \quad i = 1, 2.
\]

In view of (58) \( \varrho_i / dt > 0 \), so we use the second equality in (55) to state the correct initial condition for (60)

\[
z_i(x, t) \bigg|_{x=\varrho_i(0)} = \sqrt{2} \, a_i^{\varrho_i(0)} A_i^{(3+\mu)/2} + c_i A_i^{\varrho_i(0)} A_i^{(1+\mu)/2}(t), \quad t > 0,
\]

\[
z_i(x, t) \bigg|_{t=0} = z_i^{0}(x), \quad x \leq x_{\varrho_i(0)}
\]

where \( c_i = a_i(3 - \mu)(1 + \mu)/(2a_i(5 - \mu)) \), \( z_i^{0}(x) \) is an arbitrary smooth function, which satisfies the consistency condition

\[
z_i^{0}(x_{\varrho_i(0)}) = \left[ \sqrt{2} \, a_i^{\varrho_i(0)} A_i^{(3+\mu)/2} + c_i A_i^{\varrho_i(0)} A_i^{(1+\mu)/2} \right]_{t=0}.
\]

We should note that the nonlinearity \( u^\varepsilon \) in (1) can require the inequality \( u \geq 0 \). To this end, we will assume

\[
A_i(t) > 0, \quad z_i(x, t) \bigg|_{x=\varrho_i(0)} \geq 0 \quad \text{for} \quad t \geq 0.
\]

Furthermore, equating zero the coefficients of \( \delta(x - x^*) \) and \( \varepsilon \delta(x - x^*) \) yield (29), (30) again. Consequently, the condition \( \theta \leq \theta^* \) guaranties the existence of \( S, \varrho_i \) with the properties of (14), (15). In particular
where

\[ q = \min\{1, \gamma\}, \quad \lambda(\sigma) = a_i^2 \int_{-\infty}^{\infty} \omega(\eta) \omega(\eta_\omega) d\eta. \]  

The last step of the construction is the determination of \( \Theta(x), i = 1, 2 \). By setting the coefficients of \( \epsilon \delta(x - x') \) zero, we obtain

\[ \frac{\partial}{\partial t} \phi_i(\epsilon \delta(x - x')) = \mathcal{F}_i \left|_{\epsilon = \epsilon'} \right., \]

\[ \mathcal{F}_i = -\sum \left[ a_i \psi_i(x', t) \frac{\partial}{\partial x} - \psi_i(x', t) \frac{\partial}{\partial t} \right] \left|_{\epsilon' = \epsilon} \right. + c_i \mathcal{R}(\epsilon) \left( \sigma, \gamma \right) \left| \epsilon = \epsilon' \right., \]

\[ a_i = a_i/(2a_i), \quad c_i = 2a_i \psi_i(x', t) \theta, \quad \lambda(\sigma) = a_i^2 \int_{-\infty}^{\infty} \omega(\eta) \omega(\eta_\omega) d\eta. \]

Calculating the determinant \( \Delta \) of the matrix in the left-hand part of (67) and using (65), we conclude

\[ \Delta = (G_2 - G_1 + \lambda(G_1 - \theta G_2)) \left|_{\epsilon = \epsilon'} \right. = f_\sigma \left( 1 - \theta^r - \lambda(\theta - \theta^r) - \lambda^2 \theta^r(1 + O(\theta^r)) \right) \left| \epsilon = \epsilon' \right.. \]

Obviously, \( \Delta \neq 0 \) for sufficiently small \( \theta \). Since the right-hand sides \( \mathcal{F}_i \) belong to the Schwartz space, the functions \( \varphi_i \) exist and satisfy the assumption (48).

Henceforth, we pass to the final result:

**Theorem 4.** Let \( \theta \) be sufficiently small and let the assumptions (50), (63), and (64), if it is necessary, be fulfilled. Then, the self-similar two-wave weak asymptotic mod \( O(\epsilon^r) \) solution of the equation (1) exists and has the form (46).

Let us finally stress that the self-similarity implies a special choice of the initial data: for the classical asymptotics in the C-sense, there appears a very restrictive condition for small correction of the soliton \( A(0)\omega((x - x_0)/\epsilon) \) (see [6, 17]), and for weak asymptotics, there appears the restriction (63). If it is violated, then the perturbed soliton generates a rapidly oscillating tail of the amplitude \( o(1) \) ("radiation") instead of the smooth tails \( \epsilon u(x, t) \) (see [21] and numerical results [14, 15, 17]). Nowadays, this radiation phenomenon can be described analytically only for integrable equations, so that we should use self-similar approximation for essentially nonintegrable equations. However, the smooth tail \( \epsilon u(x, t) \), which can be treated
as an average of the radiation, describes sufficiently well the tendency of the radiation amplitude behavior, see graphics depicted in Figures 3 and 4, and other numerical results in [15, 17]).
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