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Sub-Manifolds of a Riemannian Manifold

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Abstract

In this chapter, we introduce the theory of sub-manifolds of a Riemannian manifold. The fundamental notations are given. The theory of sub-manifolds of an almost Riemannian product manifold is one of the most interesting topics in differential geometry. According to the behaviour of the tangent bundle of a sub-manifold, with respect to the action of almost Riemannian product structure of the ambient manifolds, we have three typical classes of sub-manifolds such as invariant sub-manifolds, anti-invariant sub-manifolds and semi-invariant sub-manifolds. In addition, slant, semi-slant and pseudo-slant sub-manifolds are introduced by many geometers.

Keywords: Riemannian product manifold, Riemannian product structure, integral manifold, a distribution on a manifold, real product space forms, a slant distribution

1. Introduction

Let \( i : M \to \tilde{M} \) be an immersion of an \( n \)-dimensional manifold \( M \) into an \( m \)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\). Denote by \( g = i^\ast \tilde{g} \) the induced Riemannian metric on \( M \). Thus, \( i \) becomes an isometric immersion and \( M \) is also a Riemannian manifold with the Riemannian metric \( \tilde{g}(X, Y) = \tilde{g}(X, Y) \) for any vector fields \( X, Y \) in \( M \). The Riemannian metric \( g \) on \( M \) is called the induced metric on \( M \). In local components, \( \tilde{g}_{ij} = \tilde{g}_{AB} \tilde{B}_i^A \) with \( \tilde{g} = \tilde{g}_{ij} dx^i dx^j \) and \( \tilde{g} = g_{AB} dU^B dU^A \).

If a vector field \( \xi_p \) of \( \tilde{M} \) at a point \( p \in M \) satisfies

\[ \tilde{g}(X_p, \xi_p) = 0 \]  

for any vector \( X_p \) of \( M \) at \( p \), then \( \xi_p \) is called a normal vector of \( M \) in \( \tilde{M} \) at \( p \). A unit normal vector field of \( M \) in \( \tilde{M} \) is called a normal section on \( M \) [3].

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By $T^1\mathcal{M}$, we denote the vector bundle of all normal vectors of $\mathcal{M}$ in $\mathcal{M}$. Then, the tangent bundle of $\mathcal{M}$ is the direct sum of the tangent bundle $TM$ of $\mathcal{M}$ and the normal bundle $T^1\mathcal{M}$ of $\mathcal{M}$ in $\mathcal{M}$, i.e.,

$$TM = TM \oplus T^1\mathcal{M}.$$  \hspace{1cm} (2)

We note that if the sub-manifold $\mathcal{M}$ is of codimension one in $\mathcal{M}$ and they are both orientiable, we can always choose a normal section $\xi$ on $\mathcal{M}$, i.e.,

$$g(X, \xi) = 0, \quad g(\xi, \xi) = 1,$$  \hspace{1cm} (3)

where $X$ is any arbitrary vector field on $\mathcal{M}$.

By $\bar{\nabla}$, denote the Riemannian connection on $\mathcal{M}$ and we put

$$\bar{\nabla}X Y = \nabla X Y + h(X, Y)$$  \hspace{1cm} (4)

for any vector fields $X, Y$ tangent to $\mathcal{M}$, where $\nabla X Y$ and $h(X, Y)$ are tangential and the normal components of $\bar{\nabla}X Y$, respectively. Formula (4) is called the Gauss formula for the sub-manifold $\mathcal{M}$ of a Riemannian manifold $(\mathcal{M}, \bar{g})$.

**Proposition 1.1.** $\nabla$ is the Riemannian connection of the induced metric $\tilde{g} = i^*\bar{g}$ on $\mathcal{M}$ and $h(X, Y)$ is a normal vector field over $\mathcal{M}$, which is symmetric and bilinear in $X$ and $Y$.

**Proof:** Let $\alpha$ and $\beta$ be differentiable functions on $\mathcal{M}$. Then, we have

$$\bar{\nabla}_\alpha X (\beta Y) = X\{\alpha(\beta Y) + \beta \tilde{\nabla} X Y\}$$

$$= X\{\alpha(\beta Y) + \beta \tilde{\nabla} X Y + \beta h(X, Y)\}$$

$$\nabla_\alpha X Y + h(\alpha X, \beta Y) = \alpha \beta \tilde{\nabla} X Y + \alpha X(\beta Y) + \alpha \beta h(X, Y)$$  \hspace{1cm} (5)

This implies that

$$\nabla_\alpha X (\beta Y) = \alpha X(\beta Y) + \alpha \beta \tilde{\nabla} X Y$$  \hspace{1cm} (6)

and

$$h(\alpha X, \beta Y) = \alpha \beta h(X, Y).$$  \hspace{1cm} (7)

Eq. (6) shows that $\nabla$ defines an affine connection on $\mathcal{M}$ and Eq. (4) shows that $h$ is bilinear in $X$ and $Y$ since additivity is trivial [1].

Since the Riemannian connection $\tilde{\nabla}$ has no torsion, we have

$$0 = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \tilde{\nabla}_X Y + h(X, Y) - \tilde{\nabla}_Y X - h(Y, X) + h(Y, X) - [X, Y].$$  \hspace{1cm} (8)

By comparing the tangential and normal parts of the last equality, we obtain
\[ \nabla_X Y - \nabla_Y X = [X, Y] \] 

(9)

and

\[ h(X, Y) = h(Y, X). \] 

(10)

These equations show that \( \nabla \) has no torsion and \( h \) is a symmetric bilinear map. Since the metric \( \tilde{g} \) is parallel, we can easily see that

\[
(\nabla_X \tilde{g})(Y, Z) = (\tilde{\nabla}_X \tilde{g})(Y, Z) \\
= \tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(Y, \tilde{\nabla}_X Z) \\
= \tilde{g}(\nabla_X Y + h(X, Y), Z) + \tilde{g}(Y, \nabla_X Z + h(X, Z)) \\
= \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z) \\
= g(\nabla_X Y, Z) + g(Y, \nabla_X Z)
\]

(11)

for any vector fields \( X, Y, Z \) tangent to \( M \), that is, \( \nabla \) is also the Riemannian connection of the induced metric \( g \) on \( M \).

We recall \( h \) the second fundamental form of the sub-manifold \( M \) (or immersion \( i \)), which is defined by

\[ h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(T^2M). \]

(12)

If \( h = 0 \) identically, then sub-manifold \( M \) is said to be totally geodesic, where \( \Gamma(T^2M) \) is the set of the differentiable vector fields on normal bundle of \( M \).

Totally geodesic sub-manifolds are simplest sub-manifolds.

**Definition 1.1.** Let \( M \) be an \( n \)-dimensional sub-manifold of an \( m \)-dimensional Riemannian manifold \( (\tilde{M}, \tilde{g}) \). By \( h \) we denote the second fundamental form of \( M \) in \( \tilde{M} \).

\( H = \frac{1}{n} \text{trace}(h) \) is called the mean curvature vector of \( M \) in \( \tilde{M} \). If \( H = 0 \), the sub-manifold is called minimal.

On the other hand, \( M \) is called pseudo-umbilical if there exists a function \( \lambda \) on \( M \), such that

\[
\tilde{g}\left(h(X, Y), H\right) = \lambda g(X, Y)
\]

(13)

for any vector fields \( X, Y \) on \( M \) and \( M \) is called totally umbilical sub-manifold if

\[ h(X, Y) = g(X, Y)H. \]

(14)

It is clear that every minimal sub-manifold is pseudo-umbilical with \( \lambda = 0 \). On the other hand, by a direct calculation, we can find \( \lambda = \tilde{g}(H, H) \) for a pseudo-umbilical sub-manifold. So, every
totally umbilical sub-manifold is a pseudo-umbilical and a totally umbilical sub-manifold is totally geodesic if and only if it is minimal [2].

Now, let $M$ be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$ and $V$ be a normal vector field on $M$, $X$ be a vector field on $M$. Then, we decompose

$$\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V,$$  

(15)

where $A_V X$ and $\nabla^\perp_X V$ denote the tangential and the normal components of $\nabla^\perp_X V$, respectively. We can easily see that $A_V X$ and $\nabla^\perp_X V$ are both differentiable vector fields on $M$ and normal bundle of $M$, respectively. Moreover, Eq. (15) is also called Weingarten formula.

**Proposition 1.2.** Let $M$ be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then

(a) $A_V X$ is bilinear in vector fields $V$ and $X$. Hence, $A_V X$ at point $p \in M$ depends only on vector fields $V_p$ and $X_p$.

(b) For any normal vector field $V$ on $M$, we have

$$g(A_V X, Y) = g(h(X, Y), V).$$  

(16)

**Proof:** Let $a$ and $\beta$ be any two functions on $M$. Then, we have

$$\tilde{\nabla}_aX(\beta V) = a\tilde{\nabla}_X(\beta V)
= a\{X(\beta)V + \beta\tilde{\nabla}_X V\}
-A_\beta aX + \nabla^\perp_{aX} \beta V = aX(\beta)V - a\beta A_V X + a\beta \nabla^\perp_X V.$$  

(17)

This implies that

$$A_\beta aX = a\beta A_V X$$  

(18)

and

$$\nabla^\perp_{aX} \beta V = aX(\beta)V + a\beta \nabla^\perp_X V.$$  

(19)

Thus, $A_V X$ is bilinear in $V$ and $X$. Additivity is trivial. On the other hand, since $g$ is a Riemannian metric,

$$X_{\tilde{g}}(Y, V) = 0,$$  

(20)

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Eq. (12) implies that

$$\tilde{g}(\tilde{\nabla}_X Y, V) + \tilde{g}(Y, \tilde{\nabla}_X V) = 0.$$  

(21)

By means of Eqs. (4) and (15), we obtain
The proof is completed [3]. Let \( M \) be a sub-manifold of a Riemannian manifold \( (\tilde{M}, \tilde{g}) \), and \( h \) and \( A_V \) denote the second fundamental form and shape operator of \( M \), respectively. The covariant derivative of \( h \) and \( A_V \) is, respectively, defined by

\[
\tilde{\nabla}_X h(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)
\]

and

\[
(\nabla_X A)VY = \nabla_X (A_V Y) - A_V \nabla_X Y - A_V \nabla_X Y
\]

for any vector fields \( X, Y \) tangent to \( M \) and any vector field \( V \) normal to \( M \). If \( \nabla_X h = 0 \) for all \( X \), then the second fundamental form of \( M \) is said to be parallel, which is equivalent to \( \nabla_X A = 0 \). By direct calculations, we get the relation

\[
g\left((\nabla_X h)(Y, Z), V\right) = g\left((\nabla_X A)VY, Z\right).
\]

**Example 1.1.** We consider the isometric immersion

\[
\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^4,
\]

\[
\phi(x_1, x_2) = (x_1, \sqrt{x_1^2-1}, x_2, \sqrt{x_2^2-1})
\]

we note that \( M = \phi(\mathbb{R}^2) \subset \mathbb{R}^4 \) is a two-dimensional sub-manifold of \( \mathbb{R}^4 \) and the tangent bundle is spanned by the vectors

\[
TM = sp\{e_1 = \left(\sqrt{x_1^2-1}, x_1, 0, 0\right), e_2 = \left(0, 0, \sqrt{x_2^2-1}, x_2\right)\}
\]

and the normal vector fields

\[
T^\perp M = sp\{w_1 = \left(-x_1, \sqrt{x_1^2-1}, 0, 0\right), w_2 = \left(0, 0, -x_1, \sqrt{x_2^2-1}\right)\}.
\]

By \( \tilde{\nabla} \), we denote the Levi-Civita connection of \( \mathbb{R}^4 \), the coefficients of connection, are given by

\[
\tilde{\nabla}_{e_1} e_1 = \frac{2x_1 \sqrt{x_1^2-1}}{2x_1^2-1} e_1 - \frac{1}{2x_1^2-1} w_1,
\]

\[
\tilde{\nabla}_{e_2} e_2 = \frac{2x_2 \sqrt{x_2^2-1}}{2x_2^2-1} e_2 - \frac{1}{2x_2^2-1} w_2
\]
and

$$\nabla_{e_1} e_1 = 0. \quad (31)$$

Thus, we have $h(e_1, e_1) = -\frac{1}{2x_1^2} w_1$, $h(e_2, e_2) = -\frac{1}{2x_1^2} w_2$ and $h(e_2, e_1) = 0$. The mean curvature vector of $M = \phi(\mathbb{R}^2)$ is given by

$$H = -\frac{1}{2}(w_1 + w_2). \quad (32)$$

Furthermore, by using Eq. (16), we obtain

$$g(A_{u_2} e_1, e_1) = g\left(h(e_1, e_1), w_1\right) = -\frac{1}{2x_1^2} (x_1^2 + x_1^2 - 1) = -1,$$

$$g(A_{u_2} e_2, e_2) = g\left(h(e_2, e_2), w_1\right) = -\frac{1}{2x_1^2} g(w_1, w_2) = 0,$$

and

$$g(A_{u_2} e_1, e_2) = 0.$$

Thus, we have

$$A_{u_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{u_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \quad (35)$$

Now, let $M$ be a sub-manifold of a Riemannian manifold $(\tilde{M}, g)$, $\tilde{R}$ and $R$ be the Riemannian curvature tensors of $\tilde{M}$ and $M$, respectively. From then the Gauss and Weingarten formulas, we have

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z$$

$$= \tilde{\nabla}_X \left(h(Y, Z)\tilde{\nabla}_Y Z + h(X, Z)\tilde{\nabla}_Y h(Y, Z)\right) - \tilde{\nabla}_Y \left(h(X, Z)\tilde{\nabla}_X Z + h(Y, Z)\tilde{\nabla}_X h(Y, Z)\right)$$

$$+ \tilde{\nabla}_{X,Y} Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$+ \nabla_{X,Y} Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$+ \nabla_{X,Y} Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X, Y)Z + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + A_{h(X,Y)} Z - A_{h(Y,Z)} X \quad (36)$$
Next, we will define the curvature tensor $R(X, Y)Z = R(X, Y)Z + A_{h[X, Z]}Y - A_{h[Y, Z]}X + (V_X h)(Y, Z) - (V_Y h)(X, Z)$, (37)

for any vector fields $X, Y$ and $Z$ tangent to $M$. For any vector field $W$ tangent to $M$, Eq. (37) gives the Gauss equation

$$g(\tilde{R}(X, Y)Z, W) = g(\tilde{R}(X, Y)Z, W) + g(h(Y, W), h(X, Z)) - g(h(Y, Z), h(X, W)).$$  (38)

On the other hand, the normal component of Eq. (37) is called equation of Codazzi, which is given by

$$\left(\tilde{R}(X, Y)Z\right)^{\perp} = (V_X h)(Y, Z) - (V_Y h)(X, Z).$$  (39)

If the Codazzi equation vanishes identically, then sub-manifold $M$ is said to be curvature-invariant sub-manifold [4].

In particular, if $\tilde{M}$ is of constant curvature, $\tilde{R}(X, Y)Z$ is tangent to $M$, that is, sub-manifold is curvature-invariant. Whereas, in Kenmotsu space forms, and Sasakian space forms, this is not true.

Next, we will define the curvature tensor $R^{\perp}$ of the normal bundle of the sub-manifold $M$ by

$$R^{\perp}(X, Y)V = \nabla_X^{\perp}V - \nabla_Y^{\perp}V = -\nabla_X h(Y, X) + \nabla_Y h(X, Y)$$

for any vector fields $X, Y$ tangent to sub-manifold $M$, and any vector field $V$ normal to $M$. From the Gauss and Weingarten formulas, we have

$$\tilde{R}(X, Y)V = \tilde{\nabla}_X \tilde{\nabla}_Y V - \tilde{\nabla}_Y \tilde{\nabla}_X V - \tilde{\nabla}_{[X, Y]} V$$

$$= \tilde{\nabla}_X (-AV_Y + \nabla_Y V) - \tilde{\nabla}_Y (-AV_X + \nabla_X V) + AV_{[X, Y]} - \nabla_{[X, Y]} V$$

$$= -\tilde{\nabla}_X AV_Y + \tilde{\nabla}_Y AV_X + \tilde{\nabla}_X \nabla_Y V - \tilde{\nabla}_Y \nabla_X V + AV_{[X, Y]} - \nabla_{[X, Y]} V$$

$$= -\nabla_X h(Y, A_Y) + \nabla_Y h(X, A_Y) + \nabla_X h(Y, A_Y) + h(Y, A_Y)$$

$$+ \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - A_{V^2_Y} \nabla_X V + A_{V^2_X} \nabla_Y V + AV_{[X, Y]} - \nabla_{[X, Y]} V$$

$$= \nabla_X h(Y, A_Y) + \nabla_Y h(X, A_Y) + h(Y, A_Y)$$

$$- \nabla_X h(Y, A_Y) + \nabla_Y h(X, A_Y) + h(Y, A_Y)$$

$$- \nabla_X h(Y, A_Y) + \nabla_Y h(X, A_Y) + h(Y, A_Y)$$

$$= R^{\perp}(X, Y)V + h(A_V X, Y) - h(X, A_Y) - (\nabla_X h)_Y + (\nabla_Y h)_X.$$  (41)

For any normal vector $U$ to $M$, we obtain
\[
g\left(\tilde{R}(X,Y)V, U\right) = g\left(R^\perp(X,Y)V, U\right) + g\left(h(A_VX,Y), U\right) - g\left(h(X,A_VY), U\right)
\]
\[
= g\left(R^\perp(X,Y)V, U\right) + g(A_UY, A_VX) - g(A_VY, A_UX)
\]
\[
= g\left(R^\perp(X,Y)V, U\right) + g(A_VA_UY, X) - g(A_UA_VY, X)
\]
\[(42)\]

Since \([A_U, A_V] = A_UA_V - A_VA_U\), Eq. (42) implies
\[
g\left(\tilde{R}(X,Y)V, U\right) = g\left(R^\perp(X,Y)V, U\right) + g([A_U, A_V]Y, X).
\]
\[(43)\]

Eq. (43) is also called the Ricci equation.

If \(R^\perp = 0\), then the normal connection of \(M\) is said to be flat [2].

When \(\left(\tilde{R}(X,Y)V\right)^\perp = 0\), the normal connection of the sub-manifold \(M\) is flat if and only if the second fundamental form \(M\) is commutative, i.e. \([A_U, A_V] = 0\) for all \(U, V\). If the ambient space \(\tilde{M}\) is real space form, then \(\left(\tilde{R}(X,Y)V\right)^\perp = 0\) and hence the normal connection of \(M\) is flat if and only if the second fundamental form is commutative. If \(\tilde{R}(X,Y)Z\) tangent to \(M\), then equation of codazzi Eq. (37) reduces to
\[
(\nabla_Xh)(Y, Z) = (\nabla_Yh)(X, Z)
\]
\[(44)\]

which is equivalent to
\[
(\nabla_XA)_Y = (\nabla_YA)_X.
\]
\[(45)\]

On the other hand, if the ambient space \(\tilde{M}\) is a space of constant curvature \(c\), then we have
\[
\tilde{R}(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y]
\]
\[(46)\]

for any vector fields \(X, Y\) and \(Z\) on \(\tilde{M}\).

Since \(\tilde{R}(X,Y)Z\) is tangent to \(M\), the equation of Gauss and the equation of Ricci reduce to
\[
g\left(R(X,Y)Z, W\right) = c[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]
\]
\[
+ g\left(h(Y,Z), h(X,W)\right) - g\left(h(Y,W), h(X,Z)\right)
\]
\[(47)\]

and
\[
g\left(R^\perp(X,Y)V, U\right) = g([A_U, A_V]X, Y),
\]
\[(48)\]

respectively.
Proposition 1.3. A totally umbilical sub-manifold $M$ in a real space form $\tilde{M}$ of constant curvature $c$ is also of constant curvature.

Proof: Since $M$ is a totally umbilical sub-manifold of $\tilde{M}$ of constant curvature $c$, by using Eqs. (14) and (46), we have

$$g\left(R(X, Y)Z, W\right) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} + \sum_{\alpha} [g(A_{\alpha}Y, Z)g(A_{\alpha}X, W) - g(A_{\alpha}X, Z)g(A_{\alpha}Y, W)]$$

This shows that the sub-manifold $M$ is of constant curvature $c + \|H\|^2$ for $n > 2$. If $n = 2$, $\|H\|^2 = $ constant follows from the equation of Codazzi [3].

This proves the proposition.

On the other hand, for any orthonormal basis $\{e_i\}$ of normal space, we have

$$g(Y, Z)g(X, W) - g(X, Z)g(Y, W) = \sum_a g(h(Y, Z), e_a)g(h(X, W), e_a)$$

Thus, Eq. (45) can be rewritten as

$$g\left(R(X, Y)Z, W\right) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} + \sum_{\alpha} [g(A_{\alpha}Y, Z)g(A_{\alpha}X, W) - g(A_{\alpha}X, Z)g(A_{\alpha}Y, W)]$$

By using $A_{e_i}$, we can construct a similar equation to Eq. (47) for Eq. (23).

Now, let $S$ be the Ricci tensor of $M$. Then, Eq. (47) gives us

$$S(X, Y) = c\{g(Y, X) - g(e_i, X)g(e_i, Y)\} + \sum_{\alpha} [g(A_{\alpha}e_i, e_i)g(A_{\alpha}X, Y) - g(A_{\alpha}X, e_i)g(A_{\alpha}e_i, Y)]$$

where $\{e_1, e_2, \ldots, e_n\}$ are orthonormal basis of $M$.

Therefore, the scalar curvature $r$ of sub-manifold $M$ is given by
\[ r = cn(n-1)\sum_{e} \text{Tr}^2(A_{e}) - \sum_{e} \text{Tr}(A_{e})^2 \]  

(54)

\[ \sum_{e} \text{Tr}(A_{e})^2 \] is the square of the length of the second fundamental form of \( M \), which is denoted by \( |A_{e}|^2 \). Thus, we also have

\[ \|h^2\| = \sum_{i,j=1}^{n} g(\langle e_{i}, e_{i} \rangle, \langle h(e_{i}), e_{j} \rangle) = \|A^2\|. \]  

(55)

2. Distribution on a manifold

An \( m \)-dimensional distribution on a manifold \( \tilde{M} \) is a mapping \( D \) defined on \( \tilde{M} \), which assigns to each point \( p \) of \( \tilde{M} \) an \( m \)-dimensional linear subspace \( D_p \) of \( T_{\tilde{M}}(p) \). A vector field \( X \) on \( \tilde{M} \) belongs to \( D \) if we have \( X_p \in D_p \) for each \( p \in \tilde{M} \). When this happens, we write \( X \in \Gamma(D) \). The distribution \( D \) is said to be differentiable if for any \( p \in \tilde{M} \), there exist \( m \)-differentiable linearly independent vector fields \( X_{j} \in \Gamma(D) \) in a neighborhood of \( p \).

The distribution \( D \) is said to be involutive if for all vector fields \( X, Y \in \Gamma(D) \) we have \([X,Y] \in \Gamma(D)\). A sub-manifold \( M \) of \( \tilde{M} \) is said to be an integral manifold of \( D \) if for every point \( p \in M \), \( D_p \) coincides with the tangent space to \( M \) at \( p \). If there exists no integral manifold of \( D \) which contains \( M \), then \( M \) is called a maximal integral manifold or a leaf of \( D \). The distribution \( D \) is said to be integrable if for every \( p \in \tilde{M} \), there exists an integral manifold of \( D \) containing \( p \) [2].

Let \( \tilde{\nabla} \) and \( D \) be a linear connection on \( \tilde{M} \), respectively. The distribution \( D \) is said to be parallel with respect to \( \tilde{M} \), if we have

\[ \tilde{\nabla} X Y \in \Gamma(T\tilde{M}) \text{ for all } X \in \Gamma(T\tilde{M}) \text{ and } Y \in \Gamma(D) \]  

(56)

Now, let \((\tilde{M}, \tilde{g})\) be Riemannian manifold and \( D \) be a distribution on \( \tilde{M} \). We suppose \( \tilde{M} \) is endowed with two complementary distribution \( D \) and \( D^\perp \), i.e., we have \( T\tilde{M} = D \oplus D^\perp \). Denoted by \( P \) and \( Q \) the projections of \( T\tilde{M} \) to \( D \) and \( D^\perp \), respectively.

**Theorem 2.1.** All the linear connections with respect to which both distributions \( D \) and \( D^\perp \) are parallel, are given by

\[ \nabla X Y = P\tilde{\nabla} X Y + Q\tilde{\nabla} X Y + PS(X, Y) + QS(X, Y) \]  

(57)

for any \( X, Y \in \Gamma(T\tilde{M}) \), where \( \tilde{\nabla} \) and \( S \) are, respectively, an arbitrary linear connection and arbitrary tensor field of type \((1,2)\) on \( \tilde{M} \).

**Proof:** Suppose \( \tilde{\nabla} \) is an arbitrary linear connection on \( \tilde{M} \). Then, any linear connection \( \nabla \) on \( \tilde{M} \) is given by
\[ \nabla_X Y = \nabla^*_X Y + S(X, Y) \quad (58) \]

for any \( X, Y \in \Gamma(TM) \). We can put

\[ X = PX + QX \quad (59) \]

for any \( X \in \Gamma(TM) \). Then, we have

\[ \nabla_X Y = \nabla_X (PY + QY) = \nabla_X PY + \nabla_X QY = \nabla^*_X PY + S(X, PY) + \nabla^*_X QY + PS(X, PY) + QS(X, PY) \]

\[ + PV_X QY + QV_X QY + PS(X, QY) + QS(X, QY) \quad (60) \]

for any \( X, Y \in \Gamma(TM) \).

The distributions \( D \) and \( D^\perp \) are both parallel with respect to \( \nabla \) if and only if we have

\[ \phi(\nabla_X PY) = 0 \text{ and } P(\nabla_X QY) = 0 . \quad (61) \]

From Eqs. (58) and (61), it follows that \( D \) and \( D^\perp \) are parallel with respect to \( \nabla \) if and only if

\[ QV_X PY + QS(X, PY) = 0 \text{ and } PV_X QY + PS(X, QY) = 0 . \quad (62) \]

Thus, Eqs. (58) and (62) give us Eq. (57).

Next, by means of the projections \( P \) and \( Q \), we define a tensor field \( F \) of type \((1,1)\) on \( \tilde{M} \) by

\[ F X = PX - QX \quad (63) \]

for any \( X \in \Gamma(TM) \). By a direct calculation, it follows that \( F^2 = I \). Thus, we say that \( F \) defines an almost product structure on \( \tilde{M} \). The covariant derivative of \( F \) is defined by

\[ (\nabla_X F)Y = \nabla_X FY - F \nabla_X Y \quad (64) \]

for all \( X, Y \in \Gamma(TM) \). We say that the almost product structure \( F \) is parallel with respect to the connection \( \nabla \), if we have \( \nabla_X F = 0 \). In this case, \( F \) is called the Riemannian product structure [2].

**Theorem 2.2.** Let \((\tilde{M}, \tilde{g})\) be a Riemannian manifold and \( D, D^\perp \) be orthogonal distributions on \( \tilde{M} \) such that \( TM = D \oplus D^\perp \). Both distributions \( D \) and \( D^\perp \) are parallel with respect to \( \nabla \) if and only if \( F \) is a Riemannian product structure.

**Proof:** For any \( X, Y \in \Gamma(TM) \), we can write

\[ \tilde{\nabla}_Y PX = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX \quad (65) \]

and
\[ \nabla Y X = \nabla_{PY} PX + \nabla_{PY} QX + \nabla_{QY} PX + \nabla_{QY} QX, \quad (66) \]

from which

\[ g(\nabla_{QY} PX, QZ) = QY g(PX, QZ) - g(\nabla_{QY} QZ, PX) = 0 \]

or \( g(\nabla_{QY} QZ, PX) = 0 \), \( (67) \)

that is, \( \nabla_{QY} PX \in \Gamma(D) \) and so \( P \nabla_{QY} PX = \nabla_{QY} PX \).

\[ Q \nabla_{QY} PX = 0. \quad (68) \]

In the same way, we obtain

\[ g(\nabla_{PY} QX, PZ) = PY g(QX, PZ) - g(\nabla_{PY} PZ, QX) = 0, \quad (69) \]

which implies that \( P \nabla_{PY} QX = 0 \) and \( Q \nabla_{PY} QX = \nabla_{PY} QX \).

\[ (70) \]

From Eqs. (66), (68) and (70), it follows that

\[ P \nabla_{Y} X = \nabla_{PY} PX + \nabla_{QY} PX. \quad (71) \]

By using Eqs. (64) and (71), we obtain

\[ (\nabla_{Y} P) X = \nabla_{Y} PX - P \nabla_{Y} X = \nabla_{PY} PX + \nabla_{QY} PX - \nabla_{PY} PX - \nabla_{QY} PX = 0. \quad (72) \]

In the same way, we can find \( \nabla Q = 0 \). Thus, we obtain

\[ \nabla F = \nabla (P - Q) = 0. \quad (73) \]

This proves our assertion [2].

**Theorem 2.3.** Both distributions \( D \) and \( D^\perp \) are parallel with respect to Levi-Civita connection \( \nabla \) if and only if they are integrable and their leaves are totally geodesic in \( \tilde{M} \).

**Proof:** Let us assume both distributions \( D \) and \( D^\perp \) are parallel. Since \( \nabla \) is a torsion free linear connection, we have

\[ [X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(D), \text{ for any } X, Y \in \Gamma(D), \quad (74) \]

and

\[ [U, V] = \nabla_U V - \nabla_V U \in \Gamma(D^\perp), \text{ for any } U, V \in \Gamma(D^\perp). \quad (75) \]

Thus, \( D \) and \( D^\perp \) are integrable distributions. Now, let \( M \) be a leaf of \( D \) and denote by \( h \) the second fundamental form of the immersion of \( M \) in \( \tilde{M} \). Then by the Gauss formula, we have
∇_X Y = \nabla_X Y + h(X, Y)  \tag{76}

for any \( X, Y \in \Gamma(D) \), where \( \nabla \) denote the Levi-Civita connection on \( M \). Since \( D \) is parallel from Eq. (76) we conclude \( h = 0 \), that is, \( M \) is totally in \( \tilde{M} \). In the same way, it follows that each leaf of \( D^\perp \) is totally geodesic in \( \tilde{M} \).

Conversely, suppose \( D \) and \( D^\perp \) be integrable and their leaves are totally geodesic in \( \tilde{M} \). Then by using Eq. (4), we have

\[ \nabla_X Y \in \Gamma(D) \text{ for any } X, Y \in \Gamma(D) \] \tag{77}

and

\[ \nabla_U V \in \Gamma(D^\perp) \text{ for any } U, V \in \Gamma(D^\perp). \] \tag{78}

Since \( g \) is a Riemannian metric tensor, we obtain

\[ g(\nabla_U Y, V) = -g(Y, \nabla_U V) = 0 \] \tag{79}

and

\[ g(\nabla_X Y, V) = -g(V, \nabla_X Y) = 0 \] \tag{80}

for any \( X, Y \in \Gamma(D) \) and \( U, V \in \Gamma(D^\perp) \). Thus, both distributions \( D \) and \( D^\perp \) are parallel on \( \tilde{M} \).

### 3. Locally decomposable Riemannian manifolds

Let \( (\tilde{M}, \tilde{g}) \) be \( n \)-dimensional Riemannian manifold and \( F \) be a tensor \((1,1)\)-type on \( \tilde{M} \) such that \( F^2 = \text{I} \), \( F \neq \varepsilon \text{I} \).

If the Riemannian metric tensor \( \tilde{g} \) satisfying

\[ \tilde{g}(X, Y) = \tilde{g}(FX, FY) \] \tag{81}

for any \( X, Y \in \Gamma(\tilde{T}M) \) then \( \tilde{M} \) is called almost Riemannian product manifold and \( F \) is said to be almost Riemannian product structure. If \( F \) is parallel, that is, \( (\nabla_X F)Y = 0 \), then \( \tilde{M} \) is said to be locally decomposable Riemannian manifold.

Now, let \( \tilde{M} \) be an almost Riemannian product manifold. We put

\[ P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I-F). \] \tag{82}

Then, we have
Thus, $P$ and $Q$ define two complementary distributions $P$ and $Q$ globally. Since $F^2 = I$, we easily see that the eigenvalues of $F$ are 1 and $-1$. An eigenvector corresponding to the eigenvalue 1 is in $P$ and an eigenvector corresponding to $-1$ is in $Q$. If $F$ has eigenvalue 1 of multiplicity $p$ and eigenvalue $-1$ of multiplicity $q$, then the dimension of $P$ is $p$ and that of $Q$ is $q$. Conversely, if there exist in $M$ two globally complementary distributions $P$ and $Q$ of dimension $p$ and $q$, respectively. Then, we can define an almost Riemannian product structure $F$ on $\tilde{M}$ by $F = P\tilde{Q}$ [7].

Let $(\tilde{M}, \tilde{g}, F)$ be a locally decomposable Riemannian manifold and we denote the integral manifolds of the distributions $P$ and $Q$ by $M^P$ and $M^Q$, respectively. Then we can write $\tilde{M} = M^P \times M^Q$, $(p, q > 2)$. Also, we denote the components of the Riemannian curvature $R$ of $\tilde{M}$ by $R_{\alpha\beta\gamma\delta} 1 \leq a, b, c, d \leq n = p + q$.

Now, we suppose that the two components are both of constant curvature $\lambda$ and $\mu$. Then, we have

$$R_{\alpha\beta\gamma\delta} = \lambda \{g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta}\}$$

(84) and

$$R_{\gamma\alpha\beta\delta} = \mu \{g_{\gamma\alpha}g_{\beta\delta} - g_{\gamma\beta}g_{\alpha\delta}\}.$$ 

(85)

Then, the above equations may also be written in the form

$$R_{\alpha\beta\gamma\delta} = \frac{1}{4}(\lambda + \mu)\{g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma} + (F_{\alpha}F_{\beta} - F_{\alpha\beta})\}$$

$$+ \frac{1}{4}(\lambda - \mu)\{g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma} + (F_{\alpha\beta} - g_{\alpha\beta}F)\}.$$ 

(86)

Conversely, suppose that the curvature tensor of a locally decomposable Riemannian manifold has the form

$$R_{\alpha\beta\gamma\delta} = a\{g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma} + (F_{\alpha}F_{\beta} - F_{\alpha\beta})\}$$

$$+ b\{g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma} + (F_{\alpha\beta} - g_{\alpha\beta}F)\}.$$ 

(87)

Then, we have

$$R_{\alpha\beta\gamma\delta} = 2(a + b)\{g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma}\}$$

(88) and

$$R_{\gamma\alpha\beta\delta} = 2(a - b)\{g_{\gamma\alpha}g_{\beta\delta} - g_{\gamma\beta}g_{\alpha\delta}\}.$$ 

(89)

Let $\tilde{M}$ be an $m$–dimensional almost Riemannian product manifold with the Riemannian structure $(\tilde{F}, \tilde{g})$ and $M$ be an $n$–dimensional sub–manifold of $M$. For any vector field $X$ tangent to $M$, we put
\[ FX = fX + wX, \]  
(90)

where \( fX \) and \( wX \) denote the tangential and normal components of \( FX \), with respect to \( M \), respectively. In the same way, for \( V \in \Gamma(T^1M) \), we also put

\[ FV = BV + CV, \]  
(91)

where \( BV \) and \( CV \) denote the tangential and normal components of \( FV \), respectively.

Then, we have

\[ f^2 + Bw = I, \quad Cw + wf = 0 \]  
(92)

and

\[ fB + BC = 0, \quad wB + C^2 = I. \]  
(93)

On the other hand, we can easily see that

\[ g(X, fY) = g(fX, Y) \]  
(94)

and

\[ g(X, Y) = g(fX, fY) + g(wX, wY) \]  
(95)

for any \( X, Y \in \Gamma(TM) \) [6].

If \( wX = 0 \) for all \( X \in \Gamma(TM) \), then \( M \) is said to be invariant sub-manifold in \( \tilde{M} \), i.e., \( F(T_M(p)) \subset T_M(p) \) for each \( p \in M \). In this case, \( f^2 = I \) and \( g(fX, fY) = g(X, Y) \). Thus, \( (f, g) \) defines an almost product Riemannian on \( M \).

Conversely, \( (f, g) \) is an almost product Riemannian structure on \( M \), the \( w = 0 \) and hence \( M \) is an invariant sub-manifold in \( \tilde{M} \).

Consequently, we can give the following theorem [7].

**Theorem 3.1.** Let \( M \) be a sub-manifold of an almost Riemannian product manifold \( \tilde{M} \) with almost Riemannian product structure \( (F, \tilde{g}) \). The induced structure \( (f, g) \) on \( M \) is an almost Riemannian product structure if and only if \( M \) is an invariant sub-manifold of \( \tilde{M} \).

**Definition 3.1.** Let \( M \) be a sub-manifold of an almost Riemannian product \( \tilde{M} \) with almost product Riemannian structure \( (F, \tilde{g}) \). For each non-zero vector \( X_p \in T_M(p) \) at \( p \in M \), we denote the slant angle between \( FX_p \) and \( T_M(p) \) by \( \theta(p) \). Then \( M \) said to be slant sub-manifold if the angle \( \theta(p) \) is constant, i.e., it is independent of the choice of \( p \in M \) and \( X_p \in T_M(p) \) [5].

Thus, invariant and anti-invariant immersions are slant immersions with slant angle \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \), respectively. A proper slant immersion is neither invariant nor anti-invariant.
Theorem 3.2. Let $M$ be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost product Riemannian structure $(F, \tilde{g})$. $M$ is a slant sub-manifold if and only if there exists a constant $\lambda \in (0, 1)$, such that

$$f^2 = \lambda I.$$  \hfill (96)

Furthermore, if the slant angle is $\theta$, then it satisfies $\lambda = \cos^2 \theta$ [9].

Definition 3.2. Let $M$ be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. $M$ is said to be semi-slant sub-manifold if there exist distributions $D_\theta$ and $D^\perp$ on $M$ such that

(i) $TM$ has the orthogonal direct decomposition $TM = D \oplus D^\perp$.

(ii) The distribution $D_\theta$ is a slant distribution with slant angle $\theta$.

(iii) The distribution $D^\perp$ is an invariant distribution, i.e., $F(D^\perp) \subseteq D^\perp$.

In a semi-slant sub-manifold, if $\theta = \frac{\pi}{2}$, then semi-slant sub-manifold is called semi-invariant sub-manifold [8].

Example 3.1. Now, let us consider an immersed sub-manifold $M$ in $\mathbb{R}^7$ given by the equations

$$x_1^2 + x_2^2 = x_3^2 + x_4^2, x_3 + x_4 = 0.$$  \hfill (97)

By direct calculations, it is easy to check that the tangent bundle of $M$ is spanned by the vectors

$$z_1 = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \beta \frac{\partial}{\partial x_5} + \sin \beta \frac{\partial}{\partial x_6},$$

$$z_2 = -u \sin \theta \frac{\partial}{\partial x_1} + u \cos \theta \frac{\partial}{\partial x_2}, z_3 = \frac{\partial}{\partial x_3},$$

$$z_4 = -u \sin \beta \frac{\partial}{\partial x_5} + u \cos \beta \frac{\partial}{\partial x_6}, z_5 = \frac{\partial}{\partial x_7},$$  \hfill (98)

where $\theta, \beta$ and $u$ denote arbitrary parameters.

For the coordinate system of $\mathbb{R}^7 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7)| x_i \in \mathbb{R}, 1 \leq i \leq 7\}$, we define the almost product Riemannian structure $F$ as follows:

$$F \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}, F \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial x_j}, 1 \leq i \leq 3 \text{ and } 4 \leq j \leq 7.$$  \hfill (99)

Since $F_{z_1}$ and $F_{z_2}$ are orthogonal to $M$ and $F_{z_4}, F_{z_5}, F_{z_6}$ are tangent to $M$, we can choose a $D = S_{\rho} \{z_2, z_4, z_5\}$ and $D^\perp = S_{\rho} \{z_1, z_3\}$. Thus, $M$ is a 5-dimensional semi-invariant sub-manifold of $\mathbb{R}^7$ with usual almost Riemannian product structure $(F, \langle , , \rangle)$.

Example 3.2. Let $M$ be sub-manifold of $\mathbb{R}^8$ by given

$$(u + v, u-v, u \cos \alpha, u \sin \alpha, u + v, u-v, u \cos \beta, u \sin \beta).$$  \hfill (100)
where \( u, v \) and \( \beta \) are the arbitrary parameters. By direct calculations, we can easily see that the tangent bundle of \( M \) is spanned by
\[
e_1 = \frac{\partial}{\partial x_1} + \cos \alpha \frac{\partial}{\partial x_3} + \sin \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \cos \beta \frac{\partial}{\partial x_7} + \sin \beta \frac{\partial}{\partial x_8},
\]
\[
e_2 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6},
\]
\[
e_3 = -u \sin \alpha \frac{\partial}{\partial x_3} + v \cos \alpha \frac{\partial}{\partial x_4},
\]
\[
e_4 = -u \sin \beta \frac{\partial}{\partial x_7} + v \cos \beta \frac{\partial}{\partial x_8}.
\]
(101)

For the almost Riemannian product structure \( F \) of \( \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3, F(TM) \) is spanned by vectors
\[
F_{e_1} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos \alpha \frac{\partial}{\partial x_3} + \sin \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \cos \beta \frac{\partial}{\partial x_7} - \sin \beta \frac{\partial}{\partial x_8},
\]
\[
F_{e_2} = -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6},
\]
\[
F_{e_3} = e_3 \quad \text{and} \quad F_{e_4} = -e_4.
\]
(102)

Since \( F_{e_1} \) and \( F_{e_2} \) are orthogonal to \( M \) and \( F_{e_3} \) and \( F_{e_4} \) are tangent to \( M \), we can choose \( D^1 = Sp\{e_3, e_4\} \) and \( D^2 = Sp\{e_1, e_2\} \). Thus, \( M \) is a four-dimensional semi-invariant sub-manifold of \( \mathbb{R}^8 = \mathbb{R}^3 \times \mathbb{R}^4 \) with usual Riemannian product structure \( F \).

**Definition 3.3.** Let \( M \) be a sub-manifold of an almost Riemannian product manifold \( \bar{M} \) with almost Riemannian product structure \( (F, \bar{g}) \). \( M \) is said to be pseudo-slant sub-manifold if there exist distributions \( D_0 \) and \( D_1 \) on \( M \) such that

- i. The tangent bundle \( TM = D_0 \oplus D_1 \).
- ii. The distribution \( D_0 \) is a slant distribution with slant angle \( \theta \).
- iii. The distribution \( D_1 \) is an anti-invariant distribution, i.e., \( F(D_1) \subseteq T^1M \).

As a special case, if \( \theta = 0 \) and \( \theta = \pi \), then pseudo-slant sub-manifold becomes semi-invariant and anti-invariant sub-manifolds, respectively.

**Example 3.3.** Let \( M \) be a sub-manifold of \( \mathbb{R}^6 \) by the given equation
\[
(\sqrt{3}u, v, u \sin \theta, v \cos \theta, \cos t, -\cos t)
\]
where \( u, v, s \) and \( t \) arbitrary parameters and \( \theta \) is a constant.

We can check that the tangent bundle of \( M \) is spanned by the tangent vectors
\[
e_1 = \sqrt{3} \frac{\partial}{\partial x_1},
\]
\[
e_2 = \frac{\partial}{\partial y_1} + s \sin \theta \frac{\partial}{\partial x_2} + \cos \theta \frac{\partial}{\partial y_2},
\]
\[
e_3 = \cos t \frac{\partial}{\partial x_3} - s \cos \theta \frac{\partial}{\partial y_3},
\]
\[
e_4 = -s \sin \theta \frac{\partial}{\partial x_3} + s \sin \theta \frac{\partial}{\partial y_3}.
\]
(104)

For the almost product Riemannian structure \( F \) of \( \mathbb{R}^6 \) whose coordinate systems \((x_1, y_1, x_2, y_2, x_3, y_3)\) choosing
\[ F \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial y_i}, \quad 1 \leq i \leq 3, \]

\[ F \left( \frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq 3, \]

Thus, we have

\[ F e_1 = \sqrt{3} \frac{\partial}{\partial y_1}, F e_2 = -\frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial y_2} - \cos \theta \frac{\partial}{\partial x_2}, \]

\[ F e_3 = \cos t \frac{\partial}{\partial y_3} + \cos t \frac{\partial}{\partial x_3}, F e_4 = -s \sin t \frac{\partial}{\partial y_3} - s \sin t \frac{\partial}{\partial x_3}. \]


