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Sub-Manifolds of a Riemannian Manifold

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Abstract

In this chapter, we introduce the theory of sub-manifolds of a Riemannian manifold. The fundamental notations are given. The theory of sub-manifolds of an almost Riemannian product manifold is one of the most interesting topics in differential geometry. According to the behaviour of the tangent bundle of a sub-manifold, with respect to the action of almost Riemannian product structure of the ambient manifolds, we have three typical classes of sub-manifolds such as invariant sub-manifolds, anti-invariant sub-manifolds and semi-invariant sub-manifolds. In addition, slant, semi-slant and pseudo-slant sub-manifolds are introduced by many geometers.

Keywords: Riemannian product manifold, Riemannian product structure, integral manifold, a distribution on a manifold, real product space forms, a slant distribution

1. Introduction

Let \( i : M \to \tilde{M} \) be an immersion of an \( n \)-dimensional manifold \( M \) into an \( m \)-dimensional Riemannian manifold \( (\tilde{M}, \tilde{g}) \). Denote by \( g = i^* \tilde{g} \) the induced Riemannian metric on \( M \). Thus, \( i \) becomes an isometric immersion and \( M \) is also a Riemannian manifold with the Riemannian metric \( g(X, Y) = \tilde{g}(X, Y) \) for any vector fields \( X, Y \) in \( M \). The Riemannian metric \( g \) on \( M \) is called the induced metric on \( M \). In local components, \( \tilde{g}_{AB} = g_{AB} \) with \( g = g_{ij} dx^i dx^j \) and \( \tilde{g} = \tilde{g}_{AB} dU^B dU^A \).

If a vector field \( \xi_p \) of \( \tilde{M} \) at a point \( p \in M \) satisfies
\[
\tilde{g}(X_p, \xi_p) = 0
\]
for any vector \( X_p \) of \( M \) at \( p \), then \( \xi_p \) is called a normal vector of \( M \) in \( \tilde{M} \) at \( p \). A unit normal vector field of \( M \) in \( \tilde{M} \) is called a normal section on \( M \) [3].
By $T^4 M$, we denote the vector bundle of all normal vectors of $M$ in $\tilde{M}$. Then, the tangent bundle of $\tilde{M}$ is the direct sum of the tangent bundle $TM$ of $M$ and the normal bundle $T^4 M$ of $M$ in $\tilde{M}$, i.e.,

$$T\tilde{M} = TM \oplus T^4 M. \quad (2)$$

We note that if the sub-manifold $M$ is of codimension one in $\tilde{M}$ and they are both orientiable, we can always choose a normal section $\xi$ on $M$, i.e.,

$$g(X, \xi) = 0, \quad g(\xi, \xi) = 1, \quad (3)$$

where $X$ is any arbitrary vector field on $M$.

By $\tilde{\nabla}$, denote the Riemannian connection on $\tilde{M}$ and we put

$$\tilde{\nabla} X Y = \nabla X Y + h(X, Y) \quad (4)$$

for any vector fields $X, Y$ tangent to $M$, where $\nabla X Y$ and $h(X, Y)$ are tangential and the normal components of $\tilde{\nabla} X Y$, respectively. Formula (4) is called the Gauss formula for the sub-manifold $M$ of a Riemannian manifold $(\tilde{M}, \tilde{g})$.

**Proposition 1.1.** $\nabla$ is the Riemannian connection of the induced metric $\tilde{g} = i^* \tilde{g}$ on $M$ and $h(X, Y)$ is a normal vector field over $M$, which is symmetric and bilinear in $X$ and $Y$.

**Proof:** Let $\alpha$ and $\beta$ be differentiable functions on $M$. Then, we have

$$\tilde{\nabla} \alpha X (\beta Y) = \alpha \{X(\beta) Y + \beta \tilde{\nabla} X Y\}$$

$$= \alpha \{X(\beta) Y + \beta \nabla X Y + \beta h(X, Y)\}$$

$$\nabla \alpha X Y + h(\alpha X, \beta Y) = \alpha \beta \nabla X Y + \alpha X(\beta) Y + \alpha \beta h(X, Y) \quad (5)$$

This implies that

$$\nabla \alpha X (\beta Y) = \alpha X(\beta) Y + \alpha \beta \nabla X Y \quad (6)$$

and

$$h(\alpha X, \beta Y) = \alpha \beta h(X, Y). \quad (7)$$

Eq. (6) shows that $\nabla$ defines an affine connection on $M$ and Eq. (4) shows that $h$ is bilinear in $X$ and $Y$ since additivity is trivial [1].

Since the Riemannian connection $\tilde{\nabla}$ has no torsion, we have

$$0 = \tilde{\nabla} X Y - \tilde{\nabla} Y X - [X, Y] = \nabla X Y + h(X, Y) - \nabla Y X - h(Y, X) - [X, Y]. \quad (8)$$

By comparing the tangential and normal parts of the last equality, we obtain
\[ \nabla_X Y - \nabla_Y X = [X, Y] \quad (9) \]

and

\[ h(X, Y) = h(Y, X). \quad (10) \]

These equations show that \( \nabla \) has no torsion and \( h \) is a symmetric bilinear map. Since the metric \( \tilde{g} \) is parallel, we can easily see that

\[
(\nabla X g)(Y, Z) = \tilde{g}(\nabla X Y, Z) + \tilde{g}(Y, \nabla X Z + h(X, Z))
= \tilde{g}(\nabla X Y, Z) + \tilde{g}(Y, \nabla X Z) + \tilde{g}(\nabla X Z, Y) + \tilde{g}(Y, \nabla X Z)
= h(Y, \nabla X Z) + h(\nabla X Z, Y)
= h(Y, \nabla X Z) + h(\nabla X Z, Y)
= g(\nabla X Y, Z) + g(Y, \nabla X Z) \quad (11)
\]

for any vector fields \( X, Y, Z \) tangent to \( M \), that is, \( \nabla \) is also the Riemannian connection of the induced metric \( g \) on \( M \).

We recall \( h \) the second fundamental form of the sub-manifold \( M \) (or immersion \( i \)), which is defined by

\[ h : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(T^2M). \quad (12) \]

If \( h = 0 \) identically, then sub-manifold \( M \) is said to be totally geodesic, where \( \Gamma(T^2M) \) is the set of the differentiable vector fields on normal bundle of \( M \).

Totally geodesic sub-manifolds are simplest sub-manifolds.

**Definition 1.1.** Let \( M \) be an \( n \)-dimensional sub-manifold of an \( m \)-dimensional Riemannian manifold \( (\tilde{M}, \tilde{g}) \). By \( h \) we denote the second fundamental form of \( M \) in \( \tilde{M} \).

\( H = \frac{1}{n} \text{trace}(h) \) is called the mean curvature vector of \( M \) in \( \tilde{M} \). If \( H = 0 \), the sub-manifold is called minimal.

On the other hand, \( M \) is called pseudo-umbilical if there exists a function \( \lambda \) on \( M \), such that

\[ \tilde{g}(h(X, Y), H) = \lambda g(X, Y) \quad (13) \]

for any vector fields \( X, Y \) on \( M \) and \( M \) is called totally umbilical sub-manifold if

\[ h(X, Y) = g(X, Y)H. \quad (14) \]

It is clear that every minimal sub-manifold is pseudo-umbilical with \( \lambda = 0 \). On the other hand, by a direct calculation, we can find \( \lambda = \tilde{g}(H, H) \) for a pseudo-umbilical sub-manifold. So, every
tota/ly umbilical sub-manifold is a pseudo-umbilical and a totally umbilical sub-manifold is totally geodesic if and only if it is minimal [2].

Now, let $M$ be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$ and $V$ be a normal vector field on $M$, $X$ be a vector field on $M$. Then, we decompose

$$\tilde{\nabla}_X V = -A_V X + \nabla^\bot_X V,$$  \hspace{1cm} (15)$$

where $A_V X$ and $\nabla^\bot_X V$ denote the tangential and the normal components of $\nabla^\bot_X V$, respectively. We can easily see that $A_V X$ and $\nabla^\bot_X V$ are both differentiable vector fields on $M$ and normal bundle of $M$, respectively. Moreover, Eq. (15) is also called Weingarten formula.

**Proposition 1.2.** Let $M$ be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then

(a) $A_V X$ is bilinear in vector fields $V$ and $X$. Hence, $A_V X$ at point $p \in M$ depends only on vector fields $V_p$ and $X_p$.

(b) For any normal vector field $V$ on $M$, we have

$$g(A_V X, Y) = \tilde{g}\left(h(X, Y), V\right).$$  \hspace{1cm} (16)$$

**Proof:** Let $\alpha$ and $\beta$ be any two functions on $M$. Then, we have

$$\tilde{\nabla}_{\alpha X}\beta V = \tilde{\nabla}_X (\beta V)$$

$$= \alpha \{X(\beta) V + \beta \tilde{\nabla}_X V\}$$

$$- A_{\beta V} \alpha X + \nabla^\bot_{\alpha X} \beta V = aX(\beta) V - a \beta A_V X + a \beta \nabla^\bot_X V.$$  \hspace{1cm} (17)$$

This implies that

$$a \beta A_V X = a \beta A_V X$$  \hspace{1cm} (18)$$

and

$$\nabla^\bot_{\alpha X} \beta V = aX(\beta) V + a \beta \nabla^\bot_X V.$$  \hspace{1cm} (19)$$

Thus, $A_V X$ is bilinear in $V$ and $X$. Additivity is trivial. On the other hand, since $g$ is a Riemannian metric,

$$X_{\tilde{g}}(Y, V) = 0,$$  \hspace{1cm} (20)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\bot M)$.

Eq. (12) implies that

$$\tilde{g}(\tilde{\nabla}_X Y, V) + \tilde{g}(Y, \tilde{\nabla}_X V) = 0.$$  \hspace{1cm} (21)$$

By means of Eqs. (4) and (15), we obtain
The proof is completed [3].

Let $M$ be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$, and $h$ and $A_V$ denote the second fundamental form and shape operator of $M$, respectively. The covariant derivative of $h$ and $A_V$ is, respectively, defined by

$$\tilde{\nabla}_X h(Y, Z) = \nabla_{\mathcal{N}X} h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

(23)

and

$$(\nabla_X A_V)Y = \nabla_X (A_V Y) - A_V \nabla_X Y$$

(24)

for any vector fields $X, Y$ tangent to $M$ and any vector field $V$ normal to $M$. If $\nabla_X h = 0$ for all $X$, then the second fundamental form of $M$ is said to be parallel, which is equivalent to $\nabla_X A = 0$.

By direct calculations, we get the relation

$$g(\nabla_\mathcal{N}X h(Y, Z), V) = g(\nabla_X A_V Y, Z)$$

(25)

**Example 1.1.** We consider the isometric immersion

$$\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^4,$$

$$\phi(x_1, x_2) = (x_1, \sqrt{x_1^2 - 1}, x_2, \sqrt{x_2^2 - 1})$$

(27)

we note that $M = \phi(\mathbb{R}^2) \subseteq \mathbb{R}^4$ is a two-dimensional sub-manifold of $\mathbb{R}^4$ and the tangent bundle is spanned by the vectors

$$TM = \text{span} \left\{ e_1 = \left(\sqrt{x_1^2 - 1}, x_1, 0, 0\right), e_2 = \left(0, 0, \sqrt{x_2^2 - 1}, x_2\right) \right\}$$

and the normal vector fields

$$T^cM = \text{span} \left\{ w_1 = \left(-x_1, \sqrt{x_1^2 - 1}, 0, 0\right), w_2 = \left(0, 0, -x_1, \sqrt{x_2^2 - 1}\right) \right\}.$$ 

(28)

By $\tilde{\nabla}$, we denote the Levi-Civita connection of $\mathbb{R}^4$, the coefficients of connection, are given by

$$\tilde{\nabla}_{e_1} e_1 = \frac{2x_1 \sqrt{x_1^2 - 1}}{2x_1^2 - 1} e_1 - \frac{1}{2x_1^2 - 1} w_1,$$

(29)

$$\tilde{\nabla}_{e_2} e_2 = \frac{2x_2 \sqrt{x_2^2 - 1}}{2x_2^2 - 1} e_2 - \frac{1}{2x_2^2 - 1} w_2$$

(30)
Thus, we have 
\[ h(e_1, e_1) = -\frac{1}{2\pi^2-1} w_1, \quad h(e_2, e_2) = -\frac{1}{2\pi^2-1} w_2 \]
and 
\[ h(e_2, e_1) = 0. \]
The mean curvature vector of \( M = \phi(\mathbb{R}^2) \) is given by
\[ H = -\frac{1}{2}(w_1 + w_2). \]
Furthermore, by using Eq. (16), we obtain
\begin{align*}
g(A_{w_1}e_1, e_1) &= g\left(h(e_1, e_1), w_1\right) = -\frac{1}{2\pi^2-1} (x_1^2 + x_2^2) = -1, \\
g(A_{w_1}e_2, e_2) &= g\left(h(e_2, e_2), w_1\right) = -\frac{1}{2\pi^2-1} g(w_1, w_2) = 0, \\
g(A_{w_1}e_1, e_2) &= 0, \\
g(A_{w_2}e_1, e_1) &= g\left(h(e_1, e_1), w_2\right) = 0, \\
g(A_{w_2}e_2, e_2) &= g(A_{w_2}e_2, e_2) = 1.
\end{align*}
Thus, we have
\[ A_{w_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{w_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \]
Now, let \( M \) be a sub-manifold of a Riemannian manifold \( (\tilde{M}, g) \), \( \tilde{R} \) and \( R \) be the Riemannian curvature tensors of \( \tilde{M} \) and \( M \), respectively. From then the Gauss and Weingarten formulas, we have
\begin{align*}
\tilde{R}(X, Y)Z &= \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z + \tilde{\nabla}_X [Y, Z] - \tilde{\nabla}_Y [X, Z] \\
&= \tilde{\nabla}_X \left( \nabla_Y Z + h(Y, Z) \right) - \tilde{\nabla}_Y \left( \nabla_X Z + h(X, Z) \right) - \nabla_X [Y, Z] + h(\nabla_X, Z) \\
&= \nabla_X [Y, Z] - \nabla_Y [X, Z] + h(\nabla_Y, Z) \quad \text{and} \quad \tilde{R}(X, Y)Z \\
&= R(X, Y)Z + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + A_{h(X, Z)} Y - A_{h(Y, Z)} X
\end{align*}
Next, we will define the curvature tensor curvature-invariant. Whereas, in Kenmotsu space forms, and Sasakian space forms, this not the Gauss and Weingarten formulas, we have for any vector fields $X,Y,Z$ tangent to $M$. For any vector field $W$ tangent to $M$, Eq. (37) gives the Gauss equation

$$g\left(\tilde{R}(X,Y)Z,W\right) = g\left(\tilde{R}(X,Y)Z,W\right) + g\left(h(Y,W)h(X,Z) - g(h(Y,Z),h(X,W)).\right)$$

(38)

On the other hand, the normal component of Eq. (37) is called equation of Codazzi, which is given by

$$\left(\tilde{R}(X,Y)\right)^{+} = (\nabla_{X}h)(Y,Z) - (\nabla_{Y}h)(X,Z).$$

(39)

If the Codazzi equation vanishes identically, then sub-manifold $M$ is said to be curvature-invariant sub-manifold [4].

In particular, if $\tilde{M}$ is of constant curvature, $\tilde{R}(X,Y)Z$ is tangent to $M$, that is, sub-manifold is curvature-invariant. Whereas, in Kenmotsu space forms, and Sasakian space forms, this not true.

Next, we will define the curvature tensor $R^{\perp}$ of the normal bundle of the sub-manifold $M$ by

$$R^{\perp}(X,Y)V = \nabla_{X}^{\perp}\nabla_{Y}^{\perp}V - \nabla_{Y}^{\perp}\nabla_{X}^{\perp}V - \nabla^{\perp}_{[X,Y]}V$$

(40)

for any vector fields $X,Y$ tangent to sub-manifold $M$, and any vector field $V$ normal to $M$. From the Gauss and Weingarten formulas, we have

$$\tilde{R}(X,Y)V = \tilde{\nabla}_{X}\tilde{\nabla}_{Y}V - \tilde{\nabla}_{Y}\tilde{\nabla}_{X}V - \tilde{\nabla}^{\perp}_{[X,Y]}V$$

$$= \tilde{\nabla}_{X}( -A_{VY} + \nabla_{Y}^{\perp}V) - \tilde{\nabla}_{Y}( -A_{VX} + \nabla_{X}^{\perp}V) + A_{[X,Y]}^{\perp}V$$

$$= -\tilde{\nabla}_{X}A_{VY} + \tilde{\nabla}_{Y}A_{VX} + \nabla_{X}^{\perp}V - \tilde{\nabla}_{Y}^{\perp}V + A_{[X,Y]}^{\perp}V$$

$$= -\nabla_{X}A_{VY} + A_{VY} + \nabla_{Y}A_{VX} + h(X,A_{VY}) + A_{[X,Y]}^{\perp}V$$

$$+ \nabla^{\perp}_{X}V - \nabla^{\perp}_{Y}V - A_{V}^{\perp}VX + A_{V}^{\perp}VY + A_{V}[X,Y] - \nabla^{\perp}_{[X,Y]}V$$

$$= \nabla^{\perp}_{X}V - \nabla^{\perp}_{Y}V + A_{V}^{\perp}VX - A_{V}^{\perp}VY + A_{V}[X,Y]$$

$$- \nabla_{X}A_{VY} + \nabla_{Y}A_{VX} + h(X,A_{VY}) + h(Y,A_{VX})$$

$$= R^{\perp}(X,Y)V = h(A_{V}X,Y) - h(X,A_{V}Y) - (\nabla_{X}A)_{Y}V + (\nabla_{Y}A)_{X}.\right)$$

(41)

For any normal vector $U$ to $M$, we obtain
\[ g(\tilde{R}(X,Y)V, U) = g(\tilde{R}^+(X,Y)V, U) + g(h(A_U X, Y), U) - g(h(X, A_U Y), U) \]
\[ = g(\tilde{R}^+(X,Y)V, U) + g(A_U Y, A_V X) - g(A_V Y, A_U X) = g(\tilde{R}^+(X,Y)V, U) + g(A_V A_U Y, X) - g(A_U A_V Y, X) \] (42)

Since \([A_U, A_V] = A_U A_V - A_V A_U\), Eq. (42) implies
\[ g(\tilde{R}(X,Y)V, U) = g(\tilde{R}^+(X,Y)V, U) + g([A_U, A_V] Y, X) \] (43)

Eq. (43) is also called the Ricci equation.

If \(R^+ = 0\), then the normal connection of \(M\) is said to be flat [2].

When \(\tilde{R}(X,Y)V^\perp = 0\), the normal connection of the sub-manifold \(M\) is flat if and only if the second fundamental form is commutative, i.e. \([A_U, A_V] = 0\) for all \(U, V\). If the ambient space \(\tilde{M}\) is real space form, then \(\tilde{R}(X,Y)V^\perp = 0\) and hence the normal connection of \(M\) is flat if and only if the second fundamental form is commutative. If \(\tilde{R}(X,Y)Z\) tangent to \(M\), then equation of codazzi Eq. (37) reduces to
\[ (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \] (44)
which is equivalent to
\[ (\nabla_X A)Y = (\nabla_Y A)X. \] (45)

On the other hand, if the ambient space \(\tilde{M}\) is a space of constant curvature \(c\), then we have
\[ \tilde{R}(X,Y)Z = c[g(Y, Z)X - g(X, Z)Y] \] (46)
for any vector fields \(X, Y\) and \(Z\) on \(\tilde{M}\).

Since \(\tilde{R}(X,Y)Z\) is tangent to \(M\), the equation of Gauss and the equation of Ricci reduce to
\[ g(\tilde{R}(X,Y)Z, W) = c[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + g(h(Y, Z), h(X, W)) - g(h(Y, W), h(X, Z)) \] (47)
and
\[ g(\tilde{R}^+(X,Y)V, U) = g([A_U, A_V] X, Y) \] (48)
respectively.
**Proposition 1.3.** A totally umbilical sub-manifold \( M \) in a real space form \( \tilde{M} \) of constant curvature \( c \) is also of constant curvature.

**Proof:** Since \( M \) is a totally umbilical sub-manifold of \( \tilde{M} \) of constant curvature \( c \), by using Eqs. (14) and (46), we have

\[
g\left(R(X, Y)Z, W\right) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
+ g(H, H)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
= \{c + g(H, H)\} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}.
\]

(49)

This shows that the sub-manifold \( M \) is of constant curvature \( c + \|H^2\| \) for \( n > 2 \). If \( n = 2 \), \( \|H^2\| = \text{constant} \) follows from the equation of Codazzi [3].

This proves the proposition.

On the other hand, for any orthonormal basis \( \{e_i\} \) of normal space, we have

\[
g(Y, Z)g(X, W) - g(X, Z)g(Y, W) = \sum_{a} \left[ g\left(h(Y, Z), e_a\right)g\left(h(X, W), e_a\right) - g\left(h(X, Z), e_a\right)g\left(h(Y, W), e_a\right) \right] \\
= \sum_{a} g\left(A_{a} Y, Z\right)g\left(A_{a} X, W\right) - g\left(A_{a} X, Z\right)g\left(A_{a} Y, W\right)
\]

(50)

Thus, Eq. (45) can be rewritten as

\[
g\left(R(X, Y)Z, W\right) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
+ \sum_{a} [g(A_{a} Y, Z)g(A_{a} X, W) - g(A_{a} X, Z)g(A_{a} Y, W)]
\]

(51)

By using \( A_{e_i} \), we can construct a similar equation to Eq. (47) for Eq. (23).

Now, let \( S \) be the Ricci tensor of \( M \). Then, Eq. (47) gives us

\[
S(X, Y) = c\{g(X, Y) - g(e_i, X)g(e_i, Y)\} \\
+ \sum_{e_i} [g(A_{e_i} e_i, e_i)g(A_{e_i} X, Y) - g(A_{e_i} X, e_i)g(A_{e_i} e_i, Y)] \\
= c(n-1)g(X, Y) + \sum_{e_i} [Tr(A_{e_i})g(A_{e_i} X, Y) - g(A_{e_i} X, A_{e_i} Y)].
\]

(53)

where \( \{e_1, e_2, ..., e_n\} \) are orthonormal basis of \( M \).

Therefore, the scalar curvature \( r \) of sub-manifold \( M \) is given by
\[
\sum_{\epsilon_x} Tr(A_{\epsilon_x})^2 = cn(n-1)\sum_{\epsilon_x} Tr(A_{\epsilon_x})^2 \tag{54}
\]

\[
\sum_{\epsilon_x} Tr(A_{\epsilon_x})^2 \text{ is the square of the length of the second fundamental form of } M, \text{ which is denoted by } |A_{\epsilon_x}|^2. \text{ Thus, we also have}
\]

\[
\| h^2 \| = \sum_{i,j=1}^n g(h(e_i, e_i), h(e_j, e_j)) = \| A^2 \|. \tag{55}
\]

2. Distribution on a manifold

An \( m \)-dimensional distribution on a manifold \( \tilde{M} \) is a mapping \( D \) defined on \( \tilde{M} \), which assigns
to each point \( p \) of \( \tilde{M} \) an \( m \)-dimensional linear subspace \( D_p \) of \( T_{\tilde{M}}(p) \). A vector field \( X \) on \( \tilde{M} \)
belongs to \( D \) if we have \( X_p \in D_p \) for each \( p \in \tilde{M} \). When this happens, we write \( X \in \Gamma(D) \). The
distribution \( D \) is said to be differentiable if for any \( p \in \tilde{M} \), there exist \( m \)-differentiable linearly
independent vector fields \( X_j \in \Gamma(D) \) in a neighborhood of \( p \).

The distribution \( D \) is said to be involutive if for all vector fields \( X, Y \in \Gamma(D) \) we have
\[
[X, Y] \in \Gamma(D). \]
A sub-manifold \( M \) of \( \tilde{M} \) is said to be an integral manifold of \( D \) if for every point \( p \in M \), \( D_p \) coincides with the tangent space to \( M \) at \( p \). If there exists no integral manifold of \( D \) which contains \( M \), then \( M \) is called a maximal integral manifold or a leaf of \( D \). The distribution \( D \) is said to be integrable if for every \( p \in \tilde{M} \), there exists an integral manifold of \( D \) containing \( p \) [2].

Let \( \tilde{\nabla} \) and distribution be a linear connection on \( \tilde{M} \), respectively. The distribution \( D \) is said to
be parallel with respect to \( \tilde{M} \), if we have
\[
\tilde{\nabla}_X Y = \nabla^{0}_X Y + S(X, PY) + P(\nabla^{0}_X Y + Q(X, PY)) \tag{57}
\]
for any \( X, Y \in \Gamma(TM) \), where \( \nabla^{0} \) and \( S \) are, respectively, an arbitrary linear connection and
arbitrary tensor field of type \((1, 2)\) on \( \tilde{M} \).

**Proof:** Suppose \( \tilde{\nabla} \) is an arbitrary linear connection on \( \tilde{M} \). Then, any linear connection \( \nabla \) on \( \tilde{M} \)
is given by
\[ \nabla_X Y = \nabla_X^J Y + S(X, Y) \quad (58) \]

for any \( X, Y \in \Gamma(TM) \). We can put

\[ X = PX + QX \quad (59) \]

for any \( X \in \Gamma(TM) \). Then, we have

\[
\begin{align*}
\nabla_X Y &= \nabla_X (PY + QY) = \nabla_X PY + \nabla_X QY = \nabla_X^J PY + S(X, PY) \\
&\quad + \nabla_X^J QY + S(X, QY) = \nabla_X^J PY + Q \nabla_X^J QY + P S(X, PY) + Q S(X, QY) \\
&\quad + P \nabla_X^J QY + Q \nabla_X^J QY + P S(X, QY) + Q S(X, QY) \\
&= (\nabla_X^J Y) + P \nabla_X^J QY + Q \nabla_X^J QY + P S(X, QY) + Q S(X, QY)
\end{align*}
\]

\[
(60)
\]

for any \( X, Y \in \Gamma(TM) \).

The distributions \( D \) and \( D^\perp \) are both parallel with respect to \( \nabla \) if and only if we have

\[
\phi(\nabla_X^J Y) = 0 \text{ and } P \phi(\nabla_X QY) = 0.
\]

(61)

From Eqs. (58) and (61), it follows that \( D \) and \( D^\perp \) are parallel with respect to \( \nabla \) if and only if

\[
Q \nabla_X^J QY + Q S(X, QY) = 0 \text{ and } P \nabla_X^J QY + P S(X, QY) = 0.
\]

(62)

Thus, Eqs. (58) and (62) give us Eq. (57).

Next, by means of the projections \( P \) and \( Q \), we define a tensor field \( F \) of type \((1,1)\) on \( \tilde{M} \) by

\[
FX = PX - QX
\]

(63)

for any \( X \in \Gamma(TM) \). By a direct calculation, it follows that \( F^2 = I \). Thus, we say that \( F \) defines an almost product structure on \( \tilde{M} \). The covariant derivative of \( F \) is defined by

\[
(\nabla_X F)Y = \nabla_X(FY) - F(\nabla_X Y)
\]

(64)

for all \( X, Y \in \Gamma(TM) \). We say that the almost product structure \( F \) is parallel with respect to the connection \( \nabla \), if we have \( \nabla_X F = 0 \). In this case, \( F \) is called the Riemannian product structure [2].

**Theorem 2.2.** Let \( (\tilde{M}, \tilde{g}) \) be a Riemannian manifold and \( D, D^\perp \) be orthogonal distributions on \( \tilde{M} \) such that \( TM = D \oplus D^\perp \). Both distributions \( D \) and \( D^\perp \) are parallel with respect to \( \nabla \) if and only if \( F \) is a Riemannian product structure.

**Proof:** For any \( X, Y \in \Gamma(TM) \), we can write

\[
\nabla_Y PX = \nabla_Y^J PX + \nabla_Y QY PX
\]

(65)

and
\[ \nabla_Y X = \nabla_{PY} PX + \nabla_{PY} QX + \nabla_{QY} PX + \nabla_{QY} QX, \quad (66) \]

from which

\[ g(\nabla_{QY} PX, QZ) = QYg(PX, QZ) - g(\nabla_{QY} QZ, PX) = 0 - g(\nabla_{QY} QZ, PX) = 0, \quad (67) \]

that is, \( \nabla_{QY} PX \in \Gamma(D) \) and so \( P\nabla_{QY} PX = \nabla_{QY} PX \),

\[ Q\nabla_{QY} PX = 0. \quad (68) \]

In the same way, we obtain

\[ g(\nabla_{PY} QX, PZ) = PYg(QX, PZ) - g(QX, \nabla_{PY} PZ) = 0, \quad (69) \]

which implies that

\[ P\nabla_{PY} QX = 0 \text{ and } Q\nabla_{PY} QX = \nabla_{PY} QX. \quad (70) \]

From Eqs. (66), (68) and (70), it follows that

\[ P\nabla_{Y} X = \nabla_{PY} PX + \nabla_{QY} PX, \quad (71) \]

By using Eqs. (64) and (71), we obtain

\[ (\nabla_{Y} P)X = \nabla_{PY} PX - P\nabla_{Y} X = \nabla_{PY} PX + \nabla_{QY} PX - \nabla_{QY} PX = 0. \quad (72) \]

In the same way, we can find \( \hat{\nabla}Q = 0 \). Thus, we obtain

\[ \hat{\nabla}F = \hat{\nabla}(P-Q) = 0. \quad (73) \]

This proves our assertion [2].

**Theorem 2.3.** Both distributions \( D \) and \( D^\perp \) are parallel with respect to Levi-Civita connection \( \nabla \) if and only if they are integrable and their leaves are totally geodesic in \( \hat{M} \).

**Proof:** Let us assume both distributions \( D \) and \( D^\perp \) are parallel. Since \( \nabla \) is a torsion free linear connection, we have

\[ [X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(D), \quad \text{for any } X, Y \in \Gamma(D) \quad (74) \]

and

\[ [U, V] = \nabla_U V - \nabla_V U \in \Gamma(D^\perp), \quad \text{for any } U, V \in \Gamma(D^\perp) \quad (75) \]

Thus, \( D \) and \( D^\perp \) are integrable distributions. Now, let \( M \) be a leaf of \( D \) and denote by \( h \) the second fundamental form of the immersion of \( M \) in \( \hat{M} \). Then by the Gauss formula, we have
\[ \nabla_X Y = \nabla'_X Y + h(X, Y) \quad (76) \]

for any \( X, Y \in \Gamma(D) \), where \( \nabla' \) denote the Levi-Civita connection on \( M \). Since \( D \) is parallel from Eq. (76) we conclude \( h = 0 \), that is, \( M \) is totally in \( \tilde{M} \). In the same way, it follows that each leaf of \( D^\perp \) is totally geodesic in \( \tilde{M} \).

Conversely, suppose \( D \) and \( D^\perp \) be integrable and their leaves are totally geodesic in \( \tilde{M} \). Then by using Eq. (4), we have

\[ \nabla_X Y \in \Gamma(D) \text{ for any } X, Y \in \Gamma(D) \quad (77) \]

and

\[ \nabla_U V \in \Gamma(D^\perp) \text{ for any } U, V \in \Gamma(D^\perp). \quad (78) \]

Since \( g \) is a Riemannian metric tensor, we obtain

\[ g(\nabla_X Y, V) = -g(Y, \nabla_X V) = 0 \quad (79) \]

and

\[ g(\nabla_X V, Y) = -g(V, \nabla_X Y) = 0 \quad (80) \]

for any \( X, Y \in \Gamma(D) \) and \( U, V \in \Gamma(D^\perp) \). Thus, both distributions \( D \) and \( D^\perp \) are parallel on \( \tilde{M} \).

### 3. Locally decomposable Riemannian manifolds

Let \((\tilde{M}, \tilde{\gamma})\) be \( n \)-dimensional Riemannian manifold and \( F \) be a tensor (1,1)-type on \( \tilde{M} \) such that \( F^2 = I, \ F \neq \mp I \).

If the Riemannian metric tensor \( \tilde{\gamma} \) satisfying

\[ \tilde{\gamma}(X, Y) = \tilde{\gamma}(FX, FY) \quad (81) \]

for any \( X, Y \in \Gamma(T\tilde{M}) \) then \( \tilde{M} \) is called almost Riemannian product manifold and \( F \) is said to be almost Riemannian product structure. If \( F \) is parallel, that is, \( (\nabla_X F)Y = 0 \), then \( \tilde{M} \) is said to be locally decomposable Riemannian manifold.

Now, let \( \tilde{M} \) be an almost Riemannian product manifold. We put

\[ P = \frac{1}{2}(I + F), \ Q = \frac{1}{2}(I-F). \quad (82) \]

Then, we have
\[ P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \quad \text{and} \quad F = P - Q. \quad (83) \]

Thus, \( P \) and \( Q \) define two complementary distributions \( P \) and \( Q \) globally. Since \( F^2 = I \), we easily see that the eigenvalues of \( F \) are 1 and \(-1\). An eigenvector corresponding to the eigenvalue 1 is in \( P \) and an eigenvector corresponding to \(-1\) is in \( Q \). If \( F \) has eigenvalue 1 of multiplicity \( p \) and eigenvalue \(-1\) of multiplicity \( q \), then the dimension of \( P \) is \( p \) and that of \( Q \) is \( q \). Conversely, if there exist in \( M \) two globally complementary distributions \( P \) and \( Q \) of dimension \( p \) and \( q \), respectively. Then, we can define an almost Riemannian product structure \( F \) on \( M \) by \( F = P - Q \) \[.\]

Let \((\tilde{M}, \tilde{g}, F)\) be a locally decomposable Riemannian manifold and we denote the integral manifolds of the distributions \( P \) and \( Q \) by \( M^p \) and \( M^q \), respectively. Then we can write \( \tilde{M} = M^p \times M^q \), \((p, q > 2)\). Also, we denote the components of the Riemannian curvature \( R \) of \( \tilde{M} \) by \( R_{\alpha\beta\gamma\delta} \) for \( 1 \leq \alpha, \beta, \gamma, \delta \leq n = p + q \).

Now, we suppose that the two components are both of constant curvature \( \lambda \) and \( \mu \). Then, we have

\[ R_{\alpha\beta\gamma\delta} = \lambda \{ g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\gamma\beta} \} \quad (84) \]

and

\[ R_{\gamma\alpha\beta\delta} = \mu \{ g_{\gamma\alpha} g_{\beta\delta} - g_{\gamma\delta} g_{\beta\alpha} \}. \quad (85) \]

Then, the above equations may also be written in the form

\[ R_{\alpha\beta\gamma\delta} = \frac{1}{4} (\lambda + \mu) \{ g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\gamma\beta} \} + \frac{1}{4} (\lambda - \mu) \{ F_{\alpha\beta} F_{\gamma\delta} - F_{\alpha\delta} F_{\gamma\beta} \} \quad (86) \]

Conversely, suppose that the curvature tensor of a locally decomposable Riemannian manifold has the form

\[ R_{\alpha\beta\gamma\delta} = a \{ g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\gamma\beta} \} + b \{ F_{\alpha\beta} F_{\gamma\delta} - F_{\alpha\delta} F_{\gamma\beta} \} \quad (87) \]

Then, we have

\[ R_{\alpha\beta\gamma\delta} = 2(a + b) \{ g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\gamma\beta} \} \quad (88) \]

and

\[ R_{\gamma\alpha\beta\delta} = 2(a - b) \{ g_{\gamma\alpha} g_{\beta\delta} - g_{\gamma\delta} g_{\beta\alpha} \}. \quad (89) \]

Let \( \tilde{M} \) be an \( m \)-dimensional almost Riemannian product manifold with the Riemannian structure \((\tilde{F}, \tilde{g})\) and \( M \) be an \( n \)-dimensional sub-manifold of \( M \). For any vector field \( X \) tangent to \( M \), we put
\[ FX = fX + wX, \]  
\begin{equation} \label{eq:90} \end{equation}

where \( fX \) and \( wX \) denote the tangential and normal components of \( FX \), with respect to \( M \), respectively. In the same way, for \( V \in \Gamma(T^1M) \), we also put

\[ FV = BV + CV, \]  
\begin{equation} \label{eq:91} \end{equation}

where \( BV \) and \( CV \) denote the tangential and normal components of \( FV \), respectively.

Then, we have

\[ f^2 + Bw = I, Cw + wf = 0 \]  
\begin{equation} \label{eq:92} \end{equation}

and

\[ fB + BC = 0, wB + C^2 = I. \]  
\begin{equation} \label{eq:93} \end{equation}

On the other hand, we can easily see that

\[ g(X, fY) = g(fX, Y) \]  
\begin{equation} \label{eq:94} \end{equation}

and

\[ g(X, Y) = g(fX, fY) + g(wX, wY) \]  
\begin{equation} \label{eq:95} \end{equation}

for any \( X, Y \in \Gamma(TM) \) [6].

If \( wX = 0 \) for all \( X \in \Gamma(TM) \), then \( M \) is said to be invariant sub-manifold in \( \tilde{M} \), i.e., \( F(T_M(p)) \subset T_{\tilde{M}}(p) \) for each \( p \in M \). In this case, \( f^2 = I \) and \( g(fX, fY) = g(X, Y) \). Thus, \( (f, g) \) defines an almost product Riemannian on \( M \).

Conversely, \( (f, g) \) is an almost product Riemannian structure on \( M \), the \( w = 0 \) and hence \( M \) is an invariant sub-manifold in \( \tilde{M} \).

Consequently, we can give the following theorem [7].

**Theorem 3.1.** Let \( M \) be a sub-manifold of an almost Riemannian product manifold \( \tilde{M} \) with almost Riemannian product structure \( (F, \tilde{g}) \). The induced structure \( (f, g) \) on \( M \) is an almost Riemannian product structure if and only if \( M \) is an invariant sub-manifold of \( \tilde{M} \).

**Definition 3.1.** Let \( M \) be a sub-manifold of an almost Riemannian product \( \tilde{M} \) with almost product Riemannian structure \( (F, \tilde{g}) \). For each non-zero vector \( X_p \in T_M(p) \) at \( p \in M \), we denote the slant angle between \( FX_p \) and \( T_{\tilde{M}}(p) \) by \( \theta(p) \). Then \( M \) said to be slant sub-manifold if the angle \( \theta(p) \) is constant, i.e., it is independent of the choice of \( p \in M \) and \( X_p \in T_M(p) \) [5].

Thus, invariant and anti-invariant immersions are slant immersions with slant angle \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \), respectively. A proper slant immersion is neither invariant nor anti-invariant.
Theorem 3.2. Let $M$ be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost product Riemannian structure $(\mathcal{F}, \tilde{g})$. $M$ is a slant sub-manifold if and only if there exists a constant $\lambda \in (0, 1)$, such that

$$f^2 = \lambda I.$$  \hfill (96)

Furthermore, if the slant angle is $\theta$, then it satisfies $\lambda = \cos^2 \theta$ [9].

Definition 3.2. Let $M$ be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(\mathcal{F}, \tilde{g})$. $M$ is said to be semi-slant sub-manifold if there exist distributions $D_\theta$ and $D_T$ on $M$ such that

(i) $TM$ has the orthogonal direct decomposition $TM = D \oplus D_T$.

(ii) The distribution $D_\theta$ is a slant distribution with slant angle $\theta$.

(iii) The distribution $D_T$ is an invariant distribution, i.e., $\mathcal{F}(D_T) \subseteq D_T$.

In a semi-slant sub-manifold, if $\theta = \frac{\pi}{2}$, then semi-slant sub-manifold is called semi-invariant sub-manifold [8].

Example 3.1. Now, let us consider an immersed sub-manifold $M$ in $\mathbb{R}^7$ given by the equations

$$x_1^2 + x_2^2 = x_5^2 + x_6^2, x_3 + x_4 = 0.$$  \hfill (97)

By direct calculations, it is easy to check that the tangent bundle of $M$ is spanned by the vectors

$$z_1 = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \beta \frac{\partial}{\partial x_5} + \sin \beta \frac{\partial}{\partial x_6},$$
$$z_2 = -u \sin \theta \frac{\partial}{\partial x_1} + u \cos \theta \frac{\partial}{\partial x_2}, z_3 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_4},$$
$$z_4 = -u \sin \beta \frac{\partial}{\partial x_5} + u \cos \beta \frac{\partial}{\partial x_6}, z_5 = \frac{\partial}{\partial x_7},$$  \hfill (98)

where $\theta, \beta$ and $u$ denote arbitrary parameters.

For the coordinate system of $\mathbb{R}^7 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7)\mid x_i \in \mathbb{R}, 1 \leq i \leq 7\}$, we define the almost product Riemannian structure $F$ as follows:

$$F \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}, F \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial x_j}, 1 \leq i \leq 3 \text{ and } 4 \leq j \leq 7.$$  \hfill (99)

Since $F_{z_1}$ and $F_{z_3}$ are orthogonal to $M$ and $F_{z_2}, F_{z_4}, F_{z_5}$ are tangent to $M$, we can choose a $\mathcal{D} = S_p\{z_2, z_4, z_5\}$ and $\mathcal{D}^T = S_p\{z_1, z_3\}$. Thus, $M$ is a 5-dimensional semi-invariant sub-manifold of $\mathbb{R}^7$ with usual almost Riemannian product structure $(\mathcal{F}, <, >)$.

Example 3.2. Let $M$ be sub-manifold of $\mathbb{R}^8$ by given

$$(u + v, u-\dot{v}, u \cos \alpha, u \sin \alpha, u + v, u-\dot{v}, u \cos \beta, u \sin \beta).$$  \hfill (100)
where \(u, v\) and \(\beta\) are the arbitrary parameters. By direct calculations, we can easily see that the tangent bundle of \(M\) is spanned by

\[
e_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos \alpha \frac{\partial}{\partial x_3} + \sin \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \cos \beta \frac{\partial}{\partial x_6} + \sin \beta \frac{\partial}{\partial x_7}
\]

\[
e_2 = -\sin \beta \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \frac{\partial}{\partial x_7}
\]

\[
e_3 = -\cos \beta \frac{\partial}{\partial x_1} + \sin \beta \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} - \cos \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \frac{\partial}{\partial x_7}
\]

\[
e_4 = -\sin \beta \frac{\partial}{\partial x_1} + \cos \beta \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \cos \alpha \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} \frac{\partial}{\partial x_7}
\]

(101)

For the almost Riemannian product structure \(F\) of \(\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^4\), \(F(TM)\) is spanned by vectors

\[
F_{e_1} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos \alpha \frac{\partial}{\partial x_3} + \sin \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \cos \beta \frac{\partial}{\partial x_6} + \sin \beta \frac{\partial}{\partial x_7}
\]

\[
F_{e_2} = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} \frac{\partial}{\partial x_7}
\]

(102)

Since \(F_{e_1}\) and \(F_{e_2}\) are orthogonal to \(M\) and \(F_{e_3}\) and \(F_{e_4}\) are tangent to \(M\), we can choose \(D^1 = Sp\{e_1, e_4\}\) and \(D^4 = Sp\{e_1, e_2\}\). Thus, \(M\) is a four-dimensional semi-invariant sub-manifold of \(\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4\) with usual Riemannian product structure \(F\).

**Definition 3.3.** Let \(M\) be a sub-manifold of an almost Riemannian product manifold \(\tilde{M}\) with almost Riemannian product structure \((\tilde{F}, \tilde{\gamma})\). \(M\) is said to be pseudo-slant sub-manifold if there exist distributions \(D_0\) and \(D_1\) on \(M\) such that

i. The tangent bundle \(TM = D_0 \oplus D_1\).

ii. The distribution \(D_0\) is a slant distribution with slant angle \(\theta\).

iii. The distribution \(D_1\) is an anti-invariant distribution, i.e., \(F(D_1) \subseteq T^1M\).

As a special case, if \(\theta = 0\) and \(\theta = \frac{\pi}{2}\), then pseudo-slant sub-manifold becomes semi-invariant and anti-invariant sub-manifolds, respectively.

**Example 3.3.** Let \(M\) be a sub-manifold of \(\mathbb{R}^6\) by the given equation

\[
(\sqrt{3}u, v, usin\theta, vcos\theta, scost, -scost)
\]

where \(u, v, s\) and \(t\) arbitrary parameters and \(\theta\) is a constant.

We can check that the tangent bundle of \(M\) is spanned by the tangent vectors

\[
e_1 = \sqrt{3} \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial y_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \theta \frac{\partial}{\partial y_2},
\]

\[
e_3 = \cos \theta \frac{\partial}{\partial x_3} - \cos \theta \frac{\partial}{\partial y_3}, e_4 = -\sin \theta \frac{\partial}{\partial x_4} + \sin \theta \frac{\partial}{\partial y_4}
\]

(104)

For the almost product Riemannian structure \(F\) of \(\mathbb{R}^6\) whose coordinate systems \((x_1, y_1, x_2, y_2, x_3, y_3)\) choosing
\[
F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad 1 \leq i \leq 3,
\]
\[
F\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq 3,
\]
(105)

Then, we have
\[
\begin{align*}
F_{e_1} &= \sqrt{3} \frac{\partial}{\partial y_1}, \\
F_{e_2} &= -\frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial y_2} - \cos\theta \frac{\partial}{\partial x_2}, \\
F_{e_3} &= \cos\theta \frac{\partial}{\partial y_3} + \cos\theta \frac{\partial}{\partial x_3}, \\
F_{e_4} &= -\sin\theta \frac{\partial}{\partial y_3} - \sin\theta \frac{\partial}{\partial x_3}.
\end{align*}
\]
(106)

Thus, \( D_\theta = S_p\{e_1, e_3\} \) is a slant distribution with slant angle \( \alpha = \frac{\pi}{4} \). Since \( F_{e_3} \) and \( F_{e_4} \) are orthogonal to \( M \), \( D^\perp = S_p\{e_3, e_4\} \) is an anti-invariant distribution, that is, \( M \) is a 4-dimensional proper pseudo-slant sub-manifold of \( \mathbb{R}^6 \) with its almost Riemannian product structure \( (F, <, >) \).

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