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Chapter 12

Theory of Flux Cutting for Type-II Superconducting Plates at Critical State

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1. Introduction

With the discovery in 1986 of high critical temperature superconductors $T_c \geq 77K$ –which belong to the type-II classification– efforts have been made to recognize which mechanism rules its current carrying capacity in order to expand knowledge of the vortex state and, moreover, devise new and better technological applications. Critical-state phenomenological models for such materials have been a feasible alternative for the theoretical study of the magnetic properties of high- or low-$T_c$ type-II superconductors. Here we present a brief revision of macroscopic critical-state models; following a chronological order, we will begin with the Bean model, moving on with the generalized double-critical state model, the two-velocity hydrodynamic model, and finalizing with the Elliptic Flux-Line Cutting Critical-State Model (ECSM). It will be described further the main features of type-II superconductors, the physical meaning of the critical state and the flux-line cutting phenomenon.

1.1. Type-II superconductor critical state

In 1911 Kammerlingh Onnes discovered the superconductivity of mercury at very low temperature. Nowadays, the characteristics of superconductors are well established: their electric resistance abruptly drops to zero as temperature decreases through a critical temperature value designated as $T_c$. They show the Meissner-Ochsenfeld effect, that is, they completely expel a weak magnetic field as temperature decreases through the transition point. Depending on how this diamagnetic phenomenon is destroyed, superconductors can be classified as type I or II. Type-I superconductors are perfect diamagnets below a critical field $H_c$. Because their coherence length $\xi$ exceeds the penetration length $\lambda$, it is energetically unfavorable for borders to be formed between the normal and superconductive phases. However, when a type-II superconductor is subjected to a magnetic field $H_\perp$, free energy can diminish, thus generating normal matter domains that contain trapped flux, with low-energy
borders created between the normal core and the superconductive surroundings. When the applied magnetic field exceeds the lower critical field $H_{c1}$, the magnetic flux penetrates in quantized units $\Phi_0$, forming cylindrical domains called vortices. As $H_a$ increases, vortices will overlap increasing the interior field until the material gently enters the normal state, once $H_a$ has reached the upper critical field $H_{c2}$. Between the fields $H_{c1}$ and $H_{c2}$ the superconductor state coexists with the magnetic state in a mixed state or vortex state.

Another characteristic of superconductors is the presence of a gap, just below the Fermi energy, the energy of conduction electrons. BCS superconductivity theory demonstrated that electrons in the vicinity of the Fermi level are grouped in the so-called Cooper pairs. In addition, the junction of two superconductors --separated by a thin insulating layer-- shows the DC Josephson effect, in which the superconductor current tunneling is caused by the tunneling of Cooper pairs. This effect demonstrated that the superconductor state is a coherent state, which is associated to a macroscopic uniform-phase wave function; this function corresponds to the order parameter $\kappa = \lambda / \xi$ in the Ginzburg-Landau theory. Finally, they also show the AC Josephson effect that describes the relation between the time variation of the macroscopic wave function with the voltage produced across the junction. This voltage arises from the quantized magnetic flux movement, and is identical to the macroscopic voltage observed in type-II superconductors in flux flow state.

Indeed, it is well established that type-II superconductors posses a stationary vortex spatial arrangement only if the total force over each vortex is null. If an electric current is applied with $J$ density, vortices move at a velocity $v$ with a direction determined by the Hall angle. If both the Magnus force and the Hall effect on the material are neglected, equilibrium between the Lorentz force $F_L = J \times B$ and the pinning force $F_p$ will exist:

$$ (J \times B) - F_p = 0. \tag{1} $$

In addition to being able to describe the vortex dynamics under the transport current influence, the equation (1) can be used for the time-variable external magnetic case, in absence of transport currents.

Indeed, $F_p$ opposes the magnetic flux velocity $v$ due to the local depression of the Gibbs free energy of each vortex. This potential well may be due to inhomogeneities, defects, or material grains. Therefore, magnetic flux movement will occur if the Lorentz force exceed the pinning force. Any electromotive force, even small, causes the vortices to move further into the material, inducing a local current. Initially, this superconductive current flows in regions close to the superconductor surface because pinning centers near the sample surface can catch the vortices in such a way that, in the interior, the Meissner state is preserved, thus the sample is partially penetrated. For higher external magnetic field values, vortices will completely penetrate the material.

### 1.2. Bean model for a type-II superconductor in critical state

Half a century ago, Charles Bean approached — with great physics intuition and from a macroscopic point of view — the study of the magnetic properties of superconductors made with impure metals or alloys. Bean modeled the spatial distribution of the magnetic flux, for
partial and totally penetrated states, a couple of years before having experimental evidence of the mixed or vortex state predicted by A. A. Abrikosov.

He relied on Mendelssohn sponge model to describe the magnetic behaviour of such superconductors, supposing they possessed a filamental structure capable of maintaining a maximum macroscopic current $J_c$, without energy dissipation in form of Joule heating, that he called it critical current density. Due to the $J_c$ dependence on the magnetic field, he considered that such currents were extended into the material, preserving the magnitude $J_c$.

Bean argued that the macroscopic current is a consequence of the magnetic induction gradient penetrating the material, governed by Ampère’s law $\nabla \times B = \mu_0 J$. He also argued that the current originates as the Lorentz force drives the magnetic flux into the interior of the material. Therefore, he considered that any local region where an electric field (related to heat dissipation) is perceived during the process of magnetization, it would originated a critical current density $J_c$ which flows in the electric field direction and it keep flowing even if it the electric field was null [6–9]. He synthesized these ideas in the material law:

$$J = J_c(B) \text{sign}(E),$$

which is valid for slabs or infinite cylinders subjected to a magnetic field parallel to the superconductor’s surface (this is the so-called parallel geometry).

To exemplify the Bean model, Figure (1) shows the profile evolution of magnetic induction for a PbBi plate at mixed-state, as it increased (blue curves) or decreased (red curves) the external magnetic field magnitude $H_a$ parallel to its surface. The hysteresis cycle of the average static magnetization is shown when the $H_a$ varies a full cycle, from $-0.3T$ to $0.3T$. The superconducting plate has a 8mm thickness and a penetration field $\mu_0 H_p = 0.1015T$. We have considered the reliance on $B$ of the critical current density $J_c(B)$.

Subsequently, Bean studied energy dissipation in materials subjected to a magnetic field that rotates in the specimen’s plane. He extended his arguments assuming the current density $J$ and the electric field $E$ vectors would be parallel to each other [9]. For this case, the material equation can be written as follows:

$$J = J_c(B) \frac{E}{E(J)}$$

where it is necessary to model the $E(J)$ function form. The Bean model, corresponding to the material equation (2) or (3), together with the Ampère’s law, have been used to calculate magnetic induction profiles, hysteresis of magnetization cycles, and the energy dissipation of several type-II superconductor materials. In the search for new superconducting alloys that would produce more intense magnetic fields, or a greater current conduction capacity, these materials were simultaneously subjected to a magnetic field and a transport current parallel to each other. It was desirable that the electric current density $J$ and the magnetic induction $B$ established a force-free configuration, that is, a zero Lorentz force $F_L = J \times B = 0$. However, experimental evidence showed that even if $F_L$ could be considered null, a significant voltage
Figure 1. Theoretical curves of a superconductor plate, with thickness $d = 8\text{mm}$ and penetration field $\mu_0H_p = B_p = 0.1015\text{T}$, obtained with the Bean critical-state model and considering $J_c = J_c(B)$. (Left) Evolution of magnetic induction $B$ as it increases (continuous blue lines) and decreases (red discontinuous lines) the magnitude of the external magnetic field $H_a$. When $\mu_0H_a = \mu_0H_p$, $B$ at the center of the plate is null. (Right) Average static magnetization cycle $\mu_0\langle M \rangle$ against the applied magnetic field $\mu_0H_a$. Since this material is an irreversible type-II superconductor, $\mu_0\langle M \rangle$ describes a hysteresis.

would arise from the material. Therefore, if the vortex velocity is equal to zero in a force-free configuration, what is going on in this type of configuration? This question could not be answered using the Bean model, so it was considered that another phenomenon might be occurring. The answer to this query is the so-called flux-line cutting or flux crossing, which will be discussed in the next section.

1.3. Flux-Line cutting

D.G. Wallsley [10] measured the magnetization and the axial resistance of a type-II superconductor—in mixed state and with cylindrical geometry—by subjecting it simultaneously to a magnetic axial field and a transport current, parallel to each other. The objective was to prove under what the Lorentz force density could be null. He found that, at low currents, the potential difference between the extremes of the sample was negligible (~10µV). However, when the superconducting sample conducted a sufficiently high current, it measured a voltage (or a longitudinal electric field), as well as a paramagnetic moment, that is, a positive average magnetization. He then suggested that the force-free structure could not be established on the material’s surface, which originated the flux-flow and, consequently, a voltage in the rest of the material.

He intuited that the measurement of a paramagnetic moment suggested a helicoidal vortex distribution. Nonetheless, the voltage produced by the flux flow would imply a permanent increase in the longitudinal magnetic field. For this reason, he supposed the existence of a non-stationary process in which the magnetic flux lines would continually divide each other, only to reconnect afterward.

As a solution for the flux-flow movement contradiction in a force-free configuration, Clem was the first to suggest that the helicoidal instabilities were precisely the precursors of vortex cutting or crossing; that is, considering the elastic properties of flux lines, he proposed that they could stay fixed but they would be able to bend, calling this phenomenon flux-line cutting.
Even though Josephson had already established that cutting or crossing of vortices could not occur because they were energetically too expensive, theoretical calculations done by Brandt, Clem, and Walmsley — using the London and Ginzburg-Landau theories — proved that the threshold of cutting of a pair of rigid flux lines was possible since the characteristics energies of a type-II superconductor.

Subsequently, Brandt and Sudbo extended these results for the case of a pair of twisted flux lines, they considered the tension and interaction between each flux-line or vortex. Given that cutting is energetically plausible, they found that flux-line cutting is an effective disentanglement mechanism of flux lines; and that the cutting energy barrier, in the case of twisted flux lines, is lower for the rigid two-flux-lines case [11, 12]. Several groups, for example, M.A.R. LeBlanc et. al [13–15], have done experiments in the last decades that have shown flux-cutting presence in materials with low or high $\kappa$, and low or high $T_c$. In the Clem et. al [16–18] and Brandt [19] papers, we can see flux-line cutting diagrams for the case of rigid-vortices arrangement. Furthermore, it includes diagrams for the first theoretical formulations for this often-studied and not completely understood phenomenon.

More recent theoretical studies have studied the scattering dynamics of vortices, and the resulting topology after a collision between two flux lines generated by an applied current. Using the time-dependent Ginzburg-Landau equations, numerical results yielded two generic collision types dependent on the initial angle: one local collision that induces changes in topology through recombination, and a double collision that can occur due to geometrical restrictions. The second case leads to a vortex-crossing type configuration, that is, it seems as if two vortices, while interacting, would cut themselves and join again. This can be seen in the simulations shown in paper [20]. Experiments have been proposed using a magnetic force microscope to monitor vortex-line dynamics and prove if these cut through each other when they are in a liquid-vortex phase [21].

In 2008, A. Palau et al. reported results with superconductive heterostructures subjected to an external magnetic field at a $\theta$ angle respect to the sample’s normal. For this, they designed a device made out of a thin film of low-pinning amorphous material ($Mo_{82}Si_{18}$), sandwiched between two Nb films— a material characterized for strongly pinning the vortices. They measured the critical current density $J_c$ obtained as a function of $\theta$, the applied field $\mu_0H_a$, and temperature $T$. Once obtained, they calculated the balance force between the Lorentz force, the pinning force, and a so-called breaking force. They found that the latter was necessary in order to consider vortex deformation and destabilization. Results showed that the breaking force is independent of $B$, and that the following cross joining neither limit vortex movement nor increase $J_c$. Even if a flux-line segment is strongly pinned to the area where Nb material is found, the cut induces other vortex segments to be liberated, thus reducing $J_c$ [22].

Furthermore, Campbell’s revision paper can be consulted to know the state of the art about experiments and critical state theories for flux-cutting in superconductors [23]. He included the last proposal of Clem to determine the electric field direction, for the flux transport regime in a type-II superconductor.

Here it is presented three critical state models created for the phenomenological study of type-II superconductors subject to magnetic fields that vary not only in magnitude, but also in direction. All models consider that flux pinning and flux-line cutting govern their answer. It is undeniable that both phenomena can occur in cases when a sample oscillates in presence of a static magnetic field, or when it is subjected to a DC magnetic field and a transversal sweeping magnetic field is superimposed.
2. Other critical state models

2.1. Generalized Double Critical-State Model

LeBlanc et al. proposed a model containing two critical-state equations based on their experimental observations on the magnetic response of a disc oscillating at low frequency, in presence of a magnetic field [24]:

\[ \frac{dB}{dx} = \pm \frac{F_p(B)}{B}, \quad \frac{d\alpha}{dx} = \pm k(B) \frac{dB}{dx}. \] (4)

This pair of equations is known as the Double Critical-State Model (DCSM). In their construction, the fact that the magnetic induction \( B \) and the orientation \( \alpha \) of flux-lines varied spatially was considered. They assumed as well that gradients existed in critical states. In his model, \( F_p(B) \) is a parameter that characterizes pinning intensity; \( k \) is associated to the shearing coefficient of the flux lattice in the superconductive sample.

Clem and Perez-Gonzalez extended the DCSM based on the assumption that intersection and cross-joining of adjacent non-parallel vortices generate a electric field different to the electric field \( E = B \times \mathbf{v} \). The latter field is associated to the flux flow with velocity \( \mathbf{v} \), for the case in which \( J \) is perpendicular to \( B \). For this, they proposed a pair of constitutive laws of the form:

\[ J_{\perp} = J_{c,\perp} \text{ sign } E_{\perp}, \]
\[ J_{\parallel} = J_{c,\parallel} \text{ sign } E_{\parallel}, \] (5)

where parameters \( J_{c,\perp} \) and \( J_{c,\parallel} \) correspond to the depinning and the flux-line cutting thresholds, respectively. They considered that the electric field \( E \) components obey independently the vertical laws:

\[ E_{\perp} = \begin{cases} \rho_{\perp} \left| J_{\perp} \right| - J_{c,\perp} \text{ sign } \left( J_{\perp} \right), \quad \left| J_{\perp} \right| > J_{c,\perp} \\ 0, \quad \left| J_{\perp} \right| \leq J_{c,\perp} \end{cases} \] (6)

\[ E_{\parallel} = \begin{cases} \rho_{\parallel} \left| J_{\parallel} \right| - J_{c,\parallel} \text{ sign } \left( J_{\parallel} \right), \quad \left| J_{\parallel} \right| > J_{c,\parallel} \\ 0, \quad \left| J_{\parallel} \right| \leq J_{c,\parallel} \end{cases} \] (7)

here, \( \rho_{\perp} \) and \( \rho_{\parallel} \) are the resistivities caused by flux transport and superconductor flux-line cutting, respectively. The group of equations (5)-(7) constitutes the Generalized Double Critical-State Model (GDCSM.) Clem and Perez-Gonzalez did numerical calculations considering as possible values for magnitude \( J_c \) all those defined within a rectangular region of \( J_{c,\parallel} \) and \( J_{c,\perp} \) sides. The model reproduced successfully the experimental distributions.
of magnetic induction and magnetization, when the magnetic field oscillates at great amplitudes [25].

The GDCSM was also used to try to reproduce Magnetization Collapse and Paramagnetism, phenomena encountered when a type-II superconductor is subjected first to a DC magnetic field on which an oscillating low-frequency magnetic field is superposed, perpendicular to the former.

### 2.2. Two-velocities Hydrodynamic Model (TVHM)

This macroscopic model considers that electrodynamics of a type-II superconductor depends on the translation of vortex planes and the interaction between them. It establishes two vortex subsystems, assuming they possess no elastic properties and that the flux cutting consists of the disappearance of interacting vortices, creating new vortices on a plane with an orientation different to the previous one. Gibbs energy varies through small disturbances on the vortices’ coordinates considering the following: 1) magnetic energy; 2) work done by pinning forces given the translation of the vortex network; and 3) the work done by the pinning forces to straighten a vortex after its crossing [26–28]. Thus, the model is conformed by a continuity equation for total vortex density $n(x,t)$, and the average angular distribution $\alpha(x,t)$ of the vortex planes:

\[
\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x} \left[ \frac{1}{2} \left( n \alpha \left( V_A + V_B \right) \right) \right],
\]

where $V_A = V + \frac{U}{2}, \quad V_B = V - \frac{U}{2}$

\[
\frac{\partial (n\alpha)}{\partial t} = -\frac{1}{2} \frac{\partial [n\alpha(V_A + V_B)]}{\partial x} - \frac{1}{4} \frac{\partial [n\alpha(V_A - V_B)]}{\partial x},
\]

and

\[
\frac{\partial B}{\partial x} = -\frac{\mu_0 I_{c\perp}}{2} \left[ F(V_A) + F(V_B) \right],
\]

\[
\Delta \alpha B \frac{\partial \alpha}{\partial x} + \mu_0 H_a \cos(\alpha - \alpha_0) \Delta \alpha \frac{\partial}{\partial x} \text{sign} |V_A - V_B|,
\]

\[
= -\frac{\mu_0 I_{c\perp}}{2} \left[ F(V_A) - F(V_B) + p \text{sign} |V_A - V_B| \right],
\]

correspond to the velocities of subsystems $A$ and $B$, $V(x,t)$ is the mean hydrodynamic velocity, and $U(x,t)$ is the relative velocity. The TVHM requires additionally two equations obtained from force balance conditions in a superconductor, defined for the magnetic induction gradient and for angular distribution:
here $p$ corresponds to the probability that flux-line cutting occurs. Finally, to resolve the equation system (8)-(12) for variables $V_A$, $V_B$, $B = n\Phi_0$, $\alpha$ and $\Delta\alpha$, it is required to introduce a phenomenological equation that relates $\Delta\alpha$ to the mean orientation’s spatial derivative of the form:

$$
\Delta\alpha = -l \text{sign}(V_A - V_B) \frac{\partial\alpha}{\partial x},
$$

(13)

where $l$ is the vortex mean free path between two successive cuttings or crossings.

3. Theoretical Description of the ECSM

Now we introduce the characteristics of the Elliptic Critical-State Model (ECSM) used in this chapter.

3.1. Geometrical aspects of a superconductor plate

The study system is a superconducting plate possessing an infinite surface parallel to a plane $yz$ and a finite thickness $0 \leq x \leq D$, as it is shown in Figure (2). The plate is subjected to a magnetic field $H_a$ parallel to plane $yz$ given by the expression:

$$
H_a = H_{ay}\hat{y} + H_{az}\hat{z} = H_a(\sin\alpha_a\hat{y} + \cos\alpha_a\hat{z}),
$$

(14)

where $\alpha_a$ is an angle measured relative to $z$ axis. This problem pertains to the parallel geometry. Demagnetization effects are not present, the current density $J$, electric field $E$, and magnetic induction $B$ vectors are all coplanar with their components $y$ and $z$ depending only on variable $x$ and time $t$. Given the applied magnetic field $H_a$, local magnetic induction in a superconducting sample is:
\[ \mathbf{B} = B(x,t)\mathbf{\hat{e}}_\parallel = B(\sin \alpha \mathbf{\hat{y}} + \cos \alpha \mathbf{\hat{z}}). \]  

(15)

The superconducting current circulates around planes \( x = \) constant. In region \( x \in [0, D/2) \) it circulates in the opposite direction from the one in region \( x \in (D/2, D] \).

Since we are studying flux-line cutting, ECSM postulates that the current density \( \mathbf{J} \) and electric field \( \mathbf{E} \) can possess components both parallel and perpendicular to \( \mathbf{B} \). The perpendicular component is associated to the flux-pinning effect, while parallel components are associated to flux-line cutting. Therefore, we wish to calculate how large is the force-free component parallel to \( \mathbf{B} \) of the current density, and if the latter reaches its critical value \( J_{\parallel c} \).

It is convenient to work with a reference system that rotates (follows) with the (the) magnetic induction \( \mathbf{B} \). The current density and electric field components –parallel and perpendicular to \( \mathbf{B} \)– are monitored:

\[ \mathbf{J} = J_{\parallel} \mathbf{\hat{e}}_\parallel + J_{\perp} \mathbf{\hat{e}}_\perp, \quad \mathbf{E} = E_{\parallel} \mathbf{\hat{e}}_\parallel + E_{\perp} \mathbf{\hat{e}}_\perp, \]  

(16)

where the unit vector \( \mathbf{\hat{e}}_\perp \) is built as \( \mathbf{\hat{e}}_\perp = \mathbf{\hat{e}}_\parallel \times \mathbf{\hat{x}} = \cos \alpha \mathbf{\hat{y}} - \sin \alpha \mathbf{\hat{z}} \).

3.2. Physical considerations for a Type-II superconductor

Type-II superconductor materials are found in vortex state and contain a dense distribution and random pinning centers. When applying a magnetic field \( \mathbf{H}_a \), Eq. (14), it is considered that the material is in a critic state (metastable state) when there is a balance between Lorentz force density \( \mathbf{F}_L = J_{\perp} \times \mathbf{B} \) and the average pinning force. If the perpendicular component of the current overcomes its critical value \( J_{\perp c} \), flux transport is begun until another material metastable state is reached. Also, vortex avalanches can occur; however, we will not cover here this phenomenon [29]. In addition, given that a \( \mathbf{J} \) component is parallel to \( \mathbf{B} \), magnetic flux distribution will depend on flux line cutting, only if \( J_{\parallel} \) exceeds its critical value \( J_{\parallel c} \).

Finally, the magnitude of an applied magnetic field is considered to be much larger than the first critical field \( H_{c1} \), that is, the Meissner currents on the material’s surface are neglected. We are interested in the macroscopic electrodynamics of a type-II superconductor; thus, we work with Maxwell equations, given that low-frequency magnetic fields are considered, displacement current is neglected. In order to establish the critical state, this model considers quasi-stationary electromagnetic fields. This is why first the Faraday law is used, and subsequently stationary solutions are sought. Having chosen a reference system that rotates with magnetic induction \( \mathbf{B} \), Ampère and Faraday laws are written as follows:

\[ -\frac{\partial \mathbf{B}}{\partial x} = \mu_0 J_{\perp}, \quad \frac{\partial \mathbf{B}}{\partial x} = \mu_0 J_{\parallel}. \]  

(17)
\[
\frac{\partial E_\perp}{\partial x} + E_\parallel \frac{\partial \alpha}{\partial x} = -\frac{\partial B}{\partial t}, \quad E_\perp \frac{\partial \alpha}{\partial x} - \frac{\partial E_\parallel}{\partial x} = -B \frac{\partial \alpha}{\partial t}.
\] (18)

In absence of a demagnetization factor, and considering that the applied magnetic field \( H_a \) is much larger than \( H_{c1} \), the constitutive relation between \( B \) and \( H \) is modeled with the linear approximation \( B = \mu_0 H \), where the Bean-Livingston superficial barrier is neglected. Therefore, the boundary condition is written as follows:

\[
\mu_0 H_{ay}(t) = B_y(0; t) = B_y(D; t),
\] (19)

\[
\mu_0 H_{az}(t) = B_z(0; t) = B_z(D; t).
\] (20)

### 3.3. Elliptic Flux-Cutting Critical-State Model for a type-II superconductor in critical state

The Elliptic Flux-Cutting Critical-State Model (ECSM) [1] considered the presence of an electric field in non-critical states. It models the material as a highly non-linear conductor, using a constitutive relation \( E = E(J, B) \), which completes the system of equations (17)-(18). Furthermore, it monitors the electric field’s decrease through a vertical law. The model supposes that the material reaches a critical state when the magnetic induction distribution enters a stationary state where the electric field has decreased to zero.

The material law of the elliptic model considers, in contrast to multicomponent Bean and double critical-state models, that the critical current density is a rank-2 tensor determined by the equation:

\[
J_i = (J_c)_{ik} \frac{E_k}{E}, \quad (J_c)_{ik} = J_{c,i}(B)\delta_{ik}, \quad i, k = \perp, \parallel.
\] (21)

or, by the vector relation:

\[
J = J_{c\parallel}(B) \frac{E_\parallel}{E} + J_{c\perp}(B) \frac{E_\perp}{E}.
\] (22)

In critical state, the magnitude of \( J \), that is, the critical current density \( J_c(B, \varphi) \) traces an ellipse on the plane \( J_{\parallel} - J_{\perp} \), defined by the equation:

\[
J_c(B, \varphi) = \left[ \left( \frac{\cos \varphi}{J_{c\parallel}(B)} \right)^2 + \left( \frac{\sin \varphi}{J_{c\perp}(B)} \right)^2 \right]^{-1/2},
\] (23)
where the angle $\varphi$ is measured respect to $B$. Notice that the behaviour of the vortex distribution is a function of the magnitude of $J$ and its relative orientation respect to $B$. The model postulates that $J_c(B, \varphi)$ is defined in terms of critical values $J_{c\|}$ and $J_{c\perp}$, that is, $J_c(B, \varphi)$ is restricted to the critical ellipse (23). The fact that $J_{\|}$ and $J_{\perp}$ are different produces an anisotropy on the plane $J_{\|} - J_{\perp}$ because of the external magnetic field $H_a$ and since such components are associated with different physical phenomena which are still not fully understood. Lastly, a vertical law is used to relate the current density and the electric field magnitudes as:

$$E(J) = \begin{cases} 0, & \text{for } J \leq J_c(B, \varphi), \\ \rho(J - J_c(B, \varphi)), & \text{for } J > J_c(B, \varphi). \end{cases} \hspace{1cm} (24)$$

It can be considered that an electric field can exist in the superconductor only if the current density exceeds its critical value $J_c(B, \varphi)$. Conventionally the parameter $\rho$ acts as a resistivity, however, here it serves as an auxiliary parameter that counteracts the difference $J - J_c(B, \varphi)$.

The empirical relation between the critical current density $J_c$ and magnetic induction $B$ used is known as generalized relation, it is an adaptation to such deduced by Y.B. Kim et al. [30], and was proposed by M. Xu et al. [31], its explicit form is as follows:

$$J_c(B) = J_c(0)/\left[1 + \frac{B}{B^*}\right]^n, \hspace{1cm} (25)$$

where $n$ and $B^*$ are fitting parameters, and the maximum critical current density value $J_c(B = 0)$ is defined by the following relation:

$$J_c(0) = \left[\left(1 + \frac{B_p}{B^*}\right)^{n+1}\right] \frac{2B^*}{(n+1)\mu_0 d}. \hspace{1cm} (26)$$

In our case, temperature dependency remains implicit in the parameters election.

All solutions that satisfy the system of equations formed by the Ampère’s law (17), the Faraday’s law (18), the critical current density $J_c(B, \varphi)$ (23), and the material law (24) establish the superconductor’s critical state. Such system is designated the Elliptic Flux-Cutting Critical-State Model (ECSM) [1].

4. Results using ECSM for a type-II superconductor in vortex state

Here we present theoretical curves obtained with the ECSM, corresponding to experimental results which depend on vortex dynamics. Specifically, numerical results are shown for magnetic-moment curves and magnetic-induction evolution, using type-II superconductor samples in mixed state, subject to transverse fields, or to a low-frequency rotating field.
4.1. Type-II superconductor plates subjected to transverse DC and AC fields

In this section we present the effect of an AC magnetic field on the static magnetization of a type-II superconductor plate. The purpose was to compare and explain experimental results - obtained by Fisher et al. - from YBCO samples subjected to two fields perpendicular to each other, in parallel geometry [28, 33].

In the experiments of Fisher et al., plates were cut out from the homogenous part of melt-textured YBCO ingots. This was done in such a way that the larger-sized sides were parallel to the crystallographic ab plane, which possesses isotropic properties respect to current conduction capacity. First they applied on the sample an \( H_z \) field generated by direct current, with a direction such that it was parallel to the plate’s main surface. Afterward they applied a second field \( H_y \), oscillating at low frequencies -of the order 1kOe- coplanar to the sample’s surface and perpendicular to the first field. Under this configuration, and depending on the order of magnitude for both fields, they observed phenomena denominated Magnetization Collapse and Paramagnetism.

Results presented in Figure (3) correspond to a plate with a \( d = 3 \text{mm} \) thickness that possesses a penetration field \( \mu_0 H_p = 0.0856 \text{T} \). This last value is very close to the experimental \( \mu_0 H_p \sim 0.1 \text{T} \) value for the YBCO samples [33]. The parameters used to model the perpendicular component of current critical density \( J_{c\perp}(B) \) (see Eq. 25) are: \( n_{\perp} = 0.5 \) y \( B_{\perp}^* = 0.2 \text{T} \), and a maximum value \( J_{c\perp}(0) = 5 \times 10^8 \text{A/m}^2 \) (see Eq. 26). The parallel component \( J_{c\parallel}(B) \) was
modeled just as \( J_{c\perp}(B) \). Therefore, we use \( B^\ast_\perp = B^\ast_\perp n_\perp = n_\perp \). The best fitting to the experimental curves was achieved assuming \( J_{c\parallel}(0) = 2J_{c\perp}(0) \).

Results for \( \langle M_z(H_y) \rangle \) during the first cycles of the oscillating magnetic field \( H_y \) are shown in Figure (3). One can observe in (A)-(C) graphics the symmetric reduction of the average magnetization for both the diamagnetic initial state (blue curves) and paramagnetic initial state (red curves). These three cases are characterized by the fact that static magnetic field \( H_z \) is greater than \( H_p \) (\( H_z = 5,10H_p \)), whereas the AC magnetic field oscillates at small amplitudes (\( H_{y,max} \ll H_z \)). In graphic (D) we appreciate a different behavior of the diamagnetic branch of \( \langle M_z(H_y) \rangle \) (blue curve). This is due to the order of magnitude of fields \( H_z \) and \( H_y \), which have the same order of magnitude of the sample penetration field \( H_p \). One can observe in graphics (A)-(C) that for the same number of cycles of the oscillating field \( H_y \), the magnetization reduction prevails, however, it is asymmetric. Specifically, the diamagnetic branch of the average magnetization changes sign, this behavior is the so-called paramagnetism. All theoretical results successfully agree with experiments for large (\( |H_y| \sim H_z \)) and small (\( |H_y| \ll H_z \)) values of the transverse field amplitude.

The asymmetrical suppression of average magnetization \( \langle M_z(H_y) \rangle \) — caused by an AC magnetic field for diamagnetic and paramagnetic initial states as well as the change of sign in its diamagnetic branch — is only possible to reproduce if the effects of flux-line cutting and anisotropy between the perpendicular and parallel components of \( J_c(B) \) are incorporated.

We can observe in Figure (4) the evolution of the magnetic induction profiles \( B_z(x) \) as \( H_y \) oscillates. They correspond to the paramagnetic and diamagnetic branches of the average magnetization \( \langle M_z \rangle \) (see Fig.(3). In graphics (A)-(C) we can see the reduction of magnetic

![Figure 4](image_url)

**Figure 4.** Theoretical magnetic induction curves \( B_z/\mu_0H_p \) vs \( x/d \) for a YBCO plate (in Fig. (3) specimen characteristics are specified). The red curves (blue) were calculated for an initial paramagnetic (diamagnetic) state. The magnitude of the static magnetic field \( H_z \) and the amplitude of the oscillating magnetic field \( |H_y| \) are shown. In all panels (A)-(D) it is initiated with a magnetic field \( H_y = H_{y,max} \), each profile is generated every half cycle of \( H_y \).
induction component $B_z(x)$ as the transverse field $H_y$ describes small-amplitude cycles ($|H_y| < H_p \ll H_z$). The ECSM predicts that $B_z(x)$ has a tendency towards a fixed distribution (see (A), (B)) or quasi-uniform (see (C)), regardless of the number of cycles. Close to the sample’s borders, the distribution $B_z(x)$ collapses to an almost homogeneous value ($B_z \sim B$) as $H_y$ oscillates. However, the graphic (D) shows that when the magnitude of $H_z$ and $H_y$ are comparable to $H_p$ ($|H_y| \sim H_z = H_p$), the slope of $B_z(x)$ changes (conserves) its sign close to the plate borders if the initial state is diamagnetic (paramagnetic). Moreover, $B_z$ collapse areas are not present in the region close to the plate border. Therefore, the average magnetization value $\langle M_z \rangle$, after several $H_y$ cycles becomes positive for the initial diamagnetic state.

In more recent experiments using the magneto-optic technique and Hall sensors, the effect of crossed magnetic fields in the perpendicular geometry was studied [34]. In this case, the sample is pre-magnetized in z-direction. Once the magnetic field along the z-axis is removed, an AC magnetic field, parallel to the sample’s $ab$ plane, is applied. Experimental results show that in this perpendicular geometry, given the magnitude of external magnetic fields $H_y$ and $H_z$, only the symmetrical collapse effect of the average magnetization $\langle M_z \rangle$ is present. This occurs regardless of the transverse field $H_y$ magnitude respect to the sample’s penetration field $\mu_0 H_p$. It was employed a finite-element numerical model and a power law function $E(J)$. They reproduced the experimental curves of $\langle M_z \rangle$ considering that the flux-line cutting effect is not present in the superconductor dynamics. Indeed, their results do not contradict those obtained with the ESCM, because we have reproduced symmetrical collapse of $\langle M_z \rangle$ in a type-II superconductor with the Bean multicomponent model, which does not consider flux-line cutting.

It has also studied results of magnetization collapse for dissipative states using both ECSM [1] and the extended ECSM, the latter proposed by Clem [32]. In the experiments, it was measured the remanent magnetization of a PbBi superconductor disk. First, the disk is magnetized with a single pulse of a $H_z$ field, and subsequently it is subjected to an oscillating magnetic field $H_y$. Even though the theoretical curves $\langle M_z \rangle$ vs $H_y$ obtained with each model are very similar, the profile evolution of $B_z$ obtained by using the extended ECSM contradicts the cases when the magnitude of fields $H_y$ and $H_z$ are comparable to $H_p$ [5].

4.2. Rotating type-II superconductors in presence of a magnetic field

Another instance in which flux-line cutting occurs is when a superconductor sample rotates or oscillates in presence of an external magnetic field. We will focus on experiments conducted by Cave and LeBlanc [24] where a low-$T_c$ type-II superconductor disk oscillates slowly in presence of a static magnetic field in parallel geometry. Cave and LeBlanc realized extensive research on the magnetic behavior of such sample, for a series of rotation angles, different initial states i.e. magnetic, non-magnetic, diamagnetic, paramagnetic and hybrid, and investigated the energy dissipation during oscillations.

Here we present theoretical calculations both for magnitude and direction of the magnetic induction $B$, as the sample rotates under different angles, also present the average component $\langle B_z \rangle$ and the hysteresis cycles. Results were obtained considering that the parallel and perpendicular components of current density possess a dependency on the magnetic induction of the form:
\[ I_{\perp}(B) = I_{\perp}(0) \left( 1 - \frac{B}{B_c^2} \right). \]  

(27)

We used the second critical field \( B_{c2} = 0.35T \) reported in Ref.[24], and the maximum value of critical current density \( I_{\perp}(0) = 1.6 \times 10^9 \text{A/m}^2 \). We found that, with the anisotropy parameter \( I_{\parallel}(0)/I_{\perp}(0) = 4 \), we obtain the best agreement with the experimental curves.

Figures (5) and (6) show the theoretical curves of \( \langle B_y \rangle \) and \( \langle M_z \rangle = \mu_0 H_a - \langle B_z \rangle \) for a non magnetic initial state and four rotation angles \( \theta_{\text{max}} = 45^\circ, 120^\circ, 270^\circ, 360^\circ \). All theoretical curves show the main characteristics of the corresponding experimental measurements.

**Figure 5.** Theoretical curves of \( \langle B_y \rangle \) vs \( \theta \) for a non magnetic initial state for an Nb plate with thickness \( d = 0.25 \text{mm} \), \( B_{c2} = 0.25T \), and \( I_{\perp}(0) = 1.6 \times 10^9 \text{A/m}^2 \). The relation \( I_{\parallel}(0)/I_{\perp}(0) = 4 \) was used. Curves are shown for different static external magnetic fields and rotation angles: \( \mu_0 H_a = 261 \text{mT} \) and \( \theta_{\text{max}} = 45^\circ \); \( \mu_0 H_a = 149 \text{mT} \) and \( \theta_{\text{max}} = 120^\circ \); \( \mu_0 H_a = 172 \text{mT} \) and \( \theta_{\text{max}} = 270^\circ \); \( \mu_0 H_a = 149 \text{mT} \) and \( \theta_{\text{max}} = 360^\circ \). The most outstanding phenomenon in this experiment is the magnetic flux consumption. This phenomenon is analyzed with the help of the spatial evolution distribution of \( B(x) \) and \( \alpha(x) \) predicted with the ESCM. For instance, Fig. 7 presents the results for an amplitude oscillation \( \theta_{\text{max}} = 45^\circ \) of the sample in presence of a static field \( \mu_0 H_a = 261 \text{mT} \). We can observe in the left panel that, as the sample rotates, the two U-shaped minima of the profile of \( B(x) \) move away from the surface of the sample, as well as the local magnetic flux reduction. The right panel shows how \( \alpha(x)(B) \) decreases from a maximum value at the sample’s surface to a zero value at the region \( 0.4 \lesssim x/d \lesssim 0.6 \). Therefore, the ESCM can predict the existence of three areas within the material: (1) an exterior area where both flux cutting and flux transport occur, \( \alpha \neq 0 \) y \( dB/dx \neq 0 \); (2) an internal area where only flux transport occurs
(α = 0 y dB/dx ≠ 0) and (3) a central area were neither transport flux nor flux cutting occur (α = dB/dx = 0).

Figure 6. Theoretical curves of $-\mu_0\langle M_z \rangle$ vs $\theta$ corresponding to the graphs presented in Fig.(5).

Figure 7. Theoretical curves for the evolution of $B(x)$ vs $x/d$ and $\theta(x)$ vs $x/d$ corresponding to $\mu_0 H_a = 261$ mT y $\theta_{\text{max}} = 45^\circ$. 
5. Conclusions and expectations

As J.R. Clem [32] emphasized, in contrast with other critical-state models, the Elliptic Flux-line Cutting Critical-state Model (ECSM) contains important new physical characteristics: it considers that the depinning threshold decreases as magnitude $J_\parallel$ increases and, in the same way, the flux cutting threshold decreases as magnitude $J_\perp$ increases. The model also reproduces the experimentally observed soft angular dependency of $J_c(B)$. In this chapter, we have shown that the ECSM satisfactorily reproduces the response of type-II superconductors subjected to external magnetic field varying in magnitude and direction — when the flux-line cutting effect is considered. Following this line of research, and based on recently results reported [32, 35, 36], the effects of the flux transport will be studied, as well as energy dissipation. This can be achieved incorporating an electric field with a well-defined direction. Our next study will consider the anisotropy in the current carrying capacity, since this characteristic is present in high-$T_c$ superconductors, and flux line cutting as well. It is also desirable to study more complex phenomena in high-$T_c$ superconductors, such as Meissner holes or the turbulent structures that occur in the perpendicular geometry when the external magnetic field is rotated. For this, it is essential to implement the ECSM for the case of finite geometries.

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