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Current-Carrying Wires and Special Relativity

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1. Introduction

This chapter introduces the main concepts of electrostatics and magnetostatics: charge and current, Coulomb's Law and the Biot-Savart Law, and electric and magnetic fields. Using linear charge distributions and currents makes it possible to do this without recourse to vector calculus. Special relativity is invoked to demonstrate that electricity and magnetism are, in a sense, two different ways of looking at the same phenomenon: in principle, from a knowledge of either electricity or magnetism and special relativity, the third theory could be derived. The three theories are shown to be mutually consistent in the case of linear currents and charge distributions.

This chapter brings together the results from a dozen or so treatments of the topic in an internally consistent manner. Certain points are emphasized that tend to be given less prominence in standard texts and articles. Where integration is used as a tool to deal with extended charge distributions, non-obvious antiderivatives are obtained from an online integrator; this is rarely encountered in textbooks, and gives the approach a more contemporary feel (admittedly, at the expense of elegance). This enables straightforward derivation of expressions for the electric and magnetic fields of radially symmetric charge and current distributions without using Gauss' or Ampère's Laws. It also allows calculation of the extent of "self-pinching" in a current-carrying wire; this appears to be a new result.

2. Electrostatics

2.1 Charge

When certain objects are rubbed together, they undergo a dramatic change. Whereas before these objects exerted no noticeable forces on their environment, they now do. For example, if you hold one of the objects near a small piece of paper, the piece of paper may jump up towards and attach itself to the object. Put this in perspective: *the entire Earth* is exerting a gravitational pull on the piece of paper, but a comparatively small object is able to exert a force big enough to overcome this pull (Arons, 1996).

If we take the standard example of rubber rods rubbed with cat fur, and glass rods rubbed with silk, we observe that all rubber rods repel each other as do all glass rods, while all rubber rods attract all glass rods. It turns out that all charged objects ever experimented on either behave like a rubber rod, or like a glass rod. This leads us to postulate that there only two types of charge state, which we call positive and negative charge for short.

As it turns out, there are also two types of charge: a positive charge as found on protons, and a negative charge as found on electrons. In this chapter, a wire will be modeled as a line of positively charged ions and negatively charged electrons; these two charge states come about through separation of one type of charge (due to electrons) from previously neutral atoms. However, the atoms themselves were electrically neutral due to equal amounts of the type of charge due to the protons in the nucleus, and the type of charge due to electrons.

Charged objects noticeably exert forces on each other when there is some distance between them. Since the 19th century, we have come to describe this behaviour in terms of electric fields. The idea is that one charged object generates a field that pervades the space around it; this field, in turn, acts on the second object.

2.2 Coulomb's Law

Late in the 18th century, Coulomb used a torsion balance to show that two small charged spheres exert a force on each other that is proportional to the inverse square of the distance between the centres of the spheres, and acts along the line joining the centres (Shamos, 1987a). He also showed that, as a consequence of this inverse square law, all charge on a conductor must reside on the surface. Moreover, by the shell theorem (Wikipedia, 2011) the forces between two perfectly spherical hollow shells are exactly as if all the charge were concentrated at the centre of each sphere. This situation is very closely approximated by two spherical insulators charged by friction, the deviation arising from a very small polarisation effect.

Coulomb also was the first person to quantify charge. For example, having completed one measurement, he halved the charge on a sphere by bringing it in contact with an identical sphere. When returning the sphere to the torsion balance, he measured that the force between the spheres had halved (Arons, 1996). When he repeated this procedure with the other sphere in the balance, the force between the spheres became one-quarter of its original value.

In modern notation, Coulomb thus found the law that bears his name: the electrostatic force $\vec{F}_E$ between two point-like objects a distance r apart, with charge Q and q respectively, is given by

$$\vec{F}_E = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r}. \tag{1}$$

In SI units, the constant of proportionality is given as $1/4\pi\epsilon_0$ for convenience in calculations. The constant ϵ_0 is called the permittivity of vacuum.

It is often useful to define the charge per unit length, called the linear charge density (symbol: λ); the charge per unit (surface) area, symbol: σ; and the charge per unit volume, symbol ρ.

We are now in a position to define the electric field $\vec{E}$ mathematically. The electric field is defined as the ratio of the force on an object and its charge. Hence, generally,

$$\vec{E} \equiv \frac{\vec{F}_E}{q}, \tag{2}$$

and for the field due to a point charge Q,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}. \tag{3}$$

Finally, experiments show that Coulomb's Law obeys the superposition principle; that is to say, the force exerted between two point-like charged objects is unaffected by the presence or absence of other point-like charged objects, and the net electrostatic force on a point-like object is found by adding all individual electrostatic forces acting on it. Of course, macroscopic objects generally are affected by other charges, for example through polarization.

2.3 An infinite line charge

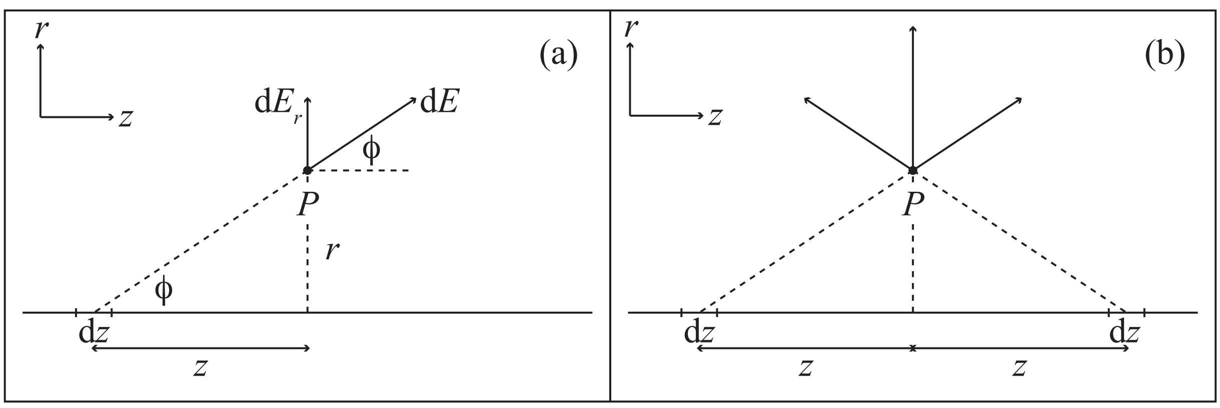

Fig. 1. Linear charges: (a) field due to a small segment of length dl, (b) net field due to two symmetrically placed segments.

Imagine an infinitely long line of uniform linear charge density λ. Take a segment of length dz, a horizontal distance z from point P which has a perpendicular distance r to the line charge. By Coulomb's Law, the magnitude of the electric field at P due this line segment is

$$\mathrm{d}E = \frac{\lambda \mathrm{d}z}{4\pi\epsilon_0(r^2+z^2)}. \tag{4}$$

A second segment of the same length dz a distance z from P (see Fig. 1b) gives rise to an electric field of the same magnitude, but pointing in a different direction. The z components cancel, leaving only the r component:

$$\mathrm{d}E_r = \frac{\lambda \mathrm{d}z \sin\phi}{4\pi\epsilon_0(r^2+z^2)}. \tag{5}$$

To find the net field at P, we add the contributions due to all line segments. This net field is thus an infinite sum, given by the integral

$$E = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}z \sin\phi}{r^2+z^2}. \tag{6}$$

The integral in (6) contains two variables, z and ϕ; we must eliminate either. It can be seen from Fig. 1a that

$$\sin\phi = \frac{\mathrm{d}E_r}{\mathrm{d}E} = \frac{r}{(r^2+z^2)^{1/2}}, \tag{7}$$

which allows us to eliminate ϕ, yielding

$$E = \frac{\lambda r}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{(r^2+z^2)^{3/2}}. \tag{8}$$

The antiderivative is readily found manually, by online integrator, or from tables; the integration yields

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{(r^2+z^2)^{3/2}} = \left.\frac{z}{r^2(r^2+z^2)^{1/2}}\right|_{-\infty}^{\infty} = \frac{2}{r^2}. \tag{9}$$

Hence, the electric field due to an infinity linear charge at a distance r from the line charge is given by

$$E = \frac{\lambda}{2\pi\epsilon_0 r}. \tag{10}$$

2.4 Electric field due to a uniformly charged hollow cylinder

Consider an infinitely long, infinitely thin hollow cylinder of radius R, with uniform surface charge density σ. A cross sectional view is given in Figure 2. What is the electric field at a point P, a distance y_0 from the centre of the cylinder axis? By analogy with the shell theorem,

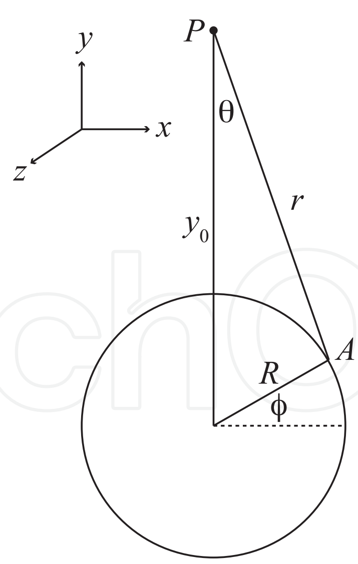

Fig. 2. Uniformly charged hollow cylinder of radius R, with auxiliary variables defined.

one might expect that the answer is the same as if all the charge were placed at the central axis. For an infinite cylinder, this turns out to be true. Think of the hollow cylinder as a collection of infinitely many parallel infinitely long line charges arranged in a circular pattern. If the angular width of each line charge is $\mathrm{d}\phi$, then each has linear charge density $\sigma R\mathrm{d}\phi$; by (10), each gives rise to an electric field of magnitude

$$\mathrm{d}E = \frac{\sigma R \mathrm{d}\phi}{2\pi\epsilon_0 r} \tag{11}$$

along the direction AP pointing away from the line charge, as shown in Figure 2.

The net field at any point P follows from superposition. We use a righthanded Cartesian coordinate system where the positive y-axis points up and the positive z-axis points out of the page. When comparing the contributions from the right half of the cylinder to the electric field with those from the left half, it is clear by symmetry that the y-components are equal and add, while the x-components are equal and subtract to yield zero. Hence

$$E = 2\int_{-\pi/2}^{\pi/2} \mathrm{d}E_y = \frac{\sigma R}{\pi\epsilon_0}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{r}\mathrm{d}\phi \tag{12}$$

The integrand in (12) contains 3 variables, r, ϕ, and θ. We may write r and $\cos\theta$ in terms of ϕ and constants:

$$\begin{cases} r = \sqrt{(R\cos\phi)^2 + (R\sin\phi - y_0)^2} = \sqrt{R^2 + y_0^2 - 2Ry_0\sin\phi} \\ \cos\theta = \frac{y_0 - R\sin\phi}{r} \end{cases}; \tag{13}$$

hence

$$E = \frac{\sigma R}{\pi\epsilon_0}\int_{-\pi/2}^{\pi/2} \frac{y_0 - R\sin\phi}{R^2 + y_0^2 - 2y_0 R\sin\phi}\mathrm{d}\phi. \tag{14}$$

When entering the integral into the Mathematica online integrator (2011), the antiderivative is given as

$$\left.\frac{-\arctan\left(\frac{R\cos x/2 - y_0\sin x/2}{y_0\cos x/2 - R\sin x/2}\right)}{2y_0} + \frac{\arctan\left(\frac{y_0\sin x/2 - R\cos x/2}{y_0\cos x/2 - R\sin x/2}\right)}{2y_0} + \frac{x}{2y_0}\right|_{-\pi/2}^{\pi/2},$$

which is admittedly ugly, but not difficult to use. Since arctan is an odd function, the first two terms are identical, and the antiderivative simplifies to

$$\left.\frac{1}{y_0}\left[\arctan\left(\frac{y_0\sin x/2 - R\cos x/2}{y_0\cos x/2 - R\sin x/2}\right) + \frac{x}{2}\right]\right|_{-\pi/2}^{\pi/2}.$$

Substitution eventually yields that the value of the integral is π/y_0. Hence Equation (14) gives for the electric field E outside the hollow cylinder:

$$E = \frac{\sigma R}{\pi\epsilon_0}\frac{\pi}{y_0}, \tag{15}$$

which, defining $\lambda = \sigma \cdot 2\pi R$, simplifies to

$$E = \frac{\lambda}{2\pi\epsilon_0 y_0}, \tag{16}$$

as expected.

2.5 Electric field due to a uniformly charged cylinder

It follows from (16) that for any cylindrical charge distribution of radius R that is a function of r only, i.e., $\rho = \rho(r)$, the electric field for $r > R$ is given by

$$E = \frac{\lambda}{2\pi\epsilon_0 r}, \tag{17}$$

where the linear charge density λ is equal to the volume charge density ρ integrated over the radial and polar coordinates.

3. Magnetic fields and current-carrying wires

3.1 Current

The flow of charge is called current. To be more precise, define a cross sectional area A through which a charge dQ flows in a time interval dt. The current I through this area is defined as

$$I \equiv \frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{18}$$

It is often convenient to define a current density J, which is the current per unit cross sectional area A:

$$J \equiv I/A. \tag{19}$$

A steady current flowing through a homogeneous wire can be modeled as a linear charge density λ moving at constant drift speed v_d. In that case, the total charge flowing through a cross sectional area in a time interval Δt is given by $\lambda v_d \Delta t$, and

$$I = \lambda v_d. \tag{20}$$

3.2 Magnetic field due to a linear current

In this chapter, we will only concern ourselves with magnetic effects due to straight current-carrying wires. Oersted found experimentally that a magnet (compass needle) gets deflected when placed near a current-carrying wire (Shamos, 1987b). As in electrostatics, we model this behaviour by invoking a field: the current in the wire creates a magnetic field B that acts on the magnet.

In subsequent decades, experiments showed that moving charged objects are affected by magnetic fields. The magnetostatic force (so called because the source of the magnetic field is steady; it is also often called the Lorentz force) is proportional to the charge q, the speed v, the field B, and the sine of the angle ϕ between v and B; it is also perpendicular to v and B. In vector notation,

$$\vec{F}_m = q\vec{v} \times \vec{B}; \tag{21}$$

in scalar notation,

$$F_m = qvB\sin\phi. \tag{22}$$

As a corollary, two parallel currents exert a magnetostatic force on each other, as the charges in each wire move in the magnetic field of the other wire.

Just as Coulomb was able to abstract from a charged sphere to a point charge, the effect of a current can be abstracted to a steady "point-current" of length $\mathrm{d}l$. (Note that a single moving point charge does not constitute a *steady* point-current.) In fact, there is a close analogy between the electric field due to a line of static charges and the magnetic field due to a line segment of moving charges – i.e., a steady linear current. The Biot-Savart law states that the magnetic field at a point P due to a steady point current is given by

$$\mathrm{d}B = \frac{\mu_0 I}{2\pi} \frac{\mathrm{d}z \sin\phi}{R^2}, \tag{23}$$

where μ_0 is a constant of proportionality called the permeability of vacuum, I is the current, $\mathrm{d}z$ is the length of an infinitesimal line segment, ϕ is the angle between the wire and the line connecting the segment to point P, the length of which is R; see Figure 3. Maxwell (1865) showed that μ_0 and ϵ_0 are related; their product is equal to $1/c^2$, where c is the speed of light in vacuum.

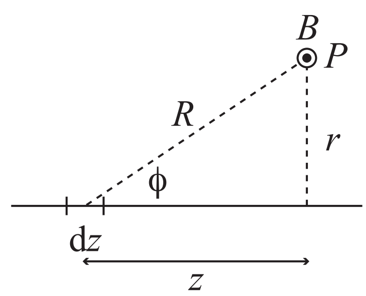

Fig. 3. The Biot-Savart law: magnetic field due to a small segment carrying a current I. The direction of the magnetic field is out of the page.

The magnetostatic force at point P due to an infinitely long straight current-carrying wire is then

$$B = \frac{\mu_0 I}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}z \sin\phi}{r^2 + z^2}, \tag{24}$$

which has the exact same form as (6).

Because the current distribution must have radial symmetry, all conclusions reached from (6) can be applied here. Thus, the magnetic field due to a steady current I in an infinitely long wire, hollow cylinder, or solid cylinder where the current density only depends on the distance from the centre of the wire, varies with the distance r as

$$B = \frac{\mu_0 I}{2\pi r} \tag{25}$$

outside the wire.

4. Special relativity

4.1 Relativity in Newtonian mechanics

Newton's laws of motion were long assumed to be valid for all inertial reference frames. In Newton's model, an observer in one reference frame measures the position x of an object at various times t. An observer in a second reference frame moves with speed v relative to the first frame, with identical, synchronized clocks and metre sticks. Time intervals and lengths are assumed to be same for both observers.

The second observer sees the first observer move away at speed v. The distance between the two observers at a time t' is given by vt'. Hence, the second observer can use the measurements of the first observer, provided the following changes are made:

$$x' = x - vt \tag{26}$$

$$t' = t \tag{27}$$

Equations (26) and (27) are known as a Galilean transformation. It is easy to see that if Newton's second law holds for one observer, it automatically holds for the other. For an object moving at speed u we find that

$$u' \equiv \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x'}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - v = u - v, \tag{28}$$

so we get

$$a' = \frac{\mathrm{d}^2x'}{\mathrm{d}t'^2} = \frac{\mathrm{d}^2x'}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = a. \tag{29}$$

Hence, in both reference frames, the accelerations are the same, and hence the forces are the same, too.

4.2 The wave equation in two inertial reference frames

A problem occurs when we consider light waves. The transformation (28) implies that, in a rest frame travelling at the speed of light c with respect to an emitter, light would be at rest – it is not clear how that could be.

To put this problem on a firmer mathematical footing, we derive the general linear transformation of the wave equation; we then substitute in the Galilean transformation. For an electromagnetic wave, the electric field E satisfies, in one reference frame,

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0. \tag{30}$$

We can express the derivative with respect to x in terms of variables used in another reference frame, x' and t', by using the chain rule:

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x}. \tag{31}$$

The second derivative contains five terms:

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial x'^2}\left(\frac{\partial x'}{\partial x}\right)^2 + 2\frac{\partial^2 E}{\partial x'\partial t'}\frac{\partial x'}{\partial x}\frac{\partial t'}{\partial x} + \frac{\partial^2 x'}{\partial x^2}\frac{\partial E}{\partial x'} + \frac{\partial^2 E}{\partial t'^2}\left(\frac{\partial t'}{\partial x}\right)^2 + \frac{\partial^2 t'}{\partial x^2}\frac{\partial E}{\partial t'}. \tag{32}$$

For linear transformations, the third and fifth terms are zero. Hence we obtain:

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial x'^2}\left(\frac{\partial x'}{\partial x}\right)^2 + 2\frac{\partial^2 E}{\partial x'\partial t'}\frac{\partial x'}{\partial x}\frac{\partial t'}{\partial x} + \frac{\partial^2 E}{\partial t'^2}\left(\frac{\partial t'}{\partial x}\right)^2. \tag{33}$$

The second derivative with respect to time is, likewise:

$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x'^2}\left(\frac{\partial x'}{\partial t}\right)^2 + 2\frac{\partial^2 E}{\partial x'\partial t'}\frac{\partial x'}{\partial t}\frac{\partial t'}{\partial t} + \frac{\partial^2 E}{\partial t'^2}\left(\frac{\partial t'}{\partial t}\right)^2. \tag{34}$$

Substituting all this back into the wave equation, and grouping judiciously, we obtain

$$\frac{\partial^2 E}{\partial x'^2}\left[\left(\frac{\partial x'}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial x'}{\partial t}\right)^2\right] - \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2}\left[\left(\frac{\partial t'}{\partial t}\right)^2 - c^2\left(\frac{\partial t'}{\partial x}\right)^2\right] = 2\frac{\partial^2 E}{\partial x'\partial t'}\left[\frac{1}{c^2}\frac{\partial x'}{\partial t}\frac{\partial t'}{\partial t} - \frac{\partial x'}{\partial x}\frac{\partial t'}{\partial x}\right]. \tag{35}$$

To retain the wave equation (30), it is clear that the right-hand side of this equation must be zero while the terms in square brackets on the left-hand side must be equal. This is not true for the Galilean transformation, since we obtain:

$$\frac{\partial^2 E}{\partial x'^2}\left(1 - \frac{v^2}{c^2}\right) - \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2} = -\frac{2v}{c^2}\frac{\partial^2 E}{\partial x'\partial t'}. \tag{36}$$

4.3 Principles of special relativity

Einstein's theory of special relativity resolved the problem. In special relativity, velocities measured in two different reference frames can no longer be added as Newton did, because one observer disagrees with the time intervals and lengths measured by the other observer. As a result, the wave equation has the same form to all inertial observers, with the same value for the speed of light, c. Newton's laws of motion are modified in such a way that in all situations they were originally developed for (e.g., uncharged objects moving at speeds much smaller than the speed of light), the differences are so small as to be practically immeasurable. However, when we look at currents it turns out that these very small differences do matter in everyday situations.

In special relativity, all inertial frames are equivalent – meaning that all laws of physics are the same, as they are in Galilean relativity. However, rather than postulating that time and space are the same ("invariant") for all inertial observers, it is postulated that the speed of light c is invariant: it is measured to be the same in all reference frames by all inertial observers. As a consequence, measurements of time and space made in one reference frame that is moving with respect to another are different – even though the measurements may be made in the exact same way as seen from within each reference system. Seen from one reference system, a clock travelling at constant speed appears to be ticking more slowly, and appears contracted in the direction of motion. Also, if there is more than one clock at different locations, the clocks can only be synchronized according to one observer, but not simultaneously to another observer in a different reference frame.

These ideas can be investigated with an imaginary device – a light clock. Because both observers agree that light travels at speed c in both reference frames, this allows us to compare measurements in the two reference frames. Both observers agree that their own light clock consists of two mirrors mounted on a ruler a distance l_0 apart, and that it takes a light pulse a time interval Δt_0 for a round trip. Both observers agree that l_0 and Δt_0 are related by

$$l_0 = c \cdot \frac{1}{2} \Delta t_0, \tag{37}$$

and both agree that this is true irrespective of the orientation of the light clock.

However, when comparing each other's measurements, the observers are in for some surprises. As motion in one direction is independent from motion in an orthogonal direction, it makes sense to distinguish between lengths parallel and perpendicular to the relative motion of the two reference frames. A very useful sequence of looking at the light clock was given by Mermin (1989):

1. length perpendicular to motion
2. time intervals
3. length parallel to motion
4. synchronization of clocks

4.4 Lengths perpendicular to motion are unaffected

In the first thought experiments, each observer has a light clock. They are parallel to each other, and perpendicular to their relative motion (see Figure 4a). We can imagine that a piece of chalk is attached to each end of each clock, so that when the two clocks overlap, each makes a mark on the other.

We arrive at a result by reductio ad absurdum. Say that observer 1 sees clock 2 contract (but his own does not, of course – the observed contraction would be due purely to relative motion). Both observers would agree on the marks made by the pieces of chalk on clock 2 – they are inside the ends of clock 1. They would also both agree that the ends of clock 1 do not mark clock 2. Special relativity demands that the laws of physics are the same for both observers: so observer 2 must see clock 1 shrink by the same factor, clock 2 retains the same length; and hence the chalk marks on clock 2 are inside the ends, while there are no marks on clock 1. Thus, we arrive at a contradiction. Assuming one observer sees the other's clocks expand lead to the same conundrum. The only possible conclusion: both observers agree that both clocks have length l_0 in both frames.

4.5 Time intervals: moving clocks run more slowly

In the same set-up, observer 1 sees the light pulse in his clock move vertically, while the light pulse in clock 2 moves diagonally (see Figure 4a). Observer 1 uses his measurements only, plus the information that clock 2 has length l_0 and that the light pulse of clock 2 moves at speed c, also as measured by observer 1. Observer 1 measures that a pulse in clock 2 goes from the bottom mirror to the top and back again in an interval Δt, which must be greater than Δt_0, as the light bouncing between the mirrors travels further at the same speed. Thus, as seen by observer 1, clock 2 takes longer to complete a tick, and runs slow; clock 1 has already started a second cycle when clock 2 completes its first.

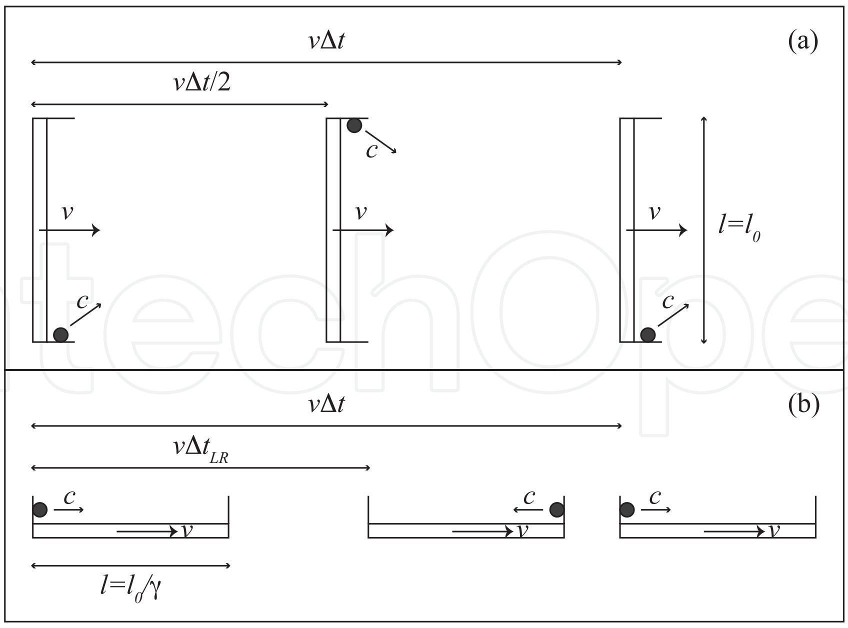

Fig. 4. A light clock in frame 2 as measured by observer 1, when the light clock is (a) perpendicular and (b) parallel to the relative motion of the two frames at speed v. The dot indicates a photon travelling at speed c in the direction indicated by the arrow. Other variables are defined in the text.

This reasoning can be quantified using Pythagoras' Theorem. Observer 1 sees that

$$\left(c \cdot \frac{1}{2}\Delta t\right)^2 - \left(v \cdot \frac{1}{2}\Delta t\right)^2 = l_0^2. \tag{38}$$

The time interval Δt can be related to the time on clock 1, Δt, because $l_0 = c \cdot \frac{1}{2}\Delta t_0$; hence

$$\Delta t^2 \left(1 - \frac{v^2}{c^2}\right) = \Delta t_0^2. \tag{39}$$

Now, defining

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{40}$$

substituting, taking the square root and dividing by γ, we conclude that

$$\Delta t = \gamma \Delta t_0. \tag{41}$$

Since all processes in frame 2 are in sync with clock 2, observer 1 sees all processes in frame 2 run slower than those in frame 1 by a factor γ. Conversely, to observer 2, everything is normal in frame 2; but observer 2 sees all processes in frame 1 run slow by the same factor γ.

4.6 Lengths parallel to relative motion are contracted

Now both observers turn their clocks through 90 degrees, so the light travels parallel to their relative motion, as shown in Figure 4b. Within their own reference frames, the clocks still run at the same rate; hence each observer sees the other's clock run slow by a factor γ as before.

However, to observer 1, after the light pulse leaves the left mirror of clock 2, that whole clock travels to the right. The pulse thus travels by a distance $l + v \cdot \Delta t_{LR}$ at speed c during a time interval Δt_{LR} before it hits the right mirror, where l is the length of the clock as measured by observer 1. Hence

$$l + v \cdot \Delta t_{LR} = c \cdot \Delta t_{LR}. \tag{42}$$

Similarly, after the pulse reflects it travels a distance $l - v \cdot \Delta t_{RL}$ before it hits the left mirror again. Observer 1 finds for the total time Δt

$$\Delta t = \Delta t_{LR} + \Delta t_{RL} = \frac{l}{c - v} + \frac{l}{c + v} = \frac{2l}{c} \cdot \frac{1}{1 - \frac{v^2}{c^2}} = \gamma^2 \frac{2l}{c}. \tag{43}$$

The time interval Δt can be linked to the time interval in frame 1, Δt_0, by (41), which, in turn, is linked to the length in frame 1, Δl_0, by (37). Straightforward substitution yields

$$l = \frac{l_0}{\gamma}. \tag{44}$$

Thus, as measured by observer 1, all lengths in frame 2 parallel to the motion are shorter than in frame 1 by a factor γ (but both perpendicular lengths are the same). As seen by observer 2, everything is normal in frame 2, but all parallel lengths in frame 1 are contracted by the same factor γ. When the two observers investigate each other's metre sticks, they both agree on how many atoms there are in the each stick, but disagree on the spacing between them.

4.7 Synchronization of clocks is only possible in one frame at a time

As it stands, it is hard to see how the observations in both frames can be reconciled. How can *both* observers see the other clocks run slowly, and the other's lengths contracted? The answer lies in synchronization. Without going into much detail, we outline some key points here.

Measuring the length of an object requires, in principle, the determination of two locations (the ends of the object) at the same time. However, when two clocks are synchronized in frame 1 according to observer 1, they are not according to observer 2. As the frames move with respect to each other, observer 2 concludes that observer 1 moved his ruler while determining the position of each end of the object. In the end, each observer can explain all measurements in a consistent fashion. For an accessible yet rigorous in-depth discussion see Mermin (1989). The end result is the transformation laws

$$x' = \gamma(x - vt) \tag{45}$$

$$t' = \gamma(t - vx/c^2) \tag{46}$$

4.8 Transformation of forces and invariance of the wave equation in special relativity

Substituting the transformations of special relativity into the wave equation (35) shows that the wave equation has the same form in both frames: the two factors in square brackets are equal to 1, and the right hand side is equal to zero.

However, Newton's Second Law does not transform in special relativity. In the situations under discussion in Section 5.1, all forces are perpendicular to the relative speed v. In that case, a force of magnitude F_0 in the rest frame is measured by an observer in a moving frame to have magnitude F given by

$$F = F_0/\gamma. \tag{47}$$

An operational definition for a transverse force is given by Martins (1982). For the sake of completeness we note that a parallel force transforms as $F = F_0/\gamma^3$.

5. Electric fields, magnetic fields, and special relativity

The considerations of the three previous sections can be brought together quite neatly. We model a current-carrying wire as a rigid lattice of ions, and a fluid of electrons that are free to move through the lattice. In the reference frame of the ions, then, the electrons move with a certain drift speed, v_d. But, by the same token, in the frame of the electrons, the ions move with a speed v_d.

We will consider four cases:

1. An infinitely thin current-carrying wire;
2. A current-carrying wire of finite width;
3. An charged object moving parallel to a current-carrying wire at speed v_d;
4. Two parallel current-carrying wires.

5.1 Length contraction in a current carrying wire

Experimental evidence shows that a stationary charge is not affected by the presence of a current-carrying wire. This absence of a net electrostatic force implies that the ion and electron charge densities in a current-carrying wire must have the same magnitude. This statement is more problematic than it may seem at first glance.

Consider the case of zero current. Call the linear charge density of the ions λ_0. By charge neutrality, the linear charge density of the electrons must be equal to $-\lambda_0$. Now let the electrons move at drift speed v_d relative to the ions, causing a current I. Experimentally, both linear charge densities remain unchanged, since a stationary charged object placed near the wire does not experience a net force. So, as seen in the ion frame, the linear electron charge density is given by:

$$\lambda_- = -\lambda_0. \tag{48}$$

In the electron frame, the linear charge density of the electrons must be

$$\lambda'_- = -\lambda_0/\gamma, \tag{49}$$

so that

$$\lambda_- = \gamma\lambda'_- = \gamma \cdot (-\lambda_0/\gamma) = -\lambda_0. \tag{50}$$

Moreover, in the electron frame, the ions are moving, and hence their linear charge density is

$$\lambda'_+ = \gamma\lambda_0. \tag{51}$$

The net charge density in the electron frame, λ', is then given by

$$\lambda' = \lambda'_+ + \lambda'_- = \gamma\lambda_0 - \lambda_0/\gamma = \lambda_0\gamma\left(1 - 1/\gamma^2\right) = \lambda_0\gamma v_d^2/c^2. \tag{52}$$

Thus, in the electron frame, the wire is charged. We cannot, however, simply assume that Coulomb's Law (6) holds; that law was obtained from experiments on stationary charges, while the ions are moving in the electron frame. In fact, the magnitude of the electric field dE due to a point charge λdz moving at speed v_d is given by (French, 1968; Purcell, 1984)

$$\mathrm{d}E = \frac{\lambda \mathrm{d}z\left(1 - v_d^2/c^2\right)}{4\pi\epsilon_0(r^2+z^2)\left(1 - \frac{v_d^2}{c^2}\frac{r^2}{r^2+z^2}\right)}, \tag{53}$$

using the notation of Figure 1a. However, when we integrate the radial component of this electric field, we *do* obtain the same result; switching to primed coordinates to denote the electron frame,

$$E' = \frac{\lambda'}{2\pi\epsilon_0 r'} = \frac{\lambda_0\gamma v_d^2}{2\pi\epsilon_0 r}. \tag{54}$$

We have used the fact that lengths perpendicular to motion do not contract; hence $r = r'$.

5.2 Current and charge distribution within a wire

Now consider a wire of finite radius, R. We can model this as an infinite number of parallel infinitely thin wires placed in a circle. Assume that each wire starts out as discussed above.

As seen in the ion frame, there are many electron currents in the same direction; each current will set up a magnetostatic field, the net effect of which will be an attraction towards the centre. However, once the electrons start to migrate towards the centre, a net negative charge is created in the centre of the wire; equilibrium is established when the two cancel (Gabuzda, 1993; Matzek & Russell, 1968).

As seen in the electron frame, there is a current of positive ions, but since the ion frame is assumed perfectly rigid, no redistribution of charge occurs as a result. However, due to length contraction there is also a net positive charge density, which will attract the electrons towards the centre of the wire (Gabuzda, 1993). This must occur in such a way that the net electric field is zero; this, in turn, can only happen if the net volume charge density is zero. Consequently, the linear electron density is distorted: within a radius a, a uniform electron volume charge density is established that is equal to the ion volume charge density; between a and R, the electron density is zero.

The magnitude of this effect can be calculated easily. The uniform ion volume charge density is given by

$$\rho'_+ = \frac{\lambda'_+}{\pi R^2} = \frac{\gamma\lambda_0}{\pi R^2}; \tag{55}$$

this must be equal to (minus) the uniform electron volume density over a radius a. Hence we obtain

$$\rho'_- = \frac{\lambda'_-}{\pi a^2} = -\frac{\lambda_0}{\gamma\pi a^2}. \tag{56}$$

Combining the two yields

$$a = R/\gamma. \tag{57}$$

Thus, the wire is electrically neutral between 0 and R/γ, and has a positive volume charge density given by (55) between R/γ and R. Because in practice the outer shell is very thin, it can be approximated as a surface density:

$$\sigma' = \rho'_+ \cdot 2\pi R = 2\gamma\lambda_0/R. \tag{58}$$

As we have seen, lengths perpendicular to the motion do not change. Hence in the ion frame the electrons comprise a uniform line of electrons moving at speed v_d; in other words, there is a "self-pinched" uniform current density given by

$$J = \frac{I}{\pi a^2} = \frac{\gamma^2 I}{\pi R^2} \tag{59}$$

between 0 and R/γ, and zero current density between R/γ and R.

Figure 5 shows some relevant current and volume charge densities in both reference frames. Note that, by the considerations of Section 2.4, for $r > R$ we may treat the wire as if all current and charge were located on the central axis of the wire.

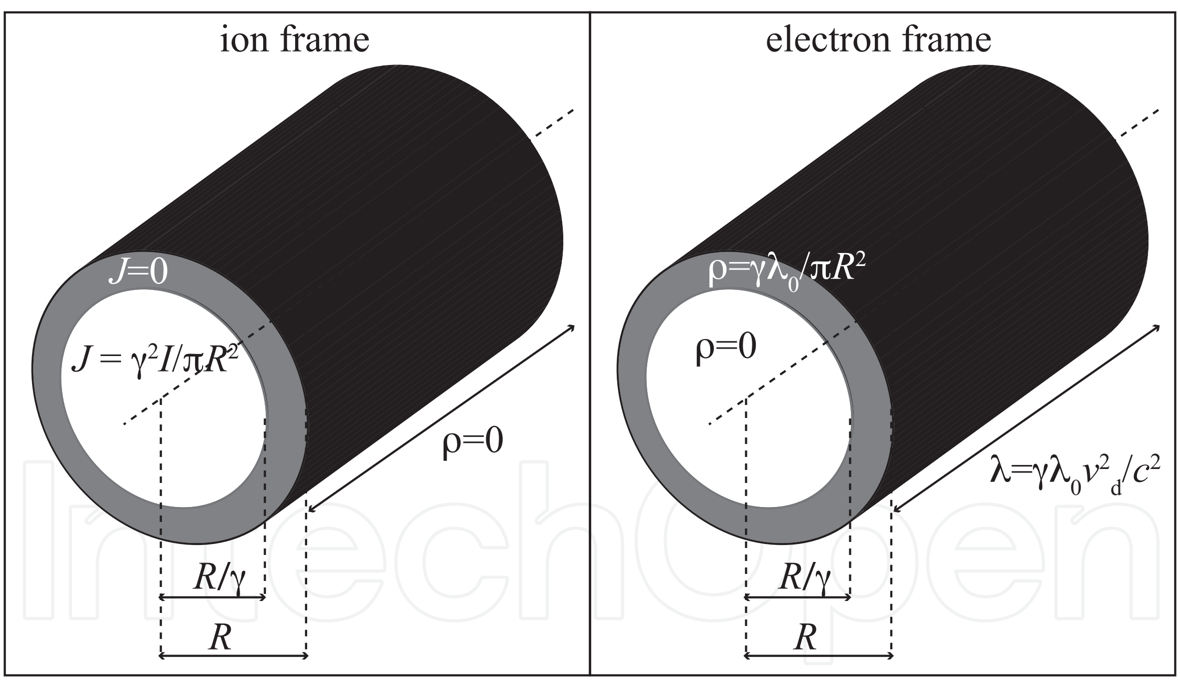

Fig. 5. Current and charge densities in a current-carrying wire. $\gamma = 1/\sqrt{1 - v_d^2/c^2}$, where v_d is the relative speed of the ions and electrons, and λ_0 is the linear ion density as seen in the ion frame.

5.3 A charged object near a current carrying wire

We have established that in the ion frame, a current-carrying wire does not exert an electrostatic force; but in the electron frame, it does. There is nothing wrong with this, but we must make sure that the effect on charges near the wire is the same in both frames; otherwise, the principle of relativity would be violated.

First, consider a point-like object of charge q stationary in the ion frame. There is no electrostatic force on the object, since there is no net charge; nor is there a magnetostatic force, because the speed of the object is zero. In the electron frame, the electrostatic force F'_e is given by

$$F'_e = qE' = \frac{q\lambda_0 \gamma v_d^2}{2\pi\epsilon_0 c^2 r'}. \tag{60}$$

There is also a magnetostatic force, F'_m, as in the electron frame the object is moving with speed v_d parallel to a current of positive ions; hence

$$F'_m = qv_d B' = \frac{qv_d \mu_0 I}{2\pi r'}. \tag{61}$$

The two forces are readily shown to be equal, as $I = \lambda'_+ v_d = \gamma\lambda_0 v_d$ and $\mu_0 = 1/\epsilon_0 c^2$. The forces cancel, because a current of positive ions attracts a positively charged object while the positive charge density repels it.

As a second case, a point-like object of charge q, moving parallel to a current carrying wire at speed v_d in the ion frame, experiences a purely magnetostatic force due to the electrons in the wire, given by:

$$F = qv_d B = \frac{qv_d \mu_0 I}{2\pi r} = \frac{qv_d^2 \lambda_0}{2\pi\epsilon_0 c^2 r}, \tag{62}$$

since $I = \lambda_- v_d = \lambda_0 v_d$. In the electron frame, the speed of q is zero so the force is purely electrostatic:

$$F' = qE' = \frac{q\lambda'}{2\pi\epsilon_0 r} = \frac{\gamma q v_d^2 \lambda_0}{2\pi\epsilon_0 c^2 r}. \tag{63}$$

The electron frame is the rest frame for q; hence (47) becomes

$$F = F'/\gamma, \tag{64}$$

which is obviously satisfied. Hence, what appears as a purely magnetostatic force in the ion frame appears as a purely electrostatic force in the electron frame.

In the general case, where a point-like object of charge q is moving at any speed v (say, in the ion frame), the ions and electrons are contracted by different factors, but always in such a way that the resulting net electrostatic force is balanced by the net magnetostatic force between the charge q and both on and electron currents. This case is discussed in detail by Gabuzda (1987).

5.4 Two parallel wires

As a final case, consider two parallel wires, each carrying a current I. When considering the effect of wire 1 on wire 2, we must consider the electrons and ions in wire 2 separately, as they have no common rest frame (Redžić, 2010). We cannot reify one segment of length l in one frame and transform it as a whole, even though coincidentally the same formulae can be obtained (van Kampen, 2008; 2010).

One convenient way of looking at the problem is by considering a segment of wire 2 of length l, as measured in the ion frame. This consists of a segment of ions of length l/γ as seen in the electron frame, and a segment of electrons of length γl as seen in the electron frame. The transformation leaves the total charge unaltered: it is $\lambda_0 l$.

In the ion frame, we can find the total force on the ion segment by dividing it up into point-like segments of charge density λ_0, and integrating over the entire length l. An identical procedure holds for the electrons. In the electron frame, we integrate ion segments of charge density $\gamma\lambda_0$ over a length l/γ, and electron segments of charge density λ_0/γ over a length γl. The net result is that all expressions found in the previous paragraph hold, if we replace q with $\lambda_0 l$:

$$F_{e+} = F_{m+} = F_{e-} = F'_{m-} = 0 \tag{65}$$

$$F'_{e+} = F'_{e-} = F'_{m+} = \frac{\gamma\lambda_0^2 v_d^2 l}{2\pi\epsilon_0 c^2 r} \tag{66}$$

$$F_{m-} = \frac{\lambda_0^2 v_d^2 l}{2\pi\epsilon_0 c^2 r} \tag{67}$$

6. Conclusion

In this chapter, we have outlined how electrostatics, magnetostatics and special relativity give consistent results for a few cases involving infinitely long current carrying wires. We have used an online integrator to obtain expression for the electrostatic field due to a hollow uniformly charged cylinder, and derived expressions for a solid uniformly charged cylinder and current-carrying wires from it. We have also derived expressions for self-pinching in a current-carrying wire, by a factor γ, and the creation of a surface charge density.

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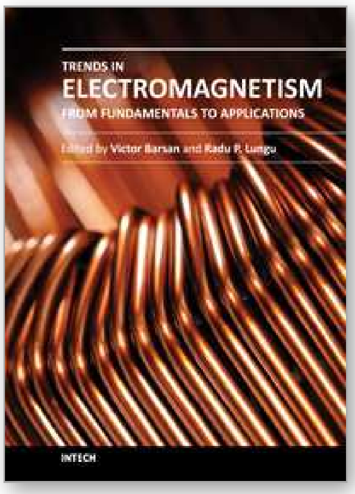

Trends in Electromagnetism - From Fundamentals to Applications

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