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## On the Fluid Queue Driven by an Ergodic Birth and Death Process

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### 1. Introduction

Fluid models are powerful tools for evaluating the performance of packet telecommunication networks. By masking the complexity of discrete packet based systems, fluid models are in general easier to analyze and yield simple dimensioning formulas. Among fluid queuing systems, those with arrival rates modulated by Markov chains are very efficient to capture the burst structure of packet arrivals, notably in the Internet because of bulk data transfers. By exploiting the Markov property, very efficient numerical algorithms can be designed to estimate performance metrics such as the overflow probability, the delay of a fluid particle or the duration of a busy period.

In the last decade, stochastic fluid models and in particular Markov driven fluid queues, have received a lot of attention in various contexts of system modeling, e.g. manufacturing systems (see Aggarwal et al. (2005)), communication systems (in particular TCP modeling; see vanForeest et al. (2002)) or more recently peer to peer file sharing process (see Kumar et al. (2007)) and economic systems (risk analysis; see Badescu et al. (2005)). Many techniques exist to analyze such systems.

The first studies of such queuing systems can be dated back to the works by Kosten (1984) and Anick et al. (1982), who analyzed fluid models in connection with statistical multiplexing of several identical exponential on-off input sources in a buffer. The above studies mainly focused on the analysis of the stationary regime and have given rise to a series of theoretical developments. For instance, Mitra (1987) and Mitra (1988) generalize this model by considering multiple types of exponential on-off inputs and outputs. Stern & Elwalid (1991) consider such models for separable Markov modulated rate processes which lead to a solution of the equilibrium equations expressed as a sum of terms in Kronecker product form. Igel'nik et al. (1995) derive a new approach, based on the use of interpolating polynomials, for the computation of the buffer overflow probability.

Using the Wiener-Hopf factorization of finite Markov chains, Rogers (1994) shows that the distribution of the buffer level has a matrix exponential form, and Rogers & Shi (1994) explore algorithmic issues of that factorization. Ramaswami (1999) and da Silva Soares & Latouche (2002), Ahn & Ramaswami (2003) and da Silva Soares & Latouche (2006) respectively exhibit

and exploit the similarity between stationary fluid queues in a finite Markovian environment and quasi birth and death processes.

Following the work by Sericola (1998) and that by Nabli & Sericola (1996), Nabli (2004) obtained an algorithm to compute the stationary distribution of a fluid queue driven by a finite Markov chain. Most of the above cited studies have been carried out for finite modulating Markov chains.

The analysis of a fluid queue driven by infinite state space Markov chains has also been addressed in many research papers. For instance, when the driving process is the M/M/1 queue, Virtamo & Norros (1994) solve the associated infinite differential system by studying the continuous spectrum of a key matrix. Adan & Resing (1996) consider the background process as an alternating renewal process, corresponding to the successive idle and busy periods of the M/M/1 queue. By renewal theory arguments, the fluid level distribution is given in terms of integral of Bessel functions. They also obtain the expression of Virtamo and Norros via an integral representation of Bessel functions. Barbot & Sericola (2002) obtain an analytic expression for the joint stationary distribution of the buffer level and the state of the M/M/1 queue. This expression is obtained by writing down the solution in terms of a matrix exponential and then by using generating functions that are explicitly inverted.

In Sericola & Tuffin (1999), the authors consider a fluid queue driven by a general Markovian queue with the hypothesis that only one state has a negative drift. By using the differential system, the fluid level distribution is obtained in terms of a series, which coefficients are computed by means of recurrence relations. This study is extended to the finite buffer case in Sericola (2001). More recently, Guillemin & Sericola (2007) considered a more general case of infinite state space Markov process that drives the fluid queue under some general uniformization hypothesis.

The Markov chain describing the number of customers in the M/M/1 queue is a specific birth and death process. Queueing systems with more general modulating infinite Markov chain have been studied by several authors. For instance, van Dorn & Scheinhardt (1997) studied a fluid queue fed by an infinite general birth and death process using spectral theory.

Besides the study of the stationary regime of fluid queues driven by finite or infinite Markov chains, the transient analysis of such queues has been studied by using Laplace transforms by Kobayashi & Ren (1992) and Ren & Kobayashi (1995) for exponential on-off sources. These studies have been extended to the Markov modulated input rate model by Tanaka et al. (1995). Sericola (1998) has obtained a transient solution based on simple recurrence relations, which are particularly interesting for their numerical properties. More recently, Ahn & Ramaswami (2004) use an approach based on an approximation of the fluid model by the amounts of work in a sequence of Markov modulated queues of the quasi birth and death type. When the driving Markov chain has an infinite state space, the transient analysis is more complicated. Sericola et al. (2005) consider the case of the M/M/1 queue by using recurrence relations and Laplace transforms.

In this paper, we analyze the transient behavior of a fluid queue driven by a general ergodic birth and death process using spectral theory in the Laplace transform domain. These results are applied to the stationary regime and to the busy period analysis of that fluid queue.

## 2. Model description

### 2.1 Notation and fundamental system

Throughout this paper, we consider a queue fed by a fluid traffic source, whose instantaneous transmitting bit rate is modulated by a general birth and death process  $(\Lambda_t)$  taking values in  $\mathbb{N} = \{0, 1, 2, \dots\}$ . The input rate is precisely  $r(\Lambda_t)$ , where  $r$  is a given increasing function from  $\mathbb{N}$  into  $\mathbb{R}$ .

The birth and death process  $(\Lambda_t)$  is characterized by the infinitesimal generator given by the infinite matrix

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 \dots \\ \cdot & \cdot & \cdot & \dots \end{pmatrix}, \quad (1)$$

where  $\lambda_i > 0$  for  $i \geq 0$  is the transition rate from state  $i$  to state  $i + 1$  and  $\mu_j > 0$  for  $j \geq 1$  is the transition rate from state  $j$  to state  $j - 1$ .

We assume that the birth and death process  $(\Lambda_t)$  is ergodic, which amounts to assuming (see Asmussen (1987) for instance) that

$$\sum_{i=0}^{\infty} \frac{1}{\lambda_i \pi_i} = \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \pi_i < \infty, \quad (2)$$

where the quantities  $\pi_i$  are defined by:

$$\pi_0 = 1 \quad \text{and} \quad \pi_i = \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i}, \quad \text{for } i \geq 1.$$

Under the above assumption, the birth and death process  $(\Lambda_t)$  has a unique invariant probability measure: in steady state, the probability of being in state  $i$  is

$$p(i) = \frac{\pi_i}{\sum_{j=0}^{\infty} \pi_j}.$$

Let  $p_0(i)$  denote, for  $i \geq 0$ , the probability that the birth and death process  $(\Lambda_t)$  is in state  $i$  at time 0, i.e.,  $\mathbb{P}(\Lambda_0 = i) = p_0(i)$ . Note that if  $p_0(i) = p(i)$  for all  $i \geq 0$ , then  $\mathbb{P}(\Lambda_t = i) = p(i)$  for all  $t \geq 0$  and  $i \geq 0$ .

We assume that the queue under consideration is drained at constant rate  $c > 0$ . Furthermore, we assume that  $r(i) > c$  when  $i$  is greater than a fixed  $i_0 > 0$  and that  $r(i) < c$  for  $0 \leq i \leq i_0$ . (It is worth noting that we assume that  $r(i) \neq c$  for all  $i \geq 0$  in order to exclude states with no drift and thus to avoid cumbersome special cases.) In addition, the parameters  $c$  and  $r(i)$  are such that

$$\rho = \sum_{i=0}^{\infty} \frac{r(i)}{c} p(i) < 1 \quad (3)$$

so that the system is stable. The quantity  $r_i = r(i) - c$  is either positive or negative and is the net input rate when the modulating process  $(\Lambda_t)$  is in state  $i$ .

Let  $X_t$  denote the buffer content at time  $t$ . The process  $(X_t)$  satisfies the following evolution equation: for  $t \geq 0$ ,

$$\frac{dX_t}{dt} = \begin{cases} r(\Lambda_t) - c & \text{if } X_t > 0 \text{ or } r(\Lambda_t) > c, \\ 0 & \text{if } X_t = 0 \text{ and } r(\Lambda_t) \leq c. \end{cases} \quad (4)$$

Let  $f_i(t, x)$  denote the joint probability density function defined by

$$f_i(t, x) = \frac{\partial}{\partial x} \mathbb{P}(\Lambda_t = i, X_t \leq x).$$

As shown in Sericola (1998), on top of its usual jump at point  $x = 0$ , when  $X_0 = x_0 \geq 0$ , the distribution function  $\mathbb{P}(\Lambda_t = i, X_t \leq x)$  has a jump at points  $x = x_0 + r_i t$ , for  $t$  such that  $x_0 + r_i t > 0$ , which corresponds to the case when the Markov chain  $\{\Lambda_t\}$  starts and remains during the whole interval  $[0, t)$  in state  $i$ .

We focus in the rest of the paper on the probability density function  $f_i(t, x)$  for  $x > 0$  along with its usual jump at point  $x = 0$ . A direct consequence of the evolution equation (4) is the forward Chapman-Kolmogorov equations satisfied by  $(f_i(t, x), x \geq 0, i \in \mathbb{N})$ , which form the fundamental system to be solved.

**Proposition 1** (Fundamental system). *The functions  $(x, t) \rightarrow f_i(t, x)$  for  $i \in \mathbb{N}$  satisfy the differential system (in the sense of distributions):*

$$\frac{\partial f_i}{\partial t} = -r_i \frac{\partial}{\partial x} \left( \left( \mathbb{1}_{\{i > i_0\}} + \mathbb{1}_{\{i \leq i_0\}} \mathbb{1}_{\{x > 0\}} \right) f_i \right) - (\lambda_i + \mu_i) f_i + \lambda_{i-1} f_{i-1} + \mu_{i+1} f_{i+1}, \quad (5)$$

with the convention  $\lambda_{-1} = 0, f_{-1} \equiv 0$  and  $f_i(t, x) = 0$  for  $x < 0$ .

Note that the differential system (5) holds for the density probability functions  $f_i(t, x)$ . The differential system considered in Parthasarathy et al. (2004) and van Dorn & Scheinhardt (1997) governs the probability distribution functions  $\mathbb{P}(X_t \leq x, \Lambda_t = i), i \geq 0$ . The differential system (5) is actually the equivalent of Takács' integro-differential formula for the  $M/G/1$  queue, see Kleinrock (1975). The resolution of this differential system is addressed in the next section.

## 2.2 Basic matrix Equation

Introduce the double Laplace transform

$$F_i(s, \xi) = \int_{0^-}^{\infty} \int_{0^-}^{\infty} e^{-st - \xi x} f_i(t, x) dt dx = \int_0^{\infty} e^{-st} \mathbb{E} \left( e^{-\xi X_t} \mathbb{1}_{\{\Lambda_t = i\}} \right) dt$$

and define the functions  $f_i^{(0)}(\xi)$  and  $h_i(s)$  for  $i \in \mathbb{N}$  as follows

$$f_i^{(0)}(\xi) = \int_0^{\infty} e^{-x\xi} \mathbb{P}\{\Lambda_0 = i, X_0 \in dx\}$$

$$h_i(s) = \int_0^{\infty} e^{-st} \mathbb{P}\{\Lambda_t = i, X_t = 0\} dt.$$

The functions  $f_i^{(0)}$  are related to the initial conditions of the system and are known functions. For  $i > i_0$ , we have  $\mathbb{P}\{\Lambda_t = i, X_t = 0\} = 0$ , which implies that  $h_i(s) = 0$ , for  $i > i_0$ . On the contrary, for  $i \leq i_0$ , the functions  $h_i$  are unknown and have to be determined by taking into account the dynamics of the system.

By taking Laplace transforms in Equation (5), we obtain the following result.

**Proposition 2.** *Let  $F(s, \xi)$ ,  $f^{(0)}$ , and  $h(s)$  be the infinite column vectors, which components are  $F_i(s, \xi)/\pi_i$ ,  $f_i^{(0)}/\pi_i$ , and  $h_i(s)/\pi_i$  for  $i \geq 0$ , respectively. Then, these vectors satisfy the matrix equation*

$$(s\mathbb{I} + \xi R - A)F(s, \xi) = f^{(0)}(\xi) + \xi R h(s), \tag{6}$$

where  $\mathbb{I}$  is the identity matrix,  $A$  is the infinitesimal generator of the birth and death process  $\{\Lambda_t\}$  defined by Equation (1), and  $R$  is the diagonal matrix with diagonal elements  $r_i$ ,  $i \geq 0$ .

*Proof.* Taking the Laplace transform of  $\partial f_i / \partial t$  gives rise to the term  $sF_i - f_i^{(0)}$ . In the same way, taking the Laplace transform of  $\partial(\mathbb{1}_{\{x>0\}}f_i) / \partial x$  yields the term  $\xi F_i - \xi h_i$ . Hence, taking Laplace transforms in Equation (5) and dividing all terms by  $\pi_i$  gives, for  $i \geq 0$ ,

$$s \frac{F_i}{\pi_i} - \frac{f_i^{(0)}}{\pi_i} = -r_i \xi \frac{F_i}{\pi_i} + r_i \xi \frac{h_i}{\pi_i} - (\lambda_i + \mu_i) \frac{F_i}{\pi_i} + \lambda_i \frac{F_{i+1}}{\pi_{i+1}} + \mu_i \frac{F_{i-1}}{\pi_{i-1}},$$

which can be rewritten in matrix form as Equation (6) □

When we consider the stationary regime of the fluid queue, we have to set  $f^{(0)}(\xi) \equiv 0$  and eliminate the term  $s\mathbb{I}$  in Equation (6), which then becomes

$$(\xi R - A)F(\xi) = \xi R h, \tag{7}$$

where  $h$  is the vector, which  $i$ th component is  $h_i = \lim_{t \rightarrow \infty} \mathbb{P}\{\Lambda_t = i, X_t = 0\} / \pi_i$  and  $F(\xi)$  is the vector, which  $i$ th component is  $\mathbb{E} \left[ e^{-\xi X_t} \mathbb{1}_{\{\Lambda_t = i\}} \right] / \pi_i$ . This is the Laplace transform version of Equation (12) by van Dorn & Scheinhardt (1997), which addresses the resolution of Equation (7).

### 3. Resolution of the fundamental system

In this section, we show how Equation (6) can be solved. For this purpose, we analyze the structure of this equation and in a first step, we prove that the functions  $F_i(s, \xi)$  can be expressed in terms of the function  $F_{i_0}(s, \xi)$ . (Recall that the index  $i_0$  is the greatest integer such that  $r(i) - c < 0$  and that for  $i \geq i_0 + 1$ ,  $r(i) > c$ .) The proof greatly relies on the spectral properties of some operators defined in adequate Hilbert spaces.

#### 3.1 Basic orthogonal polynomials

In the following, we use the orthogonal polynomials  $Q_i(s; x)$  defined by recursion:  $Q_0(s; x) \equiv 1$ ,  $Q_1(s; x) = (s + \lambda_0 - r_0 x) / \lambda_0$  and for  $i \geq 1$ ,

$$\frac{\lambda_i}{|r_i|} Q_{i+1}(s; x) + \left( x - \frac{s + \lambda_i + \mu_i}{|r_i|} \right) Q_i(s; x) + \frac{\mu_i}{|r_i|} Q_{i-1}(s; x) = 0. \quad (8)$$

By using Favard's criterion (see Askey (1984) for instance), it is easily checked that the polynomials  $Q_i(s; x)$  for  $i \geq 0$  form an orthogonal polynomial system.

The polynomials  $\frac{\lambda_0 \dots \lambda_{i-1}}{|r_0 \dots r_{i-1}|} Q_i(s; -z)$ ,  $i \geq 0$  are the successive denominators of the continued fraction

$$\mathcal{F}^e(s; z) = \frac{1}{z + \frac{s + \lambda_0}{|r_0|} - \frac{\frac{\mu_1 \lambda_0}{|r_0 r_1|}}{z + \frac{s + \lambda_1 + \mu_1}{|r_1|} - \frac{\frac{\mu_2 \lambda_1}{|r_2 r_1|}}{z + \frac{s + \lambda_2 + \mu_2}{|r_2|} - \dots}}$$

which is itself the even part of the continued fraction

$$\mathcal{F}(s; z) = \frac{\alpha_1(s)}{z + \frac{\alpha_2(s)}{1 + \frac{\alpha_3(s)}{z + \frac{\alpha_4(s)}{1 + \dots}}}}, \quad (9)$$

where the coefficients  $\alpha_k(s)$  are such that  $\alpha_1(s) = 1$ ,  $\alpha_2(s) = (s + \lambda_0)/|r_0|$ , and for  $k \geq 1$ ,

$$\alpha_{2k}(s)\alpha_{2k+1}(s) = \frac{\lambda_{k-1}\mu_k}{|r_{k-1}r_k|}, \quad \alpha_{2k+1}(s) + \alpha_{2(k+1)}(s) = \frac{s + \lambda_k + \mu_k}{|r_k|}. \quad (10)$$

We have the following property, which is proved in Appendix A.

**Lemma 1.** *The continued fraction  $\mathcal{F}(s; z)$  defined by Equation (9) is a converging Stieltjes fraction for all  $s \geq 0$ .*

As a consequence of the above lemma, there exists a unique bounded, increasing function  $\psi(s; x)$  in variable  $x$  such that

$$\mathcal{F}(s; z) = \int_0^\infty \frac{1}{z + x} \psi(s; dx).$$

The polynomials  $Q_n(s; x)$  are orthogonal with respect to the measure  $\psi(s; dx)$  and satisfy the orthogonality relation

$$\int_0^\infty Q_i(s; x) Q_j(s; x) \psi(s; dx) = \frac{|r_0|}{|r_i| \pi_i} \delta_{i,j} \quad (11)$$

As a consequence, it is worth noting that the polynomial  $Q_i(s; x)$  has  $i$  real, simple and positive roots.

It is possible to associate with the polynomials  $Q_i(s, x)$  a new class of orthogonal polynomials, referred to as associated polynomials and denoted by  $Q_i(i_0 + 1; s; x)$  and satisfying the

recurrence relations:  $Q_0(i_0 + 1; s; x) = 1$ ,  $Q_1(i_0 + 1; s; x) = (s + \lambda_{i_0+1+i} + \mu_{i_0+1+i} - r_{i_0+1+i}x) / \lambda_{i_0+1+i}$  and, for  $i \geq 0$ ,

$$\frac{\lambda_{i_0+1+i}}{r_{i_0+1+i}} Q_{i+1}(i_0 + 1; s; x) + \left( x - \frac{s + \lambda_{i_0+1+i} + \mu_{i_0+1+i}}{r_{i_0+1+i}} \right) Q_i(i_0 + 1; s; x) + \frac{\mu_{i_0+1+i}}{r_{i_0+1+i}} Q_{i-1}(i_0 + 1; s; x) = 0. \tag{12}$$

The polynomials  $Q_i(i_0 + 1; s; z)$  are related to the denominator of the continued fraction

$$\mathcal{F}_{i_0}^e(z) = \frac{1}{z + \frac{s + \lambda_{i_0+1} + \mu_{i_0+1}}{r_{i_0+1}} - \frac{\lambda_{i_0+1}\mu_{i_0+2}}{r_{i_0+1}r_{i_0+2}} \overbrace{\left[ z + \frac{s + \lambda_{i_0+2} + \mu_{i_0+2}}{r_{i_0+2}} - \frac{\lambda_{i_0+2}\mu_{i_0+3}}{r_{i_0+2}r_{i_0+3}} \right]} \dots}$$

which is the even part of the continued fraction  $\mathcal{F}_{i_0}(z)$  defined by

$$\mathcal{F}_{i_0}(s; z) = \frac{\beta_1(s)}{z + \frac{\beta_2(s)}{1 + \frac{\beta_3(s)}{z + \frac{\beta_4(s)}{1 + \ddots}}}}, \tag{13}$$

where the coefficients  $\beta_k(s)$  are such that

$$\beta_1(s) = 1, \quad \beta_2(s) = (s + \lambda_{i_0+1} + \mu_{i_0+1}) / |r_{i_0+1}|,$$

and for  $k \geq 1$ ,

$$\beta_{2k}(s)\beta_{2k+1}(s) = \frac{\lambda_{i_0+k}\mu_{i_0+k+1}}{r_{i_0+k}r_{i_0+k+1}}, \tag{14}$$

$$\beta_{2k+1}(s) + \beta_{2(k+1)}(s) = \frac{s + \lambda_{i_0+1+k} + \mu_{i_0+1+k}}{r_{i_0+1+k}}.$$

Since the continued fraction  $\mathcal{F}(s; z)$  is a converging Stieltjes fraction, it is quite clear that the continued fraction  $\mathcal{F}_{i_0}(s; z)$  defined by Equation (13) is a converging Stieltjes fraction for all  $s \geq 0$ . There exists hence a unique bounded, increasing function  $\psi^{[i_0]}(s; x)$  in variable  $x$  such that

$$\mathcal{F}_{i_0}(s; z) = \int_0^\infty \frac{1}{z + x} \psi^{[i_0]}(s; dx).$$

The polynomials  $Q_i(i_0 + 1; s; x)$  are orthogonal with respect to the measure  $\psi^{[i_0]}(s; dx)$  and satisfy the orthogonality relation

$$\int_0^\infty Q_i(i_0 + 1; s; x) Q_j(i_0 + 1; s; x) \psi^{[i_0]}(s; dx) = \frac{r_{i_0+1}\pi_{i_0+1}}{r_{i_0+1+i}\pi_{i_0+1+i}} \delta_{i,j}.$$



### 3.2 Resolution of the matrix equation

We show in this section how to solve the matrix Equation (6). In a first step, we solve the  $i_0 + 1$  first linear equations.

**Lemma 2.** *The functions  $F_i(s, \zeta)$ , for  $i \leq i_0$ , are related to function  $F_{i_0+1}(s, \zeta)$  as follows: for  $\zeta \neq \zeta_k(s)$ ,  $k = 0, \dots, i_0$ ,*

$$F_i(s, \zeta) = \frac{\pi_i}{r_0} \sum_{j=0}^{i_0} (f_j^{(0)}(\zeta) + r_j \zeta h_j(s)) \int_0^\infty \frac{Q_j(s; x) Q_i(s; x)}{\zeta - x} \psi_{[i_0]}(s; dx) + \mu_{i_0+1} \frac{\pi_i}{r_0} F_{i_0+1}(s, \zeta) \int_0^\infty \frac{Q_{i_0}(s; x) Q_i(s; x)}{\zeta - x} \psi_{[i_0]}(s; dx), \quad (15)$$

where the  $\zeta_k(s)$  are the roots of the polynomial  $Q_{i_0+1}(s; x)$  defined by Equation (8) and the measure  $\psi_{[i_0]}(s; dx)$  is defined by Equation (45) in Appendix A.

*Proof.* Let  $\mathbb{I}_{[i_0]}$ ,  $A_{[i_0]}$  and  $R_{[i_0]}$  denote the matrices obtained from the infinite identity matrix, the infinite matrix  $A$  defined by Equation (1) and the infinite diagonal matrix  $R$  by deleting the rows and the columns with an index greater than  $i_0$ , respectively. Denoting by  $F_{[i_0]}$ ,  $h_{[i_0]}$  and  $f_{[i_0]}$  the finite column vectors which  $i$ th components are  $F_i / \pi_i$ ,  $h_i / \pi_i$  and  $f_i^{(0)} / \pi_i$ , respectively for  $i = 0, \dots, i_0$ , Equation (6) can be written as

$$(s\mathbb{I}_{[i_0]} + \zeta R_{[i_0]} - A_{[i_0]}) F_{[i_0]} = f_{[i_0]} + \zeta R_{[i_0]} h_{[i_0]} + \frac{\lambda_{i_0}}{\pi_{i_0+1}} F_{i_0+1} e_{i_0},$$

where  $e_{i_0}$  is the column vector with all entries equal to 0 except the  $i_0$ th one equal to 1.

Since  $r(i) < c$  for all  $i \leq i_0$ , the matrix  $R_{[i_0]}$  is invertible and the above equation can be rewritten as

$$\left( \zeta \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right) F_{[i_0]} = R_{[i_0]}^{-1} f_{[i_0]} + \zeta h_{[i_0]} + \frac{\lambda_{i_0}}{r_{i_0} \pi_{i_0+1}} F_{i_0+1} e_{i_0}.$$

From Lemma 6 proved in Appendix B, we know that the operator associated with the finite matrix  $(\zeta \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s\mathbb{I}_{[i_0]} - A_{[i_0]}))$  is selfadjoint in the Hilbert space  $H_{i_0} \equiv \mathbb{C}^{i_0+1}$  equipped with the scalar product

$$(c, d)_{i_0} = \sum_{k=0}^{i_0} c_k \bar{d}_k |r_k| \pi_k.$$

The eigenvalues of the operator  $(\zeta \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s\mathbb{I}_{[i_0]} - A_{[i_0]}))$  are the quantities  $\zeta - \zeta_k(s)$  for  $k = 0, \dots, i_0$ , where the  $\zeta_k(s)$  are the roots of the polynomial  $Q_{i_0+1}(s; x)$  defined by Equation (8). Hence, for  $\zeta \notin \{\zeta_0(s), \dots, \zeta_{i_0}(s)\}$ , we have

$$F_{[i_0]} = \left( \zeta \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} R_{[i_0]}^{-1} f_{[i_0]} + \zeta \left( \zeta \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} h_{[i_0]} + \frac{\lambda_{i_0}}{r_{i_0} \pi_{i_0+1}} F_{i_0+1} \left( \zeta \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} e_{i_0}.$$

By introducing the vectors  $Q_{[i_0]}(s, \zeta_k(s))$  for  $k = 0, \dots, i_0$  defined in Appendix B, the column vector  $e_i$  with all entries equal to 0 except the  $i$ th one equal to 1 can be written as

$$e_j = \frac{|r_j|\pi_j}{|r_0|} \int_0^\infty Q_j(s, x) Q_{[i_0]}(s, x) \psi_{[i_0]}(s; dx)$$

where the measure  $\psi_{[i_0]}(s; dx)$  is defined by Equation (45). Since the vectors  $Q_{[i_0]}(s, \zeta_k(s))$  are such that

$$\left( \xi \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} Q_{[i_0]}(s, \zeta_k(s)) = \frac{1}{\xi - \zeta_k(s)} Q_{[i_0]}(s, \zeta_k(s)),$$

we deduce that

$$\left( \xi \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} e_j = \frac{|r_j|\pi_j}{|r_0|} \int_0^\infty \frac{Q_j(s, x)}{\xi - x} Q_{[i_0]}(s, x) \psi_{[i_0]}(s; dx)$$

Hence, if  $f = \sum_{j=0}^{i_0} f_j e_j$ , then

$$\left( \xi \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} f = \sum_{j=0}^{i_0} f_j \frac{|r_j|\pi_j}{|r_0|} \int_0^\infty \frac{Q_j(s, x)}{\xi - x} Q_{[i_0]}(s, x) \psi_{[i_0]}(s; dx)$$

and the  $i$ th component of the above vector is

$$\left( \left( \xi \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} f \right)_i = \sum_{j=0}^{i_0} f_j \frac{|r_j|\pi_j}{|r_0|} \int_0^\infty \frac{Q_j(s, x) Q_i(s, x)}{\xi - x} \psi_{[i_0]}(s; dx)$$

Applying the above identity to the vectors  $R_{[i_0]}^{-1}f_{[i_0]}$ ,  $h_{[i_0]}$  and  $e_{i_0}$ , Equation (15) follows. □

We now turn to the analysis of the second part of Equation (6).

**Lemma 3.** For  $s \geq 0$ , the functions  $F_i(s, \xi)$  are related to function  $F_{i_0}(s, \xi)$  by the relation: for  $i \geq 0$ ,

$$F_{i_0+i+1}(s, \xi) = \lambda_{i_0} \frac{\pi_{i_0+i+1}}{r_{i_0+1}\pi_{i_0+1}} F_{i_0}(s, \xi) \int_0^\infty \frac{Q_i(i_0 + 1; s; x)}{\xi + x} \psi^{[i_0]}(s; dx) + \frac{\pi_{i_0+i+1}}{r_{i_0+1}\pi_{i_0+1}} \sum_{j=0}^\infty f_{i_0+j+1}^{(0)}(\xi) \int_0^\infty \frac{Q_j(i_0 + 1; s; x) Q_i(i_0 + 1; s; x)}{x + \xi} \psi^{[i_0]}(s; dx), \quad (16)$$

where the measure  $\psi^{[i_0]}(s; dx)$  is the orthogonality measure of the associated polynomials  $Q_i(i_0 + 1; s; x)$ ,  $i \geq 0$ .

*Proof.* Let  $\mathbb{I}^{[i_0]}$ ,  $A^{[i_0]}$  and  $R^{[i_0]}$  denote the matrices obtained from  $\mathbb{I}$ ,  $A$  and  $R$  by deleting the first  $(i_0 + 1)$  lines and columns, respectively. The infinite matrix  $(R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]})$  induces in the Hilbert space  $H^{i_0}$  defined by

$$H^{i_0} = \left\{ (f_n) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} |f_n|^2 r_{i_0+n+1} \pi_{i_0+n+1} < \infty \right\}$$

and equipped with the scalar product

$$(f, g) = \sum_{n=0}^{\infty} f_n \bar{g}_n r_{i_0+n+1} \pi_{i_0+n+1},$$

where  $\bar{g}_n$  is the conjugate of the complex number  $g_n$ , an operator such that for  $f \in H^{i_0}$

$$\begin{aligned} ((R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]})f)_n = \\ -\frac{\mu_{i_0+1+n}}{r_{i_0+n+1}} f_{n-1} + \frac{s + \lambda_{i_0+n+1} + \mu_{i_0+1+n}}{r_{i_0+n+1}} f_n - \frac{\lambda_{i_0+n+1}}{r_{i_0+n+1}} f_{n+1}. \end{aligned}$$

The above operator is symmetric in  $H^{i_0}$ . To show that this operator is selfadjoint, we have to prove that the domains of this operator and its adjoint coincide. In Guillemin (2012), it is shown that given the special form of the operator under consideration, this condition is equivalent to the convergence of the Stieltjes fraction defined by Equation (13) and if this is the case, the spectral measure is the orthogonality measure  $\psi^{[i_0]}(s; dx)$ . Since the continued fraction  $\mathcal{F}_{i_0}(s; z)$  is a converging Stieltjes fraction, the above operator is hence selfadjoint.

Let  $Q^{[i_0]}(s; x)$  the column vector which  $i$ th entry is  $Q_i(i_0 + 1; s; x)$ . This vector is in  $H^{i_0}$  if and only if  $\|Q^{[i_0]}(s; x)\|^2 \stackrel{def}{=} (Q^{[i_0]}(s; x), Q^{[i_0]}(s; x)) < \infty$ . If it is the case, then the measure  $\psi^{[i_0]}(s; dx)$  has an atom at point  $x$  with mass  $1/\|Q^{[i_0]}(s; x)\|^2$ . Otherwise, the vector  $Q^{[i_0]}(s; x)$  is not in  $H^{i_0}$  but from the spectral theorem we have

$$H^{i_0} = \int^{\oplus} H_x^{i_0} \psi^{[i_0]}(s; dx)$$

where  $H_x^{i_0}$  is the vector space spanned by the vector  $Q^{[i_0]}(s; x)$  for  $x$  in the support of the measure  $\psi^{[i_0]}(s; dx)$ . In addition, we have the resolvent identity: For  $f, g \in H^{i_0}$  and  $\xi \in \mathbb{C}$  such that  $-\xi$  is not in the support of the measure  $\psi^{[i_0]}(s; dx)$ ,

$$\left( \left( \xi \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]}) \right)^{-1} f, g \right) = \int_0^{\infty} \frac{(f_x, g)}{\xi + x} \psi^{[i_0]}(s; dx). \quad (17)$$

where  $f_x$  is the projection on  $H_x^{i_0}$  of the vector  $f$ .

For  $i \geq 0$ , let  $e_i$  denote the column vector, which  $i$ th entry is equal to 1 and the other entries are equal to 0. Denoting by  $F^{[i_0]}$  and  $\hat{f}^{[i_0]}$  the column vectors which  $i$ th components are  $F_{i_0+1+i}/\pi_{i_0+1+i}$  and  $f_{i_0+1+i}^{(0)}/\pi_{i_0+1+i}$ , respectively, Equation (6) can be written as

$$(s\mathbb{I}^{[i_0]} + \xi R^{[i_0]} - A^{[i_0]})F^{[i_0]} = \hat{f}^{[i_0]} + \frac{\mu_{i_0+1}}{\pi_{i_0}} F_{i_0} e_0,$$

since  $h_i(s) \equiv 0$  for  $i > i_0$ .

Given that  $r_i > 0$  for  $i > i_0$ , the matrix  $R^{[i_0]}$  is invertible and the above equation can be rewritten as

$$\left( \zeta \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]}) \right) F^{[i_0]} = (R^{[i_0]})^{-1} f^{[i_0]} + \frac{\mu_{i_0+1}}{\pi_{i_0}} F_{i_0} \hat{R}^{-1} e_0,$$

The operator  $\left( \zeta \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]}) \right)$  is invertible for  $\zeta$  such that  $-\zeta$  is not in the support of the measure  $\psi^{[i_0]}(s, dx)$ , and we have

$$F^{[i_0]} = \left( \zeta \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]}) \right)^{-1} (R^{[i_0]})^{-1} f^{[i_0]} + \frac{\mu_{i_0+1}}{r_{i_0+1} \pi_{i_0}} F_{i_0} \left( \zeta \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]}) \right)^{-1} e_0.$$

By using the spectral identity (17), we can compute  $F_i$  for  $i > i_0$  as soon as  $F_{i_0}$  is known. Indeed, we have

$$F^{[i_0]} = \sum_{j=0}^{\infty} \frac{F_{i_0+1+j}}{\pi_{i_0+1+j}} e_j,$$

and then, for  $i \geq i_0 + 1$ , by using the fact that  $r_{i_0+1+i} F_{i_0+1+i} = (F^{[i_0]}, e_i)$ , we have

$$r_{i_0+1+i} F_{i_0+1+i} = \left( \left( \zeta \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]}) \right)^{-1} (R^{[i_0]})^{-1} f^{[i_0]}, e_i \right) + \frac{\mu_{i_0+1}}{r_{i_0+1} \pi_{i_0}} F_{i_0} \left( \left( \zeta \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s\mathbb{I}^{[i_0]} - A^{[i_0]}) \right)^{-1} e_0, e_i \right).$$

By using the fact that for  $j \geq 0$ ,

$$(e_j)_x = \frac{r_{i_0+j+1} \pi_{i_0+j+1}}{r_{i_0+1} \pi_{i_0+1}} Q_j(i_0 + 1; s; x) Q^{[i_0]}(s; x),$$

Equation (16) follows by using the resolvent identity (17). □

From the two above lemmas, it turns out that to determine the functions  $F_i(s, \zeta)$  it is necessary to compute the function  $h_i(s)$  for  $i = 0, \dots, i_0 + 1$ . For this purpose, let us introduce the non negative quantities  $\eta_\ell(s)$ ,  $\ell = 0, \dots, i_0$ , which are the  $(i_0 + 1)$  solution to the equation

$$1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_{i_0+1} r_0} \mathcal{F}_{i_0}(s; \zeta) \int_0^\infty \frac{Q_{i_0}(s; x)^2}{\zeta - x} \psi_{[i_0]}(s; dx) = 0. \tag{18}$$

Then, we can state the following result, which gives a means of computing the unknown functions  $h_j(s)$  for  $j = 0, \dots, i_0$ .

**Proposition 3.** The functions  $h_j(s)$ ,  $j = 0, \dots, i_0$ , satisfy the linear equations: for  $\ell = 0, \dots, i_0$ ,

$$\begin{aligned} & \frac{\lambda_{i_0} \mathcal{F}_{i_0}(s; \eta_\ell(s)) \eta_\ell(s)}{r_{i_0}} \left( \left( \eta_k(s) \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s \mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} e_{i_0}, h(s) \right)_{i_0} \\ &= \left( \left( \eta_k(s) \mathbb{I}_{[i_0]} + (R_{[i_0]})^{-1} (s \mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} e_{0}, (R_{[i_0]})^{-1} f^{[i_0]}(\eta_k(s)) \right) \\ & - \frac{\lambda_{i_0} \mathcal{F}_{i_0}(s; \eta_\ell(s))}{r_{i_0}} \left( \left( \eta_k(s) \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1} (s \mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} e_{i_0}, R_{[i_0]}^{-1} f^{[i_0]}(\eta_k(s)) \right)_{i_0}, \quad (19) \end{aligned}$$

where  $\mathcal{F}_{i_0}(s; z)$  is the continued fraction (13) and  $f^{[i_0]}(\xi)$  and  $f_{[i_0]}(\xi)$  are the vectors, which  $i$ th components are equal to  $f_{i_0+i+1}^{(0)}(\xi) / \pi_{i_0+i+1}$  and  $f_i^{(0)}(\xi) / \pi_i$ , respectively.

*Proof.* From Equation (16) for  $i = i_0 + 1$  and Equation (15) for  $i = i_0$ , we deduce that

$$\begin{aligned} & \left( 1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_0 r_{i_0+1}} \mathcal{F}_{i_0}(s; \xi) \int_0^\infty \frac{Q_{i_0}(s; x)^2}{\xi - x} \psi_{[i_0]}(s; dx) \right) F_{i_0+1}(s, \xi) = \\ & \frac{\lambda_{i_0} \pi_{i_0}}{r_0 r_{i_0+1}} \mathcal{F}_{i_0}(s; \xi) \sum_{j=0}^{i_0} (f_j^{(0)}(\xi) + r_j \xi h_j(s)) \int_0^\infty \frac{Q_j(s; x) Q_{i_0}(s; x)}{\xi - x} \psi_{[i_0]}(s; dx) \\ & + \frac{1}{r_{i_0+1}} \sum_{j=0}^\infty f_{i_0+j+1}^{(0)}(\xi) \int_0^\infty \frac{Q_j(i_0 + 1; s; x)}{x + \xi} \psi^{[i_0]}(s; dx). \quad (20) \end{aligned}$$

From equation (15), since the Laplace transform  $F_i(s, \xi)$  should have no poles for  $\xi \geq 0$ , the roots  $\zeta_k(s)$  for  $k = 0, \dots, i_0$  should be removable singularities and hence for all  $i, j, k = 0, \dots, i_0$

$$\begin{aligned} & Q_i(s; \zeta_k(s)) \left( \left( f_j^{(0)}(\zeta_k(s)) + r_j \zeta_k(s) h_j(\zeta_k(s)) \right) Q_j(s; \zeta_k(s)) \right. \\ & \left. + \mu_{i_0+1} F_{i_0+1}(s, \zeta_k(s)) Q_{i_0}(s, \zeta_k(s)) \right) = 0. \end{aligned}$$

By using the interleaving property of the roots of successive orthogonal polynomials, we have  $Q_i(s; \zeta_k(s)) \neq 0$  for all  $i, k = 0, \dots, i_0$ . Hence, the term between parentheses in the above equation is null and we deduce that the points  $\zeta_k(s)$ ,  $k = 0, \dots, i_0$ , are removable singularities in expression (20). The quantities  $h_j(s)$ ,  $j = 0, \dots, i_0$ , are then determined by using the fact that the r.h.s. of equation (20) must cancel at points  $\eta_k(s)$  for  $k = 0, \dots, i_0$ . This entails that for  $k = 0, \dots, i_0$ , the terms

$$\begin{aligned} & \sum_{j=0}^\infty f_{i_0+j+1}^{(0)}(\eta_k(s)) \int_0^\infty \frac{Q_j(i_0 + 1; s; x)}{x + \eta_k(s)} \psi^{[i_0]}(s; dx) \\ & + \frac{\lambda_{i_0} \pi_{i_0} \mathcal{F}_{i_0}(s; \eta_k(s))}{r_0} \sum_{j=0}^{i_0} v_j(s) \int_0^\infty \frac{Q_j(s; x) Q_{i_0}(s; x)}{\eta_k(s) - x} \psi_{[i_0]}(s; dx) \quad (21) \end{aligned}$$

must cancel, where

$$v_j(s) = f_j^{(0)}(\eta_k(s)) + \eta_k(s) r_j h_j(s).$$

By using the fact that

$$\int_0^\infty \frac{Q_j(s; x)Q_{i_0}(s; x)}{\eta_k(s) - x} \psi_{[i_0]}(s; dx) =$$

$$\frac{|r_0|}{|r_{i_0}| \pi_{i_0} |r_j| \pi_j} \left( \left( \eta_k(s) \mathbb{I}_{[i_0]} + R_{[i_0]}^{-1}(s \mathbb{I}_{[i_0]} - A_{[i_0]}) \right)^{-1} e_{i_0}, e_j \right)_{i_0}$$

and

$$\int_0^\infty \frac{Q_j(i_0 + 1; s; x)}{x + \eta_k(s)} \psi^{[i_0]}(s; dx) =$$

$$\frac{1}{r_{i_0+1+j} \pi_{i_0+j+1}} \left( \left( \eta_k(s) \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s \mathbb{I}^{[i_0]} - A^{[i_0]}) \right)^{-1} e_0, e_j \right),$$

Equation (19) follows. □

By solving the system of linear equations (19), we can compute the unknown functions  $h_j(s)$  for  $j = 0, \dots, i_0$ . The function  $F_{i_0+1}(s, \xi)$  is then given by

$$\begin{aligned} \left( 1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_{i_0+1} r_0} \mathcal{F}_{i_0}(s; \xi) \int_0^\infty \frac{Q_{i_0}(s; x)^2}{\xi - x} \psi_{[i_0]}(s; dx) \right) F_{i_0+1}(s, \xi) &= \\ &= \frac{1}{r_{i_0+1}} \left( \left( \xi \mathbb{I}^{[i_0]} + (R^{[i_0]})^{-1}(s \mathbb{I}^{[i_0]} - A^{[i_0]}) \right)^{-1} e_0, (R^{[i_0]})^{-1} f^{[i_0]}(\xi) \right) \\ &- \frac{\lambda_{i_0} \mathcal{F}_{i_0}(s; \xi)}{r_{i_0} r_{i_0+1}} \left( \left( \xi \mathbb{I}^{[i_0]} + R_{[i_0]}^{-1}(s \mathbb{I}^{[i_0]} - A_{[i_0]}) \right)^{-1} e_{i_0}, R_{[i_0]}^{-1} f_{[i_0]}(\xi) + \xi h(s) \right)_{i_0}, \end{aligned} \quad (22)$$

The function  $F_{i_0}(s, \xi)$  is computed by using equation (22) and equation (15) for  $i = i_0$ . The other functions  $F_i(s, \xi)$  are computed by using Lemmas 2 and 3.

The above procedure can be applied for any value  $i_0$  but expressions are much simpler when  $i_0 = 0$ , i.e., when there is only one state with negative net input rate. In that case, we have the following result, when the buffer is initially empty and the birth and death process is in state 1.

**Proposition 4.** Assume that  $r_0 < 0$  and  $r_i > 0$  for  $i > 0$ . When the buffer is initially empty and the birth and death process is in the state 1 at time 0 (i.e.,  $p_0(i) = \delta_{1,i}$  for all  $i \geq 0$ ), the Laplace transform  $h_0(s)$  is given by

$$h_0(s) = \frac{r_0 \eta_0(s) + s + \lambda_0}{\lambda_0 \eta_0(s) |r_0|} = \frac{\mu_1 \mathcal{F}_0(s; \eta_0(s))}{r_1 |r_0| \eta_0(s)}. \quad (23)$$

where  $\eta_0(s)$  is the unique positive solution to the equation

$$1 - \frac{\lambda_0 \mu_1 \mathcal{F}_0(s; \xi)}{r_1 (s + \lambda_0 + r_0 \xi)} = 0.$$

In addition,

$$F_1(s, \xi) = \frac{\frac{1}{r_1} \left( 1 + \frac{\lambda_0 \xi r_0 h_0(s)}{s + \lambda_0 + r_0 \xi} \right) \mathcal{F}_0(s; \xi)}{1 - \frac{\lambda_0 \mu_1}{r_1 (s + \lambda_0 + r_0 \xi)} \mathcal{F}_0(s; \xi)}. \quad (24)$$

*Proof.* In the case  $i_0 = 0$ , the unique root to the equation  $Q_1(s; x)$  is  $\zeta_0(s) = (s + \lambda_0)|r_0|$ . The measure  $\psi_{[0]}(s; dx)$  is given by

$$\psi_{[0]}(s; dx) = \delta_{\zeta_0(s)}(dx)$$

and Equation (18) reads

$$1 - \frac{\lambda_0 \mu_1}{r_1} \mathcal{F}_0(s; \xi) \frac{1}{s + \lambda_0 + r_0 \xi} = 0$$

which has a unique solution  $\eta_0(s) > 0$ . When the buffer is initially empty and the birth and death process is in the state 1 at time 0, we have  $f_i^{(0)}(\xi) = \delta_{1,j}$ . Then,

$$\begin{aligned} & \left( \left( \eta_0(s) \mathbb{I}^{[0]} + (R^{[0]})^{-1} (s \mathbb{I}^{[0]} - A^{[0]}) \right)^{-1} e_0, (R^{[0]})^{-1} f^{[0]}(\eta_0(s)) \right) \\ &= \frac{1}{r_1 \pi_1} \left( \left( \eta_0(s) \mathbb{I}^{[0]} + (R^{[0]})^{-1} (s \mathbb{I}^{[0]} - A^{[0]}) \right)^{-1} e_0, e_0 \right) = \int_0^\infty \frac{1}{\eta_0(s) + x} \psi^{[0]}(s; dx) \\ &= \mathcal{F}_0(s; \eta_0(s)), \end{aligned}$$

where we have used the resolvent identity (17) and the fact that  $(e_0)_x = Q^{[0]}(s; x)$ . Moreover,

$$\begin{aligned} & \left( \left( \eta_0(s) \mathbb{I}_{[0]} + R_{[0]}^{-1} (s \mathbb{I}_{[0]} - A_{[0]}) \right)^{-1} e_0, R_{[0]}^{-1} f_{[0]}(\eta_0(s)) + h(s) \right)_0 \\ &= \frac{h_0(s)}{\eta_0(s) + \frac{s + \lambda_0}{r_0}} (e_0, e_0)_0 = \frac{h_0(s) |r_0|}{\eta_0(s) + \frac{s + \lambda_0}{r_0}}. \end{aligned}$$

By using Equation (19) for  $i_0 = 0$ , Equation (23) follows. Finally, Equation (24) is obtained by using Equation (22).  $\square$

#### 4. Analysis of the stationary regime

In this section, we analyze the stationary regime. In this case, we have to take  $s = 0$  and  $f^{(0)} \equiv 0$ . To alleviate the notation, we set  $\psi_{[i_0]}(0; dx) = \psi_{[i_0]}(dx)$ ,  $\psi^{[i_0]}(0; dx) = \psi^{[i_0]}(dx)$  and  $Q_j(0; x) = Q_j(x)$  and  $Q_j(i_0 + 1; 0; x) = Q_j(i_0 + 1; x)$ . Equation (20) then reads

$$\begin{aligned} & \left( 1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_{i_0+1} r_0} \mathcal{F}_{i_0}(\xi) \int_0^\infty \frac{Q_{i_0}(x)^2}{\xi - x} \psi_{[i_0]}(dx) \right) F_{i_0+1}(\xi) \\ &= \frac{\lambda_{i_0} \pi_{i_0} \xi \mathcal{F}_{i_0}(\xi)}{r_0 r_{i_0+1}} \sum_{j=0}^{i_0} r_j h_j \int_0^\infty \frac{Q_j(x) Q_{i_0}(x)}{\xi - x} \psi_{[i_0]}(dx), \quad (25) \end{aligned}$$

where  $h_j = \lim_{t \rightarrow \infty} \mathbb{P}(\Lambda_t = j, X_t = 0)$ ,  $\mathcal{F}_{i_0}(\xi) = \mathcal{F}_{i_0}(0; \xi)$  and  $\mathcal{F}_{i_0+1}(\xi) = \mathcal{F}_{i_0+1}(0; \xi)$ .

The continued fraction  $\mathcal{F}_{i_0}(\xi)$  has the following probabilistic interpretation:

$$\mu_{i_0+1} \mathcal{F}_{i_0}(\xi) / r_{i_0+1} = \mathbb{E} \left( e^{-\xi \theta_{i_0}} \right)$$

where  $\theta_{i_0}$  is the passage time of the birth and death process with birth rates  $\lambda_n / |r_n|$  and death rates  $\mu_n / |r_n|$  from state  $i_0 + 1$  to state  $i_0$  (see Guillemin & Pinchon (1999) for details). This entails in particular that  $\mathcal{F}_{i_0}(0) = r_{i_0+1} / \mu_{i_0+1}$ .

Let us first characterize the measure  $\psi_{[i_0]}(dx)$ . For this purpose, let us introduce the polynomials of the second kind associated with the polynomials  $Q_i(x)$ . The polynomials of the second kind  $P_i(x)$  satisfy the same recursion as the polynomials  $Q_i(x)$  but with the initial conditions  $P_0(x) = 0$  and  $P_1(x) = |r_0| / \lambda_0$ . The even numerators of the continued fraction  $\mathcal{F}(z) \stackrel{\text{def}}{=} \mathcal{F}(0; z)$ , where  $\mathcal{F}(s; z)$  is defined by Equation (9), are equal to  $\frac{\lambda_0 \dots \lambda_{n-1}}{|r_0 \dots r_{n-1}|} P_n(-z)$  and the even denominators to  $\frac{\lambda_0 \dots \lambda_{n-1}}{|r_0 \dots r_{n-1}|} Q_n(-z)$ .

**Lemma 4.** *The spectral measure  $\psi_{[i_0]}(dx)$  of the non negative selfadjoint operator  $R_{[i_0]}^{-1} A_{[i_0]}$  in the Hilbert space  $H_{i_0}$  is such that*

$$\int_0^\infty \frac{1}{z-x} \psi_{[i_0]}(dx) = -\frac{P_{i_0+1}(z)}{Q_{i_0+1}(z)}. \tag{26}$$

The measure  $\psi_{[i_0]}(dx)$  is purely discrete with atoms located at the zeros  $\zeta_k$ ,  $k = 0, \dots, i_0$ , of the polynomial  $Q_{i_0+1}(z)$ .

*Proof.* Let  $P_{[i_0]}(z)$  (resp.  $Q_{[i_0]}(z)$ ) denote the column vector, which  $i$ th component for  $0 \leq i \leq i_0$  is  $P_i(z)$  (resp.  $Q_i(z)$ ). For any  $x, z \in \mathbb{C}$ , we have

$$\left( z\mathbb{I}_{[i_0]} - R_{[i_0]}^{-1} A_{[i_0]} \right) (P_{[i_0]}(z) + xQ_{[i_0]}(z)) = e_0 - \frac{\lambda_{i_0}}{|r_{i_0+1}|} (P_{i_0+1}(z) + xQ_{i_0+1}(z)) e_{i_0}.$$

Hence, if  $z \neq \zeta_i$  for  $0 \leq i \leq i_0$ , where  $\zeta_i$  is the  $i$ th zero of the polynomial  $Q_{i_0+1}(x)$ , and if we take  $x = -P_{i_0+1}(z) / Q_{i_0+1}(z)$ , we see that

$$\left( z\mathbb{I}_{[i_0]} - R_{[i_0]}^{-1} A_{[i_0]} \right)^{-1} e_0 = P_{[i_0]}(z) - \frac{P_{i_0+1}(z)}{Q_{i_0+1}(z)} Q_{[i_0]}(z).$$

From the spectral identity for the operator  $R_{[i_0]}^{-1} A_{[i_0]}$  (similar to Equation (17)), we have

$$\left( \left( z\mathbb{I}_{[i_0]} - R_{[i_0]}^{-1} A_{[i_0]} \right)^{-1} e_0, e_0 \right)_{i_0} = \int_0^\infty \frac{((e_0)_x, e_0)_{i_0}}{z-x} \psi_{[i_0]}(dx) = -\frac{P_{i_0+1}(z)}{Q_{i_0+1}(z)} |r_0|.$$

Since  $(e_0)_x = Q_{[i_0]}(x)$  because of the orthogonality relation (11), Equation (26) immediately follows. □



By using the above lemma, we can show that the smallest solution to the equation

$$1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_{i_0+1} r_0} \mathcal{F}_{i_0}(\xi) \int_0^\infty \frac{Q_{i_0}(x)^2}{\xi - x} \psi_{[i_0]}(dx) = 0 \quad (27)$$

is  $\eta_0 = 0$ . The above equation is the stationary version of Equation (18).

**Lemma 5.** *The solutions  $\eta_j$ ,  $j = 0, \dots, i_0$ , to Equation (27) are such that  $\eta_0 = 0 < \eta_1 < \dots < \eta_{i_0}$ . For  $\ell = 1, \dots, i_0$ ,  $\eta_\ell$  is solution to equation*

$$1 = \frac{\mu_{i_0+1}}{r_{i_0+1}} \mathcal{F}_{i_0}(\xi) \frac{Q_{i_0}(\xi)}{Q_{i_0+1}(\xi)}. \quad (28)$$

*Proof.* The fraction  $P_{i_0+1}(z)/Q_{i_0+1}(z)$  is a terminating fraction and from Equation (26), we have

$$\frac{P_{i_0+1}(-z)}{Q_{i_0+1}(-z)} = \int_0^\infty \frac{1}{z+x} \psi_{[i_0]}(dx).$$

On the one hand, by applying Theorem 12.11d of Henrici (1977) to this fraction, we have

$$\frac{P_{i_0+1}(-z)}{Q_{i_0+1}(-z)} - \frac{P_{i_0}(-z)}{Q_{i_0}(-z)} = \int_0^\infty \frac{Q_{i_0}(x)^2}{Q_{i_0}(-z)^2} \frac{\psi_{[i_0]}(dx)}{z+x}. \quad (29)$$

On the other hand, by using the fact that

$$\frac{P_{i_0+1}(-z)}{Q_{i_0+1}(-z)} - \frac{P_{i_0}(-z)}{Q_{i_0}(-z)} = \frac{|r_0|}{\lambda_{i_0} \pi_{i_0} Q_{i_0+1}(-z) Q_{i_0}(-z)}, \quad (30)$$

we deduce that

$$\int_0^\infty \frac{Q_{i_0}(x)^2}{x} \psi_{[i_0]}(dx) = \frac{|r_0|}{\lambda_{i_0} \pi_{i_0}},$$

since  $Q_i(0) = 1$  for all  $i \geq 0$ . In addition, by using the fact that  $\mathcal{F}_{i_0}(0) = r_{i_0+1}/\mu_{i_0+1}$ , we deduce that the smallest root of Equation (27) is  $\eta_0 = 0$ . The other roots are positive. Equation (27) can be rewritten as Equation (28) by using Equations (29) and (30).  $\square$

Note that by using the same arguments as above, we can simplify Equation (18). As a matter of fact, we have

$$\frac{P_{i_0+1}(s, -z)}{Q_{i_0+1}(s, -z)} - \frac{P_{i_0}(s, -z)}{Q_{i_0}(s, -z)} = \frac{|r_0|}{\lambda_{i_0} \pi_{i_0} Q_{i_0+1}(s, -z) Q_{i_0}(s, -z)},$$

so Equation (18) becomes

$$1 = \frac{\mu_{i_0+1}}{r_{i_0+1}} \mathcal{F}_{i_0}(s, \xi) \frac{Q_{i_0}(s, \xi)}{Q_{i_0+1}(s, \xi)}. \quad (31)$$

The quantities  $h_i$  are evaluated by using the normalizing condition  $\sum_{i=0}^{i_0} h_i = 1 - \rho$ , where  $\rho$  is defined by Equation (3), and by solving the  $i_0$  linear equations

$$\ell = 1, \dots, i_0, \quad \left( (\eta_\ell \mathbf{I} - R_{[i_0]}^{-1} A_{[i_0]})^{-1} e_{i_0}, h \right)_{i_0} = 0, \quad (32)$$

where  $h$  is the vector which  $i$ th component is  $h_i/\pi_i$ . Once the quantities  $h_i, i = 0, \dots, i_0$  are known, the function  $F_{i_0+1}(\xi)$  is computed by using relation (25). The function  $F_{i_0}(\xi)$  is computed by using the relation

$$F_{i_0+1}(\xi) = \frac{\lambda_{i_0}}{r_{i_0+1}} F_{i_0}(\xi) \mathcal{F}_{i_0}(\xi).$$

This allows us to determine the functions  $F_{i_0+1}(\xi)$  and  $F_{i_0}(\xi)$ . The functions  $F_i(\xi)$  for  $i = 0, \dots, i_0$  are computed by using Equation (15) for  $s = 0$  and  $f^{(0)} \equiv 0$ . The functions  $F_i(\xi)$  for  $i > i_0$  are computed by using Equation (16) for  $s = 0$  and  $f^{(0)} \equiv 0$ . This leads to the following result.

**Proposition 5.** *The Laplace transform of the buffer content  $X$  in the stationary regime is given by*

$$\begin{aligned} \mathbb{E} \left( e^{-\xi X} \right) &= \sum_{i=0}^{\infty} F_i(\xi) = \frac{1}{r_0} \sum_{j=0}^{i_0} r_j \xi h_j \int_0^{\infty} \frac{Q_j(x) \Pi(x)}{\xi - x} \psi_{[i_0]}(dx) \\ &+ \frac{\lambda_{i_0}}{r_{i_0+1}} F_{i_0}(\xi) \left( \frac{\mu_{i_0+1}}{r_0} \mathcal{F}_{i_0}(\xi) \int_0^{\infty} \frac{Q_{i_0}(x) \Pi(x)}{\xi - x} \psi_{[i_0]}(dx) + \frac{1}{\pi_{i_0+1}} \int_0^{\infty} \frac{\Pi_{i_0}(x)}{x + \xi} \psi^{[i_0]}(dx) \right) \end{aligned} \quad (33)$$

with

$$\begin{aligned} \Pi(x) &= \sum_{i=0}^{i_0} \pi_i Q_i(x), \\ \Pi_{i_0}(x) &= \sum_{i=0}^{\infty} \pi_{i_0+1+i} Q_i(i_0 + 1; x), \\ F_{i_0}(\xi) &= \frac{\frac{\pi_{i_0}}{r_0} \sum_{j=0}^{i_0} r_j \xi h_j \int_0^{\infty} \frac{Q_j(x) Q_{i_0}(x)}{\xi - x} \psi_{[i_0]}(dx)}{1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_0 r_{i_0+1}} \mathcal{F}_{i_0}(\xi) \int_0^{\infty} \frac{Q_{i_0}(x)^2}{\xi - x} \psi_{[i_0]}(dx)}. \end{aligned}$$

In the case when there is only one state with negative drift, the above result can be simplified as follows.

**Corollary 1.** *When there is only one state with negative drift, the Laplace transform of the buffer content is given by*

$$\mathbb{E} \left( e^{-\xi X} \right) = \frac{\xi(1 - \rho)r_0}{r_0 \xi + \lambda_0 - \frac{\lambda_0 \mu_1}{r_1} \mathcal{F}_0(\xi)} \left( 1 + \frac{\lambda_1}{r_1} \int_0^{\infty} \frac{\Pi_0(x)}{x + \xi} \psi^{[0]}(dx) \right). \quad (34)$$

*Proof.* Since  $\psi_{[0]}(dx) = \delta_{\zeta_0}(dx)$  with  $\zeta_0 = \lambda_0/|r_0|$  and  $\Pi(x) = 1$ , we have

$$\int_0^{\infty} \frac{\Pi(x)}{\xi - x} \psi_{[i_0]}(dx) = \frac{r_0}{r_0 \xi + \lambda_0}.$$

Moreover, we have  $h_0 = 1 - \rho$  and then

$$F_0(\xi) = \frac{(1 - \rho)\xi r_0}{r_0\xi + \lambda_0 - \frac{\lambda_0\mu_1}{r_1}\mathcal{F}_0(\xi)}.$$

Simple algebra then yields equation (34). □

By examining the singularities in Equation (34), it is possible to determine the tail of the probability distribution of the buffer content in the stationary regime. The asymptotic behavior greatly depends on the properties of the polynomials  $Q_i(x)$  and their associated spectral measure.

## 5. Busy period

In this section, we are interested in the duration of a busy period of the fluid reservoir. At the beginning of a busy period, the buffer is empty and the modulating process is in state  $i_0 + 1$ . More generally, let us introduce the occupation duration  $B$  which is the duration the server is busy up to an idle period. The random variable  $B$  depends on the initial conditions and we define the conditional probability distribution

$$H_i(t, x) = \mathbb{P}(B \leq t \mid \Lambda_0 = i, X_0 = x).$$

The probability distribution function of a busy period  $\beta$  of the buffer is clearly given by

$$\mathbb{P}(\beta \leq t) = H_{i_0+1}(t, 0). \quad (35)$$

It is known in Barbot et al. (2001) that for  $t > 0$  and  $x > 0$ ,  $H_i(t, x)$  satisfies the following partial differential equations

$$\frac{\partial}{\partial t} H_i(t, x) - r_i \frac{\partial}{\partial x} H_i(t, x) = -\mu_i H_{i-1}(t, x) + (\lambda_i + \mu_i) H_i(t, x) - \lambda_i H_{i+1}(t, x) \quad (36)$$

with the boundary conditions

$$H_i(t, 0) = 1 \quad \text{if } t \geq 0, r_i \leq 0,$$

$$H_i(0, x) = 0 \quad \text{if } x > 0,$$

$$H_i(0, 0) = 0 \quad \text{if } r_i > 0.$$

Define then conditional Laplace transform

$$\theta_i(u, x) = \mathbb{E} \left( e^{-uB} \mid \Lambda_0 = i, Q_0 = x \right).$$

By taking Laplace transforms in Equation (36), we have

$$r_i \frac{\partial}{\partial x} \theta_i(u, x) = u\theta_i(u, x) - \mu_i\theta_{i-1}(u, x) + (\lambda_i + \mu_i)\theta_i(u, x) - \lambda_i\theta_{i+1}(u, x)$$

By introducing the conditional double Laplace transform

$$\tilde{\theta}_i(u, \zeta) = \int_0^\infty e^{-\zeta x} \theta_i(u, x) dx.$$

we obtain for  $i \geq 0$

$$r_i \zeta \tilde{\theta}_i(u, \zeta) - r_i \theta_i(u, 0) = u \tilde{\theta}_i(u, \zeta) - \mu_i \tilde{\theta}_{i-1}(u, \zeta) + (\lambda_i + \mu_i) \tilde{\theta}_i(u, \zeta) - \lambda_i \tilde{\theta}_{i+1}(u, \zeta)$$

By introducing the infinite vector  $\Theta(u, \zeta)$ , which  $i$ th component is  $\tilde{\theta}_i(u, \zeta)$ , the above equations can be rewritten in matrix form as

$$\zeta R \Theta(u, \zeta) = RT(u) + (u \mathbb{I} - A) \Theta(u, \zeta), \tag{37}$$

where  $T(u)$  is the vector which  $i$ th component is equal to  $\theta_i(u, 0)$ . We clearly have  $\theta_i(u, 0) = 1$  for  $i = 0, \dots, i_0$ . For the moment, the functions  $\theta_i(u, 0)$  for  $i > i_0$  are unknown functions.

Equation (37) can be solved by using the same technique as in Section 3. In the following, we assume that the measure  $\psi^{[i_0]}(s; dx)$  has a discrete spectrum with atoms located at points  $\chi_k(s) > 0$  for  $k \geq 0$ . This assumption is satisfied for instance when the measure  $\psi(s; dx)$  has a discrete spectrum (see Guillemin & Pinchon (1999) for details). Under this assumption, let  $\chi_k(s) > 0$  for  $k \geq 0$  be the solutions to the equation

$$\frac{\mu_{i_0+1}}{r_{i_0+1}} \frac{Q_{i_0}(u; -\zeta)}{Q_{i_0+1}(u; -\zeta)} \mathcal{F}_{i_0}(u, -\zeta) = 1.$$

**Proposition 6.** *The Laplace transforms  $\theta_{i_0+1+j}(u, 0)$  for  $j \geq 0$  satisfy the following linear equations:*

$$\begin{aligned} \frac{1}{r_{i_0+1} \pi_{i_0+1}} \frac{Q_{i_0}(u; -\zeta)}{Q_{i_0+1}(u; -\zeta)} \sum_{j=0}^\infty r_{i_0+1+j} \pi_{i_0+1+j} \theta_{i_0+1+j}(u, 0) \int_0^\infty \frac{Q_j(i_0 + 1; u; x)}{\zeta - x} \psi^{[i_0]}(u; dx) \\ + \frac{1}{|r_0|} \sum_{j=0}^{i_0} |r_j| \pi_j \int_0^\infty \frac{Q_{i_0}(u; x) Q_j(u; x)}{\zeta + x} \psi^{[i_0]}(u; dx) = 0 \end{aligned} \tag{38}$$

for  $\zeta \in \{\chi_k(s), k \geq 0\}$ .

*Proof.* Equation (37) can be split into two parts. The first part reads

$$\left( \zeta \mathbb{I}_{[i_0]} - R_{[i_0]}^{-1} (u \mathbb{I}_{[i_0]} - A_{[i_0]}) \right) \Theta_{[i_0]} = e_{[i_0]} - \frac{\lambda_{i_0}}{r_{i_0}} \tilde{\theta}_{i_0+1}(u, \zeta) e_{i_0}, \tag{39}$$

where  $e_{[i_0]}$  is the finite vector with all entries equal to 1 for  $i = 0, \dots, i_0$  and  $\Theta_{[i_0]}$  is the finite vector, which  $i$ th entry is  $\tilde{\theta}_i(u, \zeta)$  for  $i = 0, \dots, i_0$ . The second part of the equation is

$$\left( \zeta \mathbb{I}^{[i_0]} - \left( R^{[i_0]} \right)^{-1} \left( u \mathbb{I}^{[i_0]} - A^{[i_0]} \right) \right) \Theta^{[i_0]} = T^{[i_0]} - \frac{\mu_{i_0+1}}{r_{i_0+1}} \tilde{\theta}_{i_0}(u, \zeta) e_0, \tag{40}$$

where the vector  $T^{[i_0]}$  (resp.  $\Theta^{[i_0]}$ ) has entries equal to  $\theta_{i_0+1+i}(u, 0)$  (resp.  $\tilde{\theta}_{i_0+1+i}(u, \zeta)$ ) for  $i \geq 0$ .

By adapting the proofs in Section 3, we have for  $i = 0, \dots, i_0$

$$\begin{aligned} \tilde{\theta}_i(u, \xi) = & \frac{1}{|r_0|} \sum_{j=0}^{i_0} |r_j| \pi_j \int_0^\infty \frac{Q_i(u; x) Q_j(u; x)}{\xi + x} \psi_{[i_0]}(u; dx) \\ & + \frac{\mu_{i_0+1} \pi_{i_0+1}}{|r_0|} \tilde{\theta}_{i_0+1}(u, \xi) \int_0^\infty \frac{Q_{i_0}(u; x) Q_i(s; x)}{\xi + x} \psi_{[i_0]}(u; dx), \end{aligned} \quad (41)$$

and for  $i \geq 0$

$$\begin{aligned} \tilde{\theta}_{i_0+i+1}(u, \xi) = & -\frac{\mu_{i_0+1+i}}{r_{i_0+1}} \tilde{\theta}_{i_0}(u, \xi) \int_0^\infty \frac{Q_i(i_0+1; u; x)}{\xi - x} \psi^{[i_0]}(u; dx) \\ & + \frac{1}{r_{i_0+1} \pi_{i_0+1}} \sum_{j=0}^\infty r_{i_0+1+j} \pi_{i_0+1+j} \theta_{i_0+1+j}(u, 0) \int_0^\infty \frac{Q_j(i_0+1; u; x) Q_i(i_0+1; u; x)}{\xi - x} \psi^{[i_0]}(u; dx) \end{aligned} \quad (42)$$

By using Equation 41 for  $i = i_0$  and Equation (42) for  $i = 0$ , we obtain

$$\begin{aligned} \left( 1 - \frac{\mu_{i_0+1}}{r_{i_0+1}} \frac{Q_{i_0}(u; -\xi)}{Q_{i_0+1}(u; -\xi)} \mathcal{F}_{i_0}(u, -\xi) \right) \tilde{\theta}_{i_0}(u, \xi) = \\ \frac{1}{|r_0|} \sum_{j=0}^{i_0} |r_j| \pi_j \int_0^\infty \frac{Q_{i_0}(u; x) Q_j(u; x)}{\xi + x} \psi_{[i_0]}(u; dx) \\ + \frac{1}{r_{i_0+1} \pi_{i_0+1}} \frac{Q_{i_0}(u; -\xi)}{Q_{i_0+1}(u; -\xi)} \sum_{j=0}^\infty r_{i_0+1+j} \pi_{i_0+1+j} \theta_{i_0+1+j}(u, 0) \int_0^\infty \frac{Q_j(i_0+1; u; x)}{\xi - x} \psi^{[i_0]}(u; dx) \end{aligned}$$

where we have used the fact

$$\int_0^\infty \frac{Q_{i_0}(u; x)^2}{\xi + x} \psi_{[i_0]}(u; dx) = \frac{|r_0|}{\lambda_{i_0} \pi_{i_0}} \frac{Q_{i_0}(u; -\xi)}{Q_{i_0+1}(u; -\xi)}$$

and

$$\int_0^\infty \frac{1}{\xi - x} \psi^{[i_0]}(u; dx) = -\mathcal{F}_{i_0}(u; -\xi).$$

Since the function  $\tilde{\theta}_{i_0}(u; \xi)$  shall have no poles in  $[0, \infty)$ , the result follows.  $\square$

## 6. Conclusion

We have presented in this paper a general method for computing the Laplace transform of the transient probability distribution function of the content of a fluid reservoir fed with a source, whose transmission rate is modulated by a general birth and death process. This Laplace transform can be evaluated by solving a polynomial equation (see equation (18)). Once the zeros are known, the quantities  $h_i(s)$  for  $i = 0, \dots, i_0$  are computed by solving the system of linear equations (19). These functions then completely determined the two critical functions  $F_{i_0}$  and  $F_{i_0+1}$ , which are then used for computing the functions  $F_i$  for  $i > i_0 + 1$  and  $F_i$  for  $i < i_0$

by using equations (16) and (15), respectively. Moreover, we note that the theory of orthogonal polynomials and continued fractions plays a crucial role in solving the basic equation (6).

The above method can be used for evaluating the Laplace transform of the duration of a busy period of the fluid reservoir as shown in Section 5. The results obtained in this section can be used to study the asymptotic behavior of the busy period when the service rate of the buffer becomes very large. Occupancy periods of the buffer then become rare events and one may expect that buffer characteristics converge to some limits. This will be addressed in further studies.

## 7. Appendix

### A. Proof of Lemma 1

From the recurrence relations (10), the quantities  $A_k(s)$  defined by  $A_0(s) = 1$  and for  $k \geq 1$

$$A_k(s) = |r_0 \dots r_{k-1}| \prod_{j=1}^k \alpha_{2j}(s)$$

satisfy the recurrence relation for  $k \geq 1$

$$A_{k+1}(s) = (s + \lambda_k + \mu_k)A_k(s) - \lambda_{k-1}\mu_k A_{k-1}(s).$$

It is clear that  $A_k(s)$  is a polynomial in variable  $s$ . In fact, the polynomials  $A_k(s)$  are the successive denominators of the continued fraction

$$\mathcal{G}^e(z) = \frac{1}{s + \lambda_0 - \frac{\mu_1 \lambda_0}{s + \lambda_1 + \mu_1 - \frac{\mu_2 \lambda_1}{s + \lambda_2 + \mu_2 - \dots}}}$$

which is itself the even part of the continued fraction

$$\mathcal{G}(s) = \frac{\alpha_1}{z + \frac{\alpha_2}{1 + \frac{\alpha_3}{z + \frac{\alpha_4}{1 + \dots}}}}, \tag{43}$$

where the coefficients  $\alpha_k$  are such that  $\alpha_1 = 1$ ,  $\alpha_2 = \lambda_0$ , and for  $k \geq 1$ ,

$$\alpha_{2k}\alpha_{2k+1} = \lambda_{k-1}\mu_k, \quad \alpha_{2k+1} + \alpha_{2(k+1)} = \lambda_k + \mu_k.$$

It is straightforwardly checked that  $\alpha_{2k} = \lambda_{k-1}$  and  $\alpha_{2k+1} = \mu_k$  for  $k \geq 1$ . The continued fraction  $\mathcal{G}(s)$  is hence a Stieltjes fraction and is converging for all  $s > 0$  if and only if  $\sum_{k=0}^{\infty} \alpha_k =$

$\infty$  where the coefficients  $a_k$  are defined by

$$\alpha_1 = \frac{1}{a_1}, \quad \alpha_k = \frac{1}{a_{k-1}a_k} \text{ for } k \geq 1.$$

(See Henrici (1977) for details.) It is easily checked that for  $k \geq 1$

$$a_{2k} = \frac{1}{\lambda_{k-1}\pi_{k-1}} \quad \text{and} \quad a_{2k+1} = \pi_k.$$

Since the process  $(\Lambda_t)$  is assumed to be ergodic,  $\sum_{k \geq 1} a_k = \infty$ , which shows that the continued fraction  $\mathcal{G}(s)$  is converging for all  $s > 0$  and that there exists a unique measure  $\varphi(dx)$  such that  $\mathcal{G}(s)$  is the Stieltjes transform of  $\varphi(dx)$ , that is, for all  $s \in \mathbb{C} \setminus (-\infty, 0]$

$$\mathcal{G}(s) = \int_0^\infty \frac{1}{z+x} \varphi(dx).$$

The support of  $\varphi(dx)$  is included in  $[0, \infty)$  and this measure has a mass at point  $x_0 \geq 0$  if and only if

$$\sum_{k=0}^{\infty} \frac{A_k(-x_0)^2}{\lambda_0 \dots \lambda_{k-1} \mu_1 \dots \mu_k} < \infty.$$

Since the continued fraction  $\mathcal{G}(s)$  is converging for all  $s > 0$ , we have

$$\sum_{k=0}^{\infty} \frac{A_k(s)^2}{\lambda_0 \dots \lambda_{k-1} \mu_1 \dots \mu_k} = \infty. \quad (44)$$

Since the polynomials  $A_k(s)$  are the successive denominator of the fraction  $\mathcal{G}^e(s)$ , the polynomials  $A_k(-s)$ ,  $k \geq 1$ , are orthogonal with respect to some orthogonality measure, namely the measure  $\varphi(dx)$ . From the general theory of orthogonal polynomials Askey (1984); Chihara (1978), we know that the polynomial  $A_k(-s)$  has  $k$  simple, real, and positive roots. Since the coefficient of the leading term of  $A_k(-s)$  is  $(-1)^k$ , this implies that  $A_k(s)$  can be written as  $A_k(s) = (s + s_{1,k}) \dots (s + s_{k,k})$  with  $s_{i,k} > 0$  for  $i = 1, \dots, k$ . Hence,  $A_k(s) \geq 0$  for all  $s \geq 0$  and then, for all  $k \geq 0$ ,  $\alpha_k(s) \geq 0$  for all  $s \geq 0$  and hence the continued fraction  $\mathcal{F}(s, z)$  defined by Equation (9) is a Stieltjes fraction.

The continued fraction  $\mathcal{F}(s, z)$  is converging if and only if  $\sum_{k=0}^{\infty} a_k(s) = \infty$  where the coefficients  $a_k(s)$  are defined by

$$\alpha_1(s) = \frac{1}{a_1(s)}, \quad \alpha_k(s) = \frac{1}{a_{k-1}(s)a_k(s)} \text{ for } k \geq 1.$$

(See Henrici (1977) for details.)

It is easily checked that

$$a_{2k+1}(s) = \frac{|r_k|}{|r_0|} \frac{A_k(s)^2}{\lambda_{k-1} \dots \lambda_0 \mu_k \dots \mu_1} \quad \text{and} \quad a_{2k} = |r_0| \frac{\lambda_0 \dots \lambda_{k-2} \mu_1 \dots \mu_{k-1}}{A_k(s) A_{k-1}(s)}.$$

For  $k > i_0$ ,  $r_k \geq r_{i_0+1}$  and then by taking into account Equation (44), we deduce that for all  $s > 0$ ,  $\sum_{k=0}^\infty a_k(s) = \infty$  and the continued fraction  $\mathcal{F}(s; z)$  is then converging for all  $s > 0$ . For  $s = 0$ , we have

$$a_{2k}(0) = \frac{|r_0|}{\lambda_{k-1} \pi_{k-1}}$$

and then  $\sum_{k=0}^\infty a_k(0) = \infty$  since the process  $(\Lambda_t)$  is ergodic (see Condition (2)). This shows that the Stieltjes fraction  $\mathcal{F}(s; z)$  is converging for all  $s \geq 0$ .

### B. Selfadjointness properties

We consider in this section the Hilbert space  $H_{i_0} = \mathbb{C}^{i_0+1}$  equipped with the scalar product

$$(c, d)_{i_0} = \sum_{k=0}^{i_0} c_k \bar{d}_k |r_k| \pi_k.$$

The main result of this section is the following lemma.

**Lemma 6.** *For  $s \geq 0$ , the finite matrix  $-R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]})$  defines a selfadjoint operator in the Hilbert space  $H_{i_0}$ ; the spectrum is purely point-wise and composed by the (positive) roots of the polynomial  $Q_{i_0+1}(s; x)$  defined by Equation (8), denoted by  $\zeta_k(s)$  for  $k = 0, \dots, i_0$ .*

*Proof.* The finite matrix  $-R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]})$  is given by

$$\begin{pmatrix} -\frac{s+\lambda_0}{|r_0|} & \frac{\lambda_0}{|r_0|} & 0 & \dots & \dots \\ \frac{\mu_1}{|r_1|} & -\frac{(s+\lambda_1+\mu_1)}{|r_1|} & \frac{\lambda_1}{|r_1|} & \dots & \dots \\ 0 & \frac{\mu_2}{|r_2|} & -\frac{(s+\lambda_2+\mu_2)}{|r_2|} & \frac{\lambda_2}{|r_2|} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \frac{\mu_{i_0}}{|r_{i_0}|} - \frac{s+\lambda_{i_0}+\mu_{i_0}}{|r_{i_0}|} \end{pmatrix}.$$

The symmetry of the matrix with respect to the scalar product  $(\cdot, \cdot)_{i_0}$  is readily verified by using the relation  $\lambda_k \pi_k = \mu_{k+1} \pi_{k+1}$ . Since the dimension of the Hilbert space  $H_{i_0}$  is finite, the operator associated with the matrix  $-R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]})$  is selfadjoint and its spectrum is purely point-wise.

If  $f$  is an eigenvector for the matrix  $-R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]})$  associated with the eigenvalue  $x$ , then under the hypothesis that  $f_0 = 1$ , the sequence  $f_n$  verifies the same recurrence relation as  $Q_k(s; x)$  for  $k = 0, \dots, i_0 - 1$ . This implies that  $x$  is an eigenvalue of the above matrix if and only if  $Q_{i_0+1}(s; x) = 0$ , that is,  $x$  is one of the (positive) zeros of the polynomial  $Q_{i_0+1}(s; x)$ , denoted by  $\zeta_k(s)$  for  $k = 0, \dots, i_0$ .  $\square$

Let us introduce the column vector  $Q_{[i_0]}(s, \zeta_k(s))$  for  $k = 0, \dots, i_0$ , whose  $\ell$ th component is  $Q_\ell(s, \zeta_k(s))$ . The vector  $Q_{[i_0]}(s, \zeta_k(s))$  is the eigenvector associated with the eigenvalue  $\zeta_k(s)$  of the operator  $-R_{[i_0]}^{-1}(s\mathbb{I}_{[i_0]} - A_{[i_0]})$ . From the spectral theorem, the vectors  $Q_{[i_0]}(s, \zeta_k(s))$  for



$k = 0, \dots, i_0$  form an orthogonal basis of the Hilbert space  $H_{i_0}$ . The vectors  $e_j$  for  $j = 0, \dots, i_0$  such that all entries are equal to 0 except the  $j$ th one equal to 1 form the natural orthogonal basis of the space  $H_{i_0}$ . We can moreover write for  $j = 0, \dots, i_0$

$$e_j = \sum_{k=0}^{i_0} \alpha_k^{(j)} Q_{[i_0]}(s, \zeta_k(s)).$$

By using the orthogonality of the vectors  $Q_{[i_0]}(s, \zeta_k(s))$  for  $k = 0, \dots, i_0$ , we have

$$(e_j, Q_{[i_0]}(s, \zeta_k(s)))_{i_0} = |r_j| \pi_j Q_j(s, \zeta_k(s)) = \|Q_{[i_0]}(s, \zeta_k(s))\|_{i_0}^2 \alpha_k^{(j)}$$

where for  $f \in H_{i_0}$ ,  $\|f\|_{i_0}^2 = (f, f)_{i_0}$ . We hence deduce that

$$|r_j| \pi_j \sum_{k=0}^{i_0} \frac{Q_j(s, \zeta_k(s)) Q_\ell(s, \zeta_k(s))}{\|Q_{[i_0]}(s, \zeta_k(s))\|_{i_0}^2} = \delta_{j,\ell},$$

where  $\delta_{j,\ell}$  is the Kronecker symbol. It follows that if we define the measure  $\psi_{[i_0]}(s; dx)$  by

$$\psi_{[i_0]}(s; dx) = |r_0| \sum_{k=0}^{i_0} \frac{1}{\|Q_{[i_0]}(s, \zeta_k(s))\|_{i_0}^2} \delta_{\zeta_k(s)}(dx) \quad (45)$$

the polynomials  $Q_k(s, x)$  for  $k = 0, \dots, i_0$  are orthogonal with respect to the above measure, that is, they verify

$$\int_0^\infty Q_j(s, x) Q_\ell(s, x) \psi_{[i_0]}(s; dx) = \frac{|r_0|}{|r_j| \pi_j} \delta_{j,\ell},$$

and the total mass of the measure  $\psi_{[i_0]}(s; dx)$  is equal to 1, i.e.,

$$\int_0^\infty \psi_{[i_0]}(s; dx) = 1.$$

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