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Topography Optimization of Fluid Mechanics Problems

Maatoug Hassine
ESSTH, Sousse University
Tunisia

1. Introduction

Optimal shape design problems in fluid mechanics have wide and valuable applications in aerodynamic and hydrodynamic problems such as the design of car hoods, airplane wings and inlet shapes for jet engines. One of the first studies is found in Pironneau (1974). It is devoted to determine a minimum drag profile submerged in a homogeneous, steady, viscous fluid by using optimal control theories for distributed parameter systems. Next, many shape optimization methods are introduced to determine the design of minimum drag bodies Kim and Kim (1995); Pironneau (1984), diffusers Cabuk and Modi (1992), valves Lund et al. (2002), and airfoils Cliff et al. (1998). The majority of works dealing with optimal design of flow domains fall into the category of shape optimization and are limited to determine the optimal shape of an existing boundary.

It is only recently that topological optimization has been developed and used in fluid design problems. It can be used to design features within the domain allowing new boundaries to be introduced into the design. In this context, Borvall and Petersson Borrvall and Petersson (2003) implemented the relaxed material distribution approach to minimize the power dissipated in Stokes flow. To approximate the no-slip condition along the solid-fluid interface they used a generalized Stokes problem to model fluid flow throughout the domain. Later, this approach was generalized by Guest and Prévost in Guest and Prévost (2006). They treated the material phase as a porous medium where fluid flow is governed by Darcy’s law. For impermeable solid material, the no-slip condition is simulated by using a small value for the material permeability to obtain negligible fluid velocities at the nodes of solid elements. The flow regularization is expressed as a system of equations; Stokes flow governs in void elements and Darcy flow governs in solid elements.

In this work, we propose a new topological optimization method. Our approach is based on topological sensitivity analysis Amstutz (2005); Amstuts and Masmoudi (2003); Garreau et al. (2001); Guillaume and Hassine (2007); Guillaume and Sid Idris (2004); Hassine et al. (2007); Hassine and Masmoudi (2004); Masmoudi (2002); Sokolowski and Zochowski (1999). The optimal domain is constructed through the insertion of some obstacles in the initial one. The problem leads to optimize the obstacles location. The main idea is to compute the topological asymptotic expansion of a cost function \( j \) with respect to the insertion of a small obstacle inside the fluid flow domain. The obstacle is modeled as a small hole \( O_{z,\varepsilon} \) around a point \( z \) having an homogeneous condition on the boundary \( \partial O_{z,\varepsilon} \). The best location \( z \) of \( O_{z,\varepsilon} \) is given by the most negative value of a scalar function \( \delta j \), called the topological gradient.
In practice, this approach leads to a simple, fast and accurate topological optimization algorithm. The final domain is obtained using an iterative process building a sequence of geometries \( (\Omega_k) \) starting with the initial fluid flow domain \( \Omega_0 = \Omega \). Knowing \( \Omega_k \), the new domain \( \Omega_{k+1} \) is obtained by inserting an obstacle \( O_k \) in the domain \( \Omega_k \): \( \Omega_{k+1} = \Omega_k \setminus O_k \). The location and the shape of \( O_k \) are defined by a level set curve of the topological gradient \( \delta j_k \)

\[
O_k = \{ x \in \Omega_k, \text{ such that } \delta j_k(x) \leq c_k \},
\]

where \( c_k \) is a scalar parameter used to control the size of the inserted obstacle. The function \( \delta j_k \) is the leading term of the variation \( j(\Omega_k \setminus O_k) - j(\Omega_k) \).

The chapter is organized as follows. In the next section, we present the topological optimization problem related to the Stokes system. The aim is to determine the fluid flow domain minimizing a given cost function. To solve this optimization problem we will use the topological sensitivity analysis method described in the Section 3. It consists in studying the variation of a cost function \( j \) with respect to a topology modification of the domain. The most simple way of modifying the topology consists in creating a small hole in the domain. In the case of structural shape optimization, creating a hole means simply removing some material. In the case of fluid dynamics where the domain represents the fluid, creating a hole means inserting a small obstacle \( O \). The topological sensitivity tools which have been developed by several authors Garreau et al. (2001); Schumacher (1995); Sokolowski and Zochowski (1999) allow to find the place where creating a small hole will bring the best improvement of the cost function. The main theoretical results are described in Sections 3.2 and 3.3. In section 3.2, we derive an asymptotic expansion for an arbitrary cost function with respect to the insertion of a small obstacle inside the fluid flow domain. In section 3.3, we derive an asymptotic expansion for two standard examples of cost functions.

As application of the proposed topological optimization method, we consider in Section 4 some engineering applications commonly found in the fluid mechanics literature. In Section 4.1, we present the optimization algorithm. In Section 4.2, we treat the shape optimization of pipes in a cavity. The aim is to determine the optimal shape of the pipes that connect the inlet to the outlets of the cavity minimizing the dissipated power in the fluid. The optimization of injectors location in an eutrophized lake is discussed in Section 4.3. Section 4.4 concerns the approximation of a wanted flow using a topological perturbation of the domain.

2. Topological optimization problem

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^d, d = 2, 3 \) with smooth boundary \( \Gamma \). We consider an incompressible fluid flow in \( \Omega \) described by the Stokes equations. The velocity field \( u \) and the pressure \( p \) satisfy the system

\[
\begin{cases}
-\nu \Delta u + \nabla p = F & \text{in } \Omega \\
d \text{div } u = 0 & \text{in } \Omega \\
 u = 0 & \text{on } \Gamma,
\end{cases}
\]

(1)

where \( \nu \) denotes the kinematic viscosity of the fluid, \( F \) is a given body force per unit of mass (gravitational force).

The aim is to determine the optimal geometry of the fluid flow domain minimizing a given design function \( j \):

\[
\min_{D \in D_{ad}} j(D), \text{ such that } |D| \leq V_{\text{desired}}.
\]
where $j$ has the form

$$j(D) = J(u_D),$$

with $u_D$ is the velocity field solution to the Stokes system in $D$ and $D_{ad}$ is a given set of admissible domains.

Here $|D|$ is the Lebesgue measure of $D$ and $V_{desired}$ denotes the target volume (weight).

To solve this shape optimization problem we shall use the topological sensitivity analysis method. It consists in studying the variation of the objective function $J$ with respect to a small topological perturbation of the domain $\Omega$.

### 2.1 Stokes equations in the perturbed domain

We denote by $\Omega \setminus \overline{O_\varepsilon}$ the perturbed domain, obtained by inserting a small obstacle $O_\varepsilon$ in $\Omega$. We suppose that the obstacle has the form $O_\varepsilon = x_0 + \varepsilon O$, where $x_0 \in \Omega$, $\varepsilon > 0$ and $O$ is a given fixed and bounded domain of $\mathbb{R}^d$, containing the origin, whose boundary $\partial O$ is connected and piecewise of class $C^1$.

In $\Omega \setminus \overline{O_\varepsilon}$, the velocity $u_\varepsilon$ and the pressure $p_\varepsilon$ are solution to

$$
\begin{aligned}
-\nu \Delta u_\varepsilon + \nabla p_\varepsilon &= F \quad \text{in } \Omega \setminus \overline{O_\varepsilon} \\
\text{div } u_\varepsilon &= 0 \quad \text{in } \Omega \setminus \overline{O_\varepsilon} \\
 u_\varepsilon &= 0 \quad \text{on } \Gamma \\
 u_\varepsilon &= 0 \quad \text{on } \partial O_\varepsilon.
\end{aligned}
$$

(2)

Note that for $\varepsilon = 0$, $\Omega_0 = \Omega$ and $(u_0, p_0)$ is solution to

$$
\begin{aligned}
-\nu \Delta u_0 + \nabla p_0 &= F \quad \text{in } \Omega_0 \\
\text{div } u_0 &= 0 \quad \text{in } \Omega_0 \\
 u_0 &= 0 \quad \text{on } \Gamma.
\end{aligned}
$$

(3)

### 2.2 Topological optimization problem

Consider now a design function $j$ of the form

$$j(\Omega \setminus \overline{O_\varepsilon}) = J_\varepsilon(u_\varepsilon),$$

(4)

where $J_\varepsilon$ is a given cost function defined on $H^1(\Omega \setminus \overline{O_\varepsilon})^d$ for $\varepsilon \geq 0$ and $u_\varepsilon$ is the velocity field solution to the Stokes system (2).

Our aim is to determine the optimal location of the obstacle $O_\varepsilon$ in the domain $\Omega$ in order to minimize the design function $j$. Then, the optimization problem we consider is given as follows:

$$\min_{O_\varepsilon \subset \Omega} j(\Omega \setminus \overline{O_\varepsilon}).$$

(5)

To this end, we will derive in the next section a topological asymptotic expansion of the function $j$ with respect to $\varepsilon$.

### 3. Topological sensitivity analysis

In this section we consider a topological sensitivity analysis for the Stokes equations. We present a topological asymptotic expansion of a design function $j$ with respect to the insertion of a small obstacle $O_\varepsilon$ inside the domain $\Omega$. The proposed approach is based on the following general adjoint method.

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3.1 General adjoint method

Let \( (V_\varepsilon)_{\varepsilon \geq 0} \) be a family of Hilbert spaces depending on the parameter \( \varepsilon \), such that, \( \forall \varepsilon \geq 0, V_\varepsilon \mapsto V_0 \). For \( \varepsilon \geq 0 \), we consider

- \( A_\varepsilon : V_\varepsilon \times V_\varepsilon \mapsto \mathbb{R} \) a bilinear, continuous and coercive form on \( V_\varepsilon \),
- \( l_\varepsilon : V_\varepsilon \mapsto \mathbb{R} \) a linear and continuous form on \( V_\varepsilon \).

For all \( \varepsilon \geq 0 \), we denote by \( u_\varepsilon \) the unique solution to the problem

\[
A_\varepsilon(u_\varepsilon, w) = l_\varepsilon(w), \quad \forall w \in V_\varepsilon. \tag{6}
\]

Consider now a cost function of the form \( j(\varepsilon) = J_\varepsilon(u_\varepsilon) \), where \( J_\varepsilon \) is defined on \( V_\varepsilon \) for \( \varepsilon \geq 0 \) and \( J_\varepsilon \) is differentiable with respect to \( u_\varepsilon \), its derivative being denoted by \( DJ_\varepsilon(u_\varepsilon) \). Our aim is to derive an asymptotic expansion of \( J \) with respect to \( \varepsilon \). We consider the following assumptions.

**Hypothesis 3.1.** There exist a real number \( \delta A \) and a scalar function \( f : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) such that \( \forall \varepsilon \geq 0 \)

\[
A_\varepsilon(u_\varepsilon - u_0, v_0) = f(\varepsilon)\delta A + o(f(\varepsilon)),
\]

\[
\lim_{\varepsilon \to 0} f(\varepsilon) = 0,
\]

where \( v_0 \in V_0 \) is the solution to the adjoint problem

\[
A_0(w, v_0) = -DJ_0(u_0)w, \quad \forall w \in V_0. \tag{7}
\]

**Hypothesis 3.2.** There exists a real number \( \delta J \) such that \( \forall \varepsilon \geq 0 \)

\[
J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = DJ_\varepsilon(u_0)(u_\varepsilon - u_0) + f(\varepsilon)\delta J + o(f(\varepsilon)).
\]

Under the assumptions 3.1 and 3.2, we have the following theorem.

**Theorem 3.1.** Hassine et al. (2008) If the assumptions 3.1 and 3.2 hold, the function \( J \) has the following asymptotic expansion

\[
j\varepsilon(\varepsilon) = j(0) + f(\varepsilon)\left(\delta A + \delta J\right) + o(f(\varepsilon)).
\]

3.2 Topological sensitivity for the Stokes problem

In this section, we derive a topological asymptotic expansion for the Stokes equations. In order to apply the adjoint method described in the previous paragraph, first we establish a variational problem associated to the Stokes system. From the weak variational formulation of (2), we deduce that \( u_\varepsilon \in V_\varepsilon \) is solution to

\[
A_\varepsilon(u_\varepsilon, w) = l_\varepsilon(w), \quad \forall w \in V_\varepsilon,
\]

where the functional space \( V_\varepsilon \), the bilinear form \( A_\varepsilon \) and the linear form \( l_\varepsilon \) are defined by

\[
V_\varepsilon = \left\{ w \in H^1_0(\Omega_\varepsilon), \ \text{div}\ w = 0 \ \text{in } \Omega_\varepsilon \right\}, \tag{8}
\]

\[
A_\varepsilon(v, w) = \nu \int_{\Omega_\varepsilon} \nabla v \cdot \nabla w \ dx, \quad \forall u, v \in V_\varepsilon, \tag{9}
\]

\[
l_\varepsilon(w) = \int_{\Omega_\varepsilon} F w \ dx, \quad \forall w \in V_\varepsilon, \tag{10}
\]

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where $\Omega_\epsilon = \Omega \setminus \overline{O}_\epsilon$.

Next we have to distinguish the cases $d = 2$ and $d = 3$, because the fundamental solutions to the Stokes equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ have an essentially different asymptotic behaviour at infinity.

### 3.2.1 The three dimensional case

Let $(U, P)$ denote a solution to

$$
\begin{cases}
-\nu \Delta U + \nabla P = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \\
\text{div } U = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \\
U \rightarrow 0 & \text{at } \infty \\
U = -u_0(x_0) & \text{on } \partial \Omega.
\end{cases}
$$

The existence of $(U, P)$ is most easily established by representing it as a single layer potential on $\partial \Omega$ (see Dautray and Lions (1987))

$$
U(y) = \int_{\partial \Omega} E(y - x) \eta(x) \, ds(x), \quad P(y) = \int_{\partial \Omega} \Pi(y - x) \eta(x) \, ds(x), \quad y \in \mathbb{R}^3 \setminus \overline{\Omega}
$$

where $(E, \Pi)$ is the fundamental solution of the Stokes equations

$$
E(y) = \frac{1}{8\pi \nu} \left( I + \epsilon_r e_r^T \right), \quad \Pi(y) = \frac{y}{4\pi \nu}
$$

with $r = ||y||$, $\epsilon_r = y/r$ and $e_r^T$ is the transposed vector of $e_r$. The function $\eta \in H^{-1/2}(\partial \Omega)^3$ is the solution to the boundary integral equation,

$$
\int_{\partial \Omega} E(y - x) \eta(x) \, ds(x) = -u_0(x_0), \quad \forall y \in \partial \Omega.
$$

One can observe that the function $\eta$ is determined up to a function proportional to the normal, hence it is unique in $H^{-1/2}(\partial \Omega)^3 / \mathbb{R}^n$.

We start the derivation of the topological asymptotic expansion with the following estimate of the $H^1(\Omega_\epsilon)$ norm of $u_\epsilon(x) - u_0(x) - U(x/\epsilon)$. This estimate plays a crucial role in the derivation of our topological asymptotic expansion. It describes the velocity perturbation caused by the presence of the small obstacle $O_\epsilon$.

**Proposition 3.1.** Guillaume and Hassine (2007); Hassine et al. (2008) There exists $c_\epsilon > 0$, independent on $\epsilon$, such that for all $\epsilon > 0$ we have

$$
\|u_\epsilon(x) - u_0(x) - U(x/\epsilon)\|_{H^1(\Omega_\epsilon)} \leq c_\epsilon
$$

The following corollary follows from Proposition 3.1. It gives the behaviour of the velocity $u_\epsilon$ when inserting an obstacle. The principal term of this perturbation is given by the function $U$, solution to (11).

**Corollary 3.1.** We have

$$
u_\epsilon(x) = u_0(x) + U(x/\epsilon) + O(\epsilon), \quad x \in \Omega_\epsilon.
$$

We are now ready to derive the topological asymptotic expansion of the cost function $j$. It consists in computing the variation $j(\Omega \setminus \overline{O}_\epsilon) - j(\Omega)$ when inserting a small obstacle inside the domain. The leading term of this variation involves the function $\eta$, the solution to the boundary integral equation (12). The main result is described by Theorem 3.2.
Theorem 3.2. Guillaume and Hassine (2007); Hassine et al. (2008) If Hypothesis 3.1 holds, the function \( j \) has the following asymptotic expansion
\[
j(\Omega \setminus \overline{\mathcal{O}}_\varepsilon) = j(\Omega) + \varepsilon \delta j(x_0) + o(\varepsilon).
\]
where the topological gradient \( \delta j \) is given by
\[
\delta j(x) = \left( - \int_{\partial \mathcal{O}} \eta(y) \, ds(y) \right) \cdot v_0(x) + \delta f(x), \quad x \in \Omega.
\]
If \( \mathcal{O} \) is the unit ball centred at the origin, \( \mathcal{O} = B(0,1) \), the density \( \eta \) is given explicitly
\[
\eta(y) = -3 \nu |u_0(x_0)|, \quad \forall y \in \partial \mathcal{O}.
\]
Corollary 3.2. If \( \mathcal{O} = B(0,1) \), under the hypotheses of theorem 3.2, we have
\[
j(\Omega \setminus \overline{\mathcal{O}}_\varepsilon) = j(\Omega) + \varepsilon \left[ 6 \pi \nu u_0(x_0) \cdot v_0(x_0) + \delta f(x_0) \right] + o(\varepsilon).
\]

3.2.2 The two dimensional case
In this paragraph, we present the topological asymptotic expansion for the Stokes equations in the two dimensional case. The result is obtained using the same technique described in the previous paragraph. The unique difference comes from the expression of the fundamental solution of the Stokes equations. In this case, \( (E, \Pi) \) is given by
\[
E(y) = \frac{1}{4 \pi \nu} \left( - \log(r) I + e_r e_r^T \right), \quad \Pi(y) = \frac{y}{2 \pi r^2}.
\]

Theorem 3.3. Guillaume and Hassine (2007); Hassine et al. (2008) Under the same hypotheses of theorem 3.2, the function \( j \) has the following asymptotic expansion
\[
j(\Omega \setminus \overline{\mathcal{O}}_\varepsilon) = j(\Omega) + \frac{-1}{\log(\varepsilon)} \delta j(x_0) + o\left( \frac{-1}{\log(\varepsilon)} \right),
\]
where the topological gradient \( \delta j \) is given by
\[
\delta j(x) = 4 \pi \nu u_0(x) \cdot v_0(x) + \delta f(x), \quad x \in \Omega.
\]

3.3 Cost function examples
We now discuss Assumption 3.2. We present two standard examples of cost functions satisfying this Assumption and we calculate their variations \( \delta f \). For the proofs one can see Guillaume and Hassine (2007) or Hassine et al. (2008).

Proposition 3.2. Let \( w_d \in H^1(\Omega) \) be a given wanted (objective) velocity field. The cost function
\[
J_\varepsilon(u) = \int_{\Omega \setminus \overline{\mathcal{O}}_\varepsilon} |u - w_d|^2 \, dx,
\]
(13)
satisfies the assumption 3.1 with
\[
D J_\varepsilon(w) = 2 \int_{\Omega} (u_0 - w_d) \cdot w \, dx, \quad \forall w \in \mathcal{V}_0, \text{ and } \delta f(x_0) = 0.
\]
Proposition 3.3. Let \( w_d \in H^2(\Omega) \). The cost function

\[
J_c(u) = v \int_{\Omega \setminus \partial \Gamma} |\nabla u - \nabla w_d|^2 \, dx,
\]

satisfies the assumption 3.1 with

\[
D J_0(w) = 2 \int \nabla (u_0 - w_d) \cdot \nabla w \, dx \quad \forall w \in V_0,
\]

\[
\delta f(x_0) = \begin{cases} \left( - \int_{\partial \Omega} \eta(y) \, dx(y) \right) \cdot u_0(x_0) & \text{if } d = 3, \\ 4\pi v |u_0(x_0)|^2 & \text{if } d = 2. \end{cases}
\]

For \( d=3 \), if \( \Omega \) is the unit ball \( B(0,1) \), we have \( \delta f = 6\pi v |u_0(x_0)|^2 \).

4. Numerical experiments

As an application of the previous theoretical results, we consider some engineering applications commonly found in the fluid mechanics literature. Our implementation is based on the following optimization algorithm.

4.1 The optimization algorithm

We apply an iterative process to build a sequence of geometries \( (\Omega_k)_{k \geq 0} \) with \( \Omega_0 = \Omega \). At the \( k^{th} \) iteration the topological gradient is denoted by \( \delta j_k \) and the new geometry \( \Omega_{k+1} \) is obtained by inserting an obstacle \( \mathcal{O}_k \) in the domain \( \Omega_k \); \( \Omega_{k+1} = \Omega_k \setminus \mathcal{O}_k \). The location and the size of the obstacle \( \mathcal{O}_k \) are chosen in such a way that \( j(\Omega_{k+1}) - j(\Omega_k) \) is negative.

Based on the last remark, the obstacle \( \mathcal{O}_k \) is defined by a level set curve of the topological gradient \( \delta j_k \)

\[
\mathcal{O}_k = \left\{ x \in \Omega_k, \text{ such that } \delta j_k(x) \leq c_k \leq 0 \right\},
\]

where \( c_k \) is chosen in such a way that \( |\mathcal{O}_k| / |\Omega_k| \) is less than a given ratio \( \delta \in ]0, 1[ \).

The algorithm : Topology optimization with volume constraint.

- Initialization: choose \( \Omega_0 = \Omega \), and set \( k = 0 \).
- Repeat until \( |\mathcal{O}_k| \leq V_{\text{desired}} \):
  - compute \( u_k \) the solution to the Stokes equations (15) in \( \Omega_k \),
  - compute \( v_k \) the solution to the associated adjoint problem (16) in \( \Omega_k \),
  - compute the topological sensitivity \( \delta j_k(z) \), \( \forall z \in \mathcal{O}_k \),
  - determine \( \Omega_{k+1} = \Omega_k \setminus \mathcal{O}_k \), where \( \mathcal{O}_k = \left\{ x \in \Omega_k, \text{ such that } \delta j_k(x) \leq c_k \leq 0 \right\} \),
  - \( k \leftarrow k + 1 \).

The topological gradient \( \delta j_k \) is defined by

\[
\delta j_k(z) = u_k(z) \cdot v_k(z) + \delta j^a_k(z), \quad \forall z \in \Omega_k,
\]

where \( u_k \) is the velocity field solution to

\[
\begin{cases} -v \Delta u_k + \nabla p_k = F & \text{in } \Omega_k, \\ \text{div } u_k = 0 & \text{in } \Omega_k, \end{cases}
\]
and \( v_k \) is the solution to the associated adjoint problem

\[
\begin{aligned}
& -v \Delta v_k + \nabla q_k = -D J(u_k) \quad \text{in } \Omega_k \\
& \text{div } v_k = 0 \quad \text{in } \Omega_k.
\end{aligned}
\]  

(16)

The discretization of the problems (15) and (16) is based on the mixed finite element method \( P_1 + \text{bubble}/P_1 \) Arnold et al. (1984). The function \( \delta f_k \) is computed piecewise constant over elements. The term \( \delta f_k \) is the variation of the considered cost function \( J \) (see Propositions 3.2 and 3.3). The constant \( c_k \) determines the volume of the obstacle \( O_k \) to be inserted. In practice, \( c_k \) is chosen in such a way that:

i- \( O_k \subset \{ x \in \Omega_k, \text{ such that } \delta f_k(x) \leq 0 \} \),

ii- the obstacle volume \( |O_k| \) is less or equal to 10% of the current domain volume \( |\Omega_k| \) i.e. \( |O_k|/|\Omega_k| \leq 0.1 \).

This algorithm can be seen as a descent method where the descent direction is determined by the topological sensitivity \( \delta f_k \) and the step length is given by the volume variation \( |\Omega_k \setminus \Omega_{k+1}| \).

4.2 Pipes shape optimization

We consider a viscous and incompressible fluid in a tank \( \Omega \) having one inlet \( \Gamma_{in} \) and some outlets \( \Gamma_{out}^i, 1 \leq i \leq m \). The aim is to determine the optimal design of the pipes that connects the inlet to the outlet of the domain minimizing the dissipated power in the fluid.

4.2.1 Comparison

In order to test the advantage of our approach, we compare our results to those obtained in Borrvall and Petersson (2003); Glowinski and Pironneau (1975). We consider two numerical examples in two dimensional (2D) case. The first one is the pipe bend example presented in Figure 1. This test case is treated by Borrvall and Pettersson in Borrvall and Petersson (2003). The second one is the double pipe shown in Figure 2. It is also considered by Borrvall and Pettersson in Borrvall and Petersson (2003) and recently by Guest and Prévost in Glowinski and Pironneau (1975). The aim here is to obtain the optimal shape minimizing the dissipated power in the fluid.

The considered design function is given by

\[
J(D) = \int_D |\nabla u_D|^2 \, dx,
\]

where \( u_D \) is the solution to the Stokes system in \( D \).

The optimization problem consists in finding the fluid flow domain solution to

\[
\min_{D \in D_{ad}} J(D), \text{ such that } |D| \leq V_{desired}
\]

where \( D_{ad} \) is the set of admissible domains defined by

\[
D_{ad} = \{ D \subset \Omega \text{ such that } \Gamma_{in} \subset \partial \Omega \cap \partial D \text{ and } \Gamma_{out}^i \subset \partial \Omega \cap \partial D \}.
\]

In both cases the inflow and the outflow conditions are given by a parabolic flow profile type with a maximum flow velocity equal to 1. Elsewhere the velocity is prescribed to be zero on the boundary of the domain.

A- Test 1 : 2D pipe bend example. We consider a cavity \( \Omega = ]0,1[ \times ]0,1[ \) having one inlet (left) and one outlet (bottom) (see figure 1(a)).
The cavity $\Omega$ is discretized using a finite elements mesh with 6561 nodes and 12800 triangular elements. The results of this example are presented in figure 1. The obtained pipe geometry is described in figure 1(b). It is computed using $V_{desired} = 0.08 \pi |\Omega|$. The prescribed volume constraint is chosen so that the optimal solution has the same volume as a quarter torus of inner radius 0.7 and outer radius 0.9 that exactly fits to the inlet and outlet. In figure 1(c) we present the velocity field computed in the final domain.

The obtained solution is nearly identical to those presented in Borrvall and Petersson (2003). However, we obtain this result in 14 iterations, where Borrvall and Petersson needed more than sixty. As it can be seen, we have a more torus shaped pipe than in Borrvall and Petersson (2003), like most pipe bends in fluid mechanics literature. As it is stated in Glowinski and Pironneau (1975), the solution in Borrvall and Petersson (2003) contains regions of artificial material and does not sufficiently take into account the adherence condition.

**B- Test 2: 2D double pipe example.** The initial domain of this example is shown in Figure 2(a). It is the rectangular $\Omega = [0, 3/2] \times [0, 1]$ with two inlets and two outlets.

The cavity $\Omega$ is discretized using a finite elements mesh with 9801 nodes and 19200 triangular elements. The results of this example are presented in figure 2. The final geometry is computed with $V_{desired} = \frac{1}{3} |\Omega|$. 

![Fig. 1. 2D pipe bend example.](image1)

![Fig. 2. The initial and the optimal domains for the 2D double pipe example.](image2)
We present in figure 2(b) the obtained geometry. The final geometry is obtained in only 12 iterations, where Borrvall and Petersson needed more than sixty. We remark that the two pipes join to form a single, wider pipe through the center of the domain. This design decreases the length of the fluid-solid interface by decreasing the power lost. As it can be seen, the optimal solution is identical to that obtained by Guest and Prévost Glowinski and Pironneau (1975), but it does not match that of Borrvall and Petersson (2003). As for the pipe bend example, the solution in Borrvall and Petersson (2003) contains regions of artificial material and does not sufficiently take into account the adherence condition.

### 4.2.2 Three dimensional case

In this section we propose an extension of the two 2D examples considered in the last section to the three dimensional case.

**A- Example 1 : 3D pipe bend example.** For the 3D pipe bend example, the initial domain is the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ having one inlet and one outlet (see figure 3). The inlet $\Gamma_{\text{in}}$ (left) and the outlet $\Gamma_{\text{out}}$ (bottom) are described by the following discs

$$
\Gamma_{\text{in}} = B(z_{\text{in}}, 0.1) \cap \{0\} \times [0, 1] \times [0, 1], \text{ and } \Gamma_{\text{out}} = B(z_{\text{out}}, 0.1) \cap [0, 1] \times [0, 1] \times \{0\},
$$

where $B(z_{\beta}, 0.1)$, $\beta = \text{in, out}$, is the ball of center $z$ and radius 0.1, with $z_{\text{in}} = (0, 0.5, 0.8)$ and $z_{\text{out}} = (0.8, 0.5, 0)$.

![Fig. 3. The initial domain](image)

For the boundary conditions, we consider a parabolic flow profile type with a maximum flow velocity equal to 1 on $\Gamma_{\text{in}}$ and $\Gamma_{\text{out}}$, and a velocity equal to zero elsewhere. The domain is discretized using 29791 nodes and 162000 tetrahedral elements.

Like in the 2D case, we aim to determine the optimal design of the pipe that connects the inlet to the outlet of the domain and minimizes the dissipated power in the fluid. We present in figure 4 the optimal pipe domains obtained for different volume constraint $V_{\text{desired}}$. The first case (figure 4(a)), corresponding to $V_{\text{desired}} = 0.50 |\Omega|$, is obtained after 7 iterations, the second one (figure 4(b)) after 11 iterations for $V_{\text{desired}} = 0.35 |\Omega|$ and the last one (figure 4(b)) needs 16 iterations to reach $V_{\text{desired}} = 0.20 |\Omega|$. We show in figure 5 a 2D cut of the velocity field corresponding to the three obtained domains.

**B- Example 2 : 3D double pipe bend example.** The initial domain is the cavity $\Omega = [0, 3/2] \times [0, 1] \times [0, 1]$ (described in figure 6). It has two inlets (left) $\Gamma_{\text{in}}^{i} = 1, 2$, and two outlets...
4.2.3 Shape optimization of tubes in a 3D cavity

In this section we treat the shape optimization of tubes in a cavity. We consider an incompressible fluid in a cavity $\Omega$ having one inlet $\Gamma_{in}$ and four outlets $\Gamma_{out}^i, i = 1, 4$. The aim here is to determine the optimal shape of the tubes that connect the inlet to the outlets of the cavity maximizing the outflow rate. It consists in inserting small obstacles in the cavity in order to maximize the outflow rate at $\Gamma_{out}^i, i = 1, 4$.
Fig. 6. The initial domain

(a) $V_{desired} = 0.40 \mid \Omega \mid$. (b) $V_{desired} = 0.10 \mid \Omega \mid$.

Fig. 7. The optimal domains (see Abdelwahed, Hassine and Masmoudi (2009)).

Fig. 8. 2D vertical cut of the velocity isovalues and field in the optimal domains.

In our numerical computation, we have used the cavity $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ with the inlet $\Gamma_{in}$:

$\Gamma_{in} = \left\{ (x, y, z) \in \Omega \text{ such that } x^2 + (y - 0.5)^2 + (z - 0.5)^2 \leq 0.04 \right\}$

and the four outlets $\Gamma^1_{out}$, $\Gamma^2_{out}$, $\Gamma^3_{out}$ and $\Gamma^4_{out}$:

$\Gamma^1_{out} = \left\{ (x, y, z) \in \Omega \text{ such that } (x - 0.75)^2 + (y - 0.5)^2 + z^2 \leq 0.0025 \right\}$,

$\Gamma^2_{out} = \left\{ (x, y, z) \in \Omega \text{ such that } (x - 0.75)^2 + (y - 0.5)^2 + (z - 1)^2 \leq 0.0025 \right\}$,

$\Gamma^3_{out} = \left\{ (x, y, z) \in \Omega \text{ such that } (x - 0.75)^2 + y^2 + (z - 0.5)^2 \leq 0.0025 \right\}$,

$\Gamma^4_{out} = \left\{ (x, y, z) \in \Omega \text{ such that } (x - 0.75)^2 + (y - 1)^2 + (z - 0.5)^2 \leq 0.0025 \right\}$.

The considered cost function measuring the outflow rate is given by

$$j(D) = \sum_{i=1}^{m} \int_{\Gamma_{out}} |u_D \cdot n| \, ds,$$
where $D \in D_{ad}$ and $u_D$ is the velocity field, solution to the Stokes equations in $D$ satisfying the following boundary conditions:

- A free surface boundary condition on the outlets

\[
\sigma(u) \cdot n = 0 \quad \text{on } \bigcup_{i=1}^{m} \Gamma_{out}^i,
\]

where $\sigma(u) = \nu (\nabla u + \nabla u^T) - pI$, $I$ is the $3 \times 3$ identity matrix and $n$ denotes the outward normal to the boundary.

- The normal component of the stress tensor is prescribed on the inlet $\Gamma_{in}$

\[
\sigma(u) \cdot n = g \quad \text{on } \Gamma_{in},
\]

- The velocity is equal to zero on $\Gamma \setminus (\bigcup_{i=1}^{m} \Gamma_{out}^i \cup \Gamma_{in})$.

![Fig. 9. The initial domain.](image)

The results of this example are described in figures 10-13. In figure 10 we present the obtained geometries for different volume constraints. We present the obtained geometry: in figure 10(a) for $V_{desired} = 0.35 |\Omega|$, in figure 10(b) for $V_{desired} = 0.25 |\Omega|$ and in figure 10(b) for $V_{desired} = 0.15 |\Omega|$. This domains are obtained respectively after 10, 14 and 19 iterations. The associated velocities fields are given in figures 11 and 12. In figure 13 we illustrate the variation of the outflow rate.

![Fig. 10. The optimal domains (see Abdelwahed, Hassine and Masmoudi (2009)).](image)
Fig. 11. 2D vertical cut of the velocity field.

Fig. 12. 2D horizontal cut of the velocity field.

Fig. 13. Variation of the outflow rate

4.3 Optimization of injectors location in an eutrophic lake

Eutrophication is a complex phenomena involving many physico-chemical parameters. Specifically in some climatic areas, the thermic factors combined to the biological and to the biochemical ones are dominant in the behavior of the aquatic ecosystems. Consequently, they generate important bio-climatology variations creating in lakes an unsteady dynamic process that decreases progressively water quality. Practically, the eutrophication in a water basin is characterized mainly by a poor dissolved oxygen concentration in water. Furthermore, this phenomena is accompanied by a stratification process dividing the water volume, during a large period of the year, into three distinct layers as depicted in Figure 14.

Three zones constitute this stratification:
Fig. 14. (a): Structure of a stratified lake, (b): average temperature curve during summer

i) at the top, the epilimnion, a layer of around 7 m depth, well mixed by the effect of drafting wind and consequently well aerated,

ii) in the middle, the thermocline, a zone with a quick decrease of temperature (27 °C to 18 °C) and of 5 m depth. This area is weakly affected by the wind action and consequently a medium concentration of oxygen is observed,

iii) at the bottom, the hypolimnion, a deeper layer beyond 12 m, having a temperature varying from 18 °C to 14 °C. This region is characterized by a low concentration of oxygen and a high concentration of toxic gas (H₂S, ammoniac, carbonic gas, etc.)

The dynamic aeration process seems to be the most promising remedial technique to treat water eutrophication. This technique consists in inserting air by the means of injectors located at the bottom of the lake in order to generate a vertical motion mixing up the water of the bottom with that in the top, thus oxygenating the lower part by bringing it in contact with the surface air.

Theoretically, the bubble flow is a multi-phase flow where the presence of free interfaces raises difficulties in both the physical and mathematical modelling. Hence, to obtain a physical and significant resolution by numerical simulation of the air injection phenomena in an eutrophised lake, one should consider a two-phase model: water-air bubble (see Ishii Ishii (1975)). This kind of modelling involves large systems of PDE’s and variables in a multi-scale frame as well as closure conditions through turbulence model and phases interface interaction. Moreover, the domain mesh size should be “small” in order to capture the significant variations of the spectrum. Therefore, the computational cost should be also addressed.

For all these reasons, we consider here, as a first approximation, only the liquid phase, which is the dominant one. The flow is described by a simplified model based on incompressible Stokes equations. The injected air is taken into account through local boundary conditions for the velocity on the injectors holes. In order to generate the best motion in the fluid with respect to the aeration purpose, the topological sensitivity analysis method is used to optimize the injectors location.

4.3.1 Optimization problem

In this section, we use the topological sensitivity analysis method to optimize the injector locations in the lake Ω in order to generate the best motion in the fluid with respect to the aeration purpose. To this end, each injector lnjk is modeled as a small hole \( B_{z_k} = z_k + \epsilon B^k \), 1 ≤ k ≤ m having an injection velocity \( u_{inj}^k \), where \( \epsilon \) is the shared diameter and \( B^k \subset \mathbb{R}^d \) are bounded and
smooth domains containing the origin. The points \( z_k \in \Omega, 1 \leq k \leq m \) determine the location of the injectors. Then, in the presence of injectors, the velocity \( u_\epsilon \) and the pressure \( p_\epsilon \) satisfy the following system

\[
\begin{aligned}
-\nu \Delta u_\epsilon + \nabla p_\epsilon &= F \quad \text{in } \Omega \setminus \bigcup_{k=1}^{m} \overline{B_{z_k,\epsilon}} \\
\text{div} u_\epsilon &= 0 \quad \text{in } \Omega \setminus \bigcup_{k=1}^{m} \overline{B_{z_k,\epsilon}} \\
u u_\epsilon &= u_d \quad \text{on } \Gamma \\
u u_\epsilon &= u_{ij}^k \quad \text{on } \omega_{k=1}^{m} \partial B_{z_k,\epsilon},
\end{aligned}
\]

(17)

where \( u_{ij}^k \) is a given injection velocity on \( \partial B_{z_k,\epsilon}, 1 \leq k \leq m. \)

Fig. 15. The geometry of the lake.

Concerning the optimization criteria, we assume that a “good” lake oxygenation can be described by a target velocity \( U_\beta \). Then, the cost function \( J_\epsilon \) to be minimized is defined by

\[
J_\epsilon(u_\epsilon) = \int_{\Omega_m} |u_\epsilon - U_\beta|^2 \, dx,
\]

(18)

where \( \Omega_m \subset \Omega \) is the measurement domain (the top layer, see Figure 15).

Consider the design function \( j \) of the form

\[
j(O_m \cup_{k=1}^{m} \overline{B_{z_k,\epsilon}}) = J_\epsilon(u_\epsilon),
\]

(19)

Our identification problem can be formulated as a topological optimization problem one. It consists in finding the optimal location of the holes \( B_{z_k,\epsilon} = z_k + \epsilon B, 1 \leq k \leq m, \) inside the domain \( \Omega \) in order to minimize the optimal design function \( j \).

To solve this optimization problem \( (O_k) \) we have used the topological sensitivity analysis method. It consists in studying the variation of the design function \( j \) with respect to the presence of a small injector \( B_{z,\epsilon} = z + \epsilon B \) in the lake \( \Omega \).

4.3.2 Numerical results

We propose an adaptation of the previous algorithm to our context. At the \( k^{th} \) iteration, the topological gradient \( \delta j_k \) is given by

\[
\delta j_k(z) = \left( u_k(x) - u_{ij}^k \right) \cdot v_k(x), \quad \forall z \in \Omega_k
\]

(20)

where \( u_k \) and \( v_k \) are, respectively, solutions to the direct and adjoint problems in \( \Omega_k \).

We consider the set \( \{ x \in \Omega_k : \delta j_k(x) < c_{k+1} \} \). Each connected component of this set is a hole created by the algorithm. Our idea is to replace each hole by an injector located at the local minimum of \( \delta j_k(x) \). The obtained results are described in figures 16 and 17.
4.4 Geometrical control of fluid flow

We consider a tank $\Omega$ filled with a viscous and incompressible fluid. The aim is to determine the optimal shape of the fluid flow domain minimizing a given objective function.

4.4.1 Approximation of a desired flow

The aim is to determine the optimal shape $O^* \subset \Omega$ of the fluid flow domain such that the velocity $u_{O^*}$, solution to the Stokes equations in $O^*$, approximate a desired flow $w_d$ defined in a fixed domain $\Omega_m$. The optimal shape $O^*$ can be characterized as the solution to the following topological optimization problem

$$\min_{D \subset \Omega} \int_{\Omega_m} |u_D - w_d|^2 \, dx,$$

where $u_D$ is the solution to the Stokes equations in $D \subset \Omega$. This test is treated in two and three dimensional cases. In 2D, the tank $\Omega = [0, 1.5] \times [0, 1]$, the domain $\Omega_m = [0, 1.5] \times [0.8, 1]$ and the velocity field $w_d$ is defined by

$$w_d = \begin{cases} 
(1, 0) & \text{in } \Omega_m, \\
(0, 0) & \text{elsewhere}.
\end{cases}$$
The numerical results are described in Figure 18. A 3D extension of this case is presented in Figure 19.

Fig. 18. Approximation of a desired flow: 2D case

4.4.2 Maximizing velocity in a fixed zone

Here the aim is to maximize the fluid flow velocity in $\Omega_m = \bigcup_k \Omega_m^k \subset \Omega$ (fixed zones) using a topological perturbation of the domain. The optimal domain of the fluid flow can be characterized as a solution to the following problem

$$\max_{\mathcal{O} \subset \Omega} \int_{\Omega_m} |u_{\mathcal{O}}|^2 \, dx,$$

where $u_{\mathcal{O}}$ is the solution to the Stokes equations in $\mathcal{O}$.

Two 3D test cases are considered. The first case is described in Figure 20. The inflow $\Gamma_{in}$ and the outflow $\Gamma_{out}$ (see Figure 20(a)) are defined by $\Gamma_{in} = [0, 1.5] \times 0 \times [0.4, 0.6]$, $\Gamma_{out} = [0, 1.5] \times 0 \times [0.4, 0.6]$. The domain $\Omega_m = \Omega_1 \cup \Omega_2^2$, with $\Omega_1 = [0, 1.5] \times [0, 1] \times [0.9, 1]$ and $\Omega_2 = [0, 1.5] \times [0, 1] \times [0, 0.1]$. The optimal domain (see Figure 20(c)) is obtained in four iterations.

The second case is described in Figure 21. Here we have used the same 3D tank considered in the last case but with different $\Gamma_{in}$, $\Gamma_{out}$ and $\Omega_m$ (see Figure 21(a)). The optimal domain (see Figure 21(c)) is obtained in five iterations.
Fig. 19. Approximation of a desired flow: 3D case

Fig. 20. Maximizing velocity in a fixed zone: first case (see Abdelwahed and Hassine (2009))
5. Conclusion

In this chapter we have proposed an accurate and fast topological optimization algorithm. The optimal domain is obtained iteratively by inserting some obstacles at each iteration. The location and size of the obstacles are described by a scalar function called the topological gradient. The topological gradient is derived as the leading term of the cost function variation with respect to the insertion of a small obstacle in the fluid flow domain.

The proposed method has two main features. The first one concerns its mathematical framework. The topological sensitivity analysis can be adapted for various operators like elasticity, Helmholtz, Maxwell, Navier Stokes, ...

The second interesting feature of the approach is that it leads to a fast and accurate numerical algorithm. Only a few iterations are needed to construct the final domain. It is easy to be implemented and can be used for many applications. At each iteration we only need to solve the direct and the adjoint problems on a fixed grid.

6. References


Whereas the field of Fluid Mechanics can be described as complicated, mathematically challenging, and esoteric, it is also imminently practical. It is central to a wide variety of issues that are important not only technologically, but also sociologically. This book highlights a cross-section of methods in Fluid Mechanics, each of which illustrates novel ideas of the researchers and relates to one or more issues of high interest during the early 21st century. The challenges include multiphase flows, compressibility, nonlinear dynamics, flow instability, changing solid-fluid boundaries, and fluids with solid-like properties. The applications relate problems such as weather and climate prediction, air quality, fuel efficiency, wind or wave energy harvesting, landslides, erosion, noise abatement, and health care.

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