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A Statistical Derivation of Non-Relativistic Quantum Theory

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1. Introduction

Quantum theory (QT) may either be defined by a set of axioms or otherwise be 'derived' from classical physics by using certain assumptions. Today, QT is frequently identified with a set of axioms defining a Hilbert space structure. This mathematical structure has been created (by von Neumann) by abstraction from the linear solution space of the central equation of QT, the Schrödinger equation. Thus, deriving Schrödinger's equation is basically the same as deriving QT. To derive the most general version of the time-dependent Schrödinger equation, describing N particles with spin in an external gauge field, means to derive essentially the whole of non-relativistic QT.

The second way of proceeding is sometimes called 'quantization'. In the standard (canonical) quantization method one starts from a classical Hamiltonian whose basic variables are then 'transformed', by means of well-known correspondence rules,

$$p \to \frac{\hbar}{\imath} \frac{\mathrm{d}}{\mathrm{d}x}, \quad E \to -\frac{\hbar}{\imath} \frac{\mathrm{d}}{\mathrm{d}t}, \tag{1}$$

into operators. Then, all relevant classical observables may be rewritten as operators acting on states of a Hilbert space etc; the details of the 'derivation' of Schrödinger's equation along this lines may be found in many textbooks. There are formal problems with this approach which have been identified many years ago, and can be expressed e.g. in terms of Groenewold's theorem, see Groenewold (1946), Gotay (1999). Even more seriously, there is no satisfactory *explanation* for this 'metamorphosis' of observables into operators. This quantization method (as well as several other mathematically more sophisticated versions of it) is just a *recipe* or, depending on one's taste, "black magic", Hall (2005). Note that the enormous success of this recipe in various contexts - including field quantization - is no substitute for an explanation.

The choice of a particular quantization procedure will be strongly influenced by the preferred interpretation of the quantum theoretical formalism. If QT is interpreted as a theory describing individual events, then the Hamiltonian of classical mechanics becomes a natural starting point. This 'individuality assumption' is an essential part of the dominating 'conventional', or 'Copenhagen', interpretation (CI) of QT. It is well-known, that QT becomes a source of mysteries and paradoxes[1] whenever it is interpreted in the sense of CI, as a (complete) theory for individual events. Thus, the canonical quantization method and the CI are similar in

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[1] I cannot report here a list, all the less a description, of all the quantum mechanical paradoxes found in the last eighty years.

two respects: both rely heavily on the concept of individual particles and both are rather mysterious.

This situation confronts us with a fundamental alternative. Should we accept the mysteries and paradoxes as inherent attributes of reality or should we not, instead, critically reconsider our assumptions, in particular the 'individuality assumption'. As a matter of fact, the dynamical numerical output of quantum mechanics consists of *probabilities*. A probability is a "deterministic" prediction which can be verified in a statistical sense only, i.e. by performing experiments on a large number of identically prepared individual systems, see Belinfante (1978), Margenau (1963). Therefore, the very structure of QT tells us that it is a theory about statistical ensembles only, see Ballentine (1970). If dogmatic or philosophical reasons 'force' us to interpret QT as a theory about individual events, we have to create complicated intellectual constructs, which are not part of the physical formalism, and lead to unsolved problems and contradictions.

The present author believes, like several other physicists [see e.g. Ali (2009); Ballentine (1970); Belinfante (1975); Blokhintsev (1964); Einstein (1936); Kemble (1929); Krüger (2004); Margenau (1935); Newman (1980); Pippard (1986); Ross-Bonney (1975); Toyozawa (1992); Tschudi (1987); Young (1980)] that QT is a purely statistical theory whose predictions can only be used to describe the behavior of statistical ensembles and not of individual particles. This statistical interpretation (SI) of QT eliminates all mysteries and paradoxes - and this shows that the mysteries and paradoxes are not part of QT itself but rather the result of a particular (mis)interpretation of QT. In view of the similarity discussed above, we adopt the statistical point of view, not only for the interpretation of QT itself, but also in our search for *quantization conditions*. The general strategy is to find a new set of (as) simple (as possible) statistical assumptions which can be understood in physical terms and imply QT. Such an approach would also provide an explanation for the correspondence rules (1).

The present paper belongs to a series of works aimed at such an explanation. Quite generally, the present work continues a long tradition of attempts, see Frieden (1989; 2004); Hall & Reginatto (2002a); Lee & Zhu (1999); Motz (1962); Rosen (1964); Schiller (1962a); Schrödinger (1926), to characterize QT by mathematical relations which can be understood in physical terms[2] (in contrast to the axiomatic approach). More specifically, it continues previous attempts to derive Schrödinger's equation with the help of statistical concepts, see Hall & Reginatto (2002b), Reginatto (1998a); Syska (2007), Klein (2009). These works, being quite different in detail, share the common feature that a statistical ensemble and not a particle Hamiltonian is used as a starting point for quantization. Finally, in a previous work, Klein (2011), of the present author an attempt has been undertaken to construct a complete statistical approach to QT with the help of a small number of very simple (statistical) assumptions. This work will be referred to as I. The present paper is a continuation and extension of I.

The quantization method reported in I is based on the following general ideas:

- QT should be a probabilistic theory in configuration space (not in phase space).

- QT should fullfil abstract versions of (i) a conservation law for probability (continuity equation) , and (ii) Ehrenfest's theorem. Such relations hold in all statistical theories no matter whether quantum or classical.

- There are no laws for particle trajectories in QT anymore. This arbitrariness, which represents a crucial difference between QT and classical statistics, should be handled by a statistical principle analogous to the principle of maximal entropy in classical statistics.

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[2] The listing given here is far from complete.

These general ideas lead to the mathematical assumptions which represent the basis for the treatment reported in I. This work was restricted to a one-dimensional configuration space (a single particle ensemble with a single spatial degree of freedom). The present work generalizes the treatment of I to a $3N-$dimensional configuration space (ensembles representing an arbitrary number N of particles allowed to move in three-dimensional space), gauge-coupling, and spin. In a first step the generalization to three spatial dimensions is performed; the properly generalized basic relations are reported in section 2. This section contains also a review of the fundamental ideas.

In section 3 we make use of a mathematical freedom, which is already contained in our basic assumptions, namely the multi-valuedness of the variable S. This leads to the appearance of potentials in statistical relations replacing the local forces of single-event (mechanical) theories. The mysterious non-local action of the vector potential (in effects of the Aharonov-Bohm type) is explained as a consequence of the statistical nature of QT. In section 4 we discuss a related question: Which constraints on admissible forces exist for the present class of statistical theories ? The answer is that only macroscopic (elementary) forces of the form of the Lorentz force can occur in nature, because only these survive the transition to QT . These forces are statistically represented by potentials, i.e. by the familiar gauge coupling terms in matter field equations. The present statistical approach provides a natural explanation for the long-standing question why potentials play an indispensable role in the field equations of physics.

In section 5 it is shown that among all statistical theories only the time-dependent Schrödinger equation follows the logical requirement of maximal disorder or minimal Fisher information. Spin one-half is introduced, in section 6, as the property of a statistical ensemble to respond to an external gauge field in two different ways. A generalized calculation, reported in sections 6 and 7, leads to Pauli's (single-particle) equation. In section 8 an alternative derivation, following Arunsalam (1970), and Gould (1995) is reported, which is particularly convenient for the generalization to arbitrary N. The latter is performed in section 9, which completes our statistical derivation of non-relativistic QT.

In section 10 the classical limit of QT is studied and it is stressed that the classical limit of QT is not classical mechanics but a classical statistical theory. In section 11 various questions related to the present approach, including the role of potentials and the interpretation of QT, are discussed. The final section 12 contains a short summary and mentions a possible direction for future research.

2. Basic equations for a class of statistical theories

In I three different types of theories have been defined which differ from each other with regard to the role of probability. We give a short review of the defining properties and supply some additional comments characterizing these theories.

The dogma underlying *theories of type 1* is determinism with regard to single events; probability does not play any role. If nature behaves according to this dogma, then measurements on identically prepared individual systems yield identical results. Classical mechanics is obviously such a deterministic type 1 theory. We shall use below (as a 'template' for the dynamics of our statistical theories) the following version of Newton's law, where the particle momentum $p_k(t)$ plays the role of a second dynamic variable besides the spatial coordinate $x_k(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_k(t) = \frac{p_k(t)}{m}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_k(t) = F_k(x, p, t). \tag{2}$$

In classical mechanics there is no restriction as regards the admissible forces. Thus, F_k is an arbitrary function of x_k, p_k, t; it is, in particular, not required that it be derivable from a potential. Note that Eqs. (2) do *not* hold in the present theory; these relations are just used to establish a correspondence between classical mechanics and associated statistical theories.

Experimental data from atomic systems, recorded since the beginning of the last century, indicate that nature does not behave according to this single-event deterministic dogma. A simple but somewhat unfamiliar idea is, to construct a theory which is deterministic only in a statistical sense. This means that measurements on identically prepared individual systems do not yield identical results (no determinism with regard to single events) but repeated measurements on ensembles [consisting each time of a large (infinite) number of measurements on individual systems] yield identical results. In this case we have 'determinism' with regard to ensembles (expectation values, or probabilities).

Note that such a theory is far from chaotic even if our macroscopic anticipation of (single-event) determinism is not satisfied. Note also that there is no reason to assume that such a statistical theory for microscopic events is incompatible with macroscopic determinism. It is a frequently observed (but not always completely understood) phenomenon in nature that systems with many (microscopic) degrees of freedom can be described by a much smaller number of variables. During this process of elimination of variables the details of the corresponding microscopic theory for the individual constituents are generally lost. In other words, there is no reason to assume that a fundamental statistical law for individual atoms and a deterministic law for a piece of matter consisting of, say, 10^{23} atoms should not be compatible with each other. This way of characterizing the relation between two physical theories is completely different from the common reductionistic point of view. Convincing arguments in favor of the former may, however, be found in Anderson (1972), Laughlin (2005).

As discussed in I two types (referred to as type 2 and type 3) of indeterministic theories may be identified. In *type 2 theories* laws for individual particles exist (roughly speaking the individuality of particles remains intact) but the initial values are unknown and are described by probabilities only. An example for such a (classical-statistical) type 2 theory is statistical thermodynamics. On the other hand, in *type 3 theories* the amount of uncertainty is still greater, insofar as no dynamic laws for individual particles exist any more. A possible candidate for this 'extreme' type of indeterministic theory is quantum mechanics.

The method used in I to construct statistical theories was based on the following three assumptions,

- A local conservation law of probability with a particular form of the probability current.

- Two differential equations which are similar in structure to the canonical equations (2) but with observables replaced by expectation values.

- A differential version (minimal Fisher information) of the statistical principle of maximal disorder.

These (properly generalized) assumptions represent also the formal basis of the present work. The first and second of these cover type 2 as well as type 3 theories, while it will be shown that the third - the requirement of maximal disorder - does only hold for a single type 3 theory, namely quantum mechanics. In this sense quantum mechanics may be considered as the *most reasonable* theory among all statistical theories defined by the first two assumptions. There is obviously an analogy between quantum mechanics and the principle of minimal Fisher information on the one hand and classical statistical mechanics and the principle of maximal entropy on the other hand; both theories are realizations of the principle of maximal disorder.

Let us now generalize the basic equations of I (see section 3 of I) with respect to the number of spatial dimensions and with respect to gauge freedom. The continuity equation takes the form

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x_k} \frac{\rho(x,t)}{m} \frac{\partial \tilde{S}(x,t)}{\partial x_k} = 0. \tag{3}$$

We use the summation convention, indices $i, k, \ldots$ run from 1 to 3 and are omitted if the corresponding variable occurs in the argument of a function. The existence of a local conservation law for the probability density $\rho(x,t)$ is a necessity for a probabilistic theory. The same is true for the fact that the probability current takes the form $j_k(x,t) = \rho(x,t)\tilde{p}_k(x,t)/m$, where $\tilde{p}_k(x,t)$ is the k−th component of the momentum probability density. The only non-trivial assumption contained in (3), is the fact that $\tilde{p}_k(x,t)$ takes the form of the gradient,

$$\tilde{p}_k(x, t) = \frac{\partial \tilde{S}(x, t)}{\partial x_k}, \tag{4}$$

of a function $\tilde{S}(x,t)$. In order to gain a feeling for the physical meaning of (4) we could refer to the fact that a similar relation may be found in the Hamilton-Jacobi formulation of classical mechanics Synge (1960); alternatively we could also refer to the fact that this condition characterizes 'irrotational flow' in fluid mechanics. Relation (4) could also be justified by means of the principle of simplicity; a gradient is the simplest way to represent a vector field, because it can be derived from a single scalar function.

In contrast to I we allow now for *multi-valued* functions $\tilde{S}(x,t)$. At first sight this seems strange since a multi-valued quantity cannot be an observable and should, consequently, not appear in equations bearing a physical meaning. However, only derivatives of $\tilde{S}(x,t)$ occur in our basic equations. Thus, this freedom is possible without any additional postulate; we just have to require that

$$\tilde{S}(x,t) \quad \text{multi-valued,} \quad \frac{\partial \tilde{S}}{\partial t}, \frac{\partial \tilde{S}}{\partial x_k} \quad \text{single-valued.} \tag{5}$$

(the quantity $\tilde{p}$ defined in (4) is not multi-valued; this notation is used to indicate that this quantity has been defined with the help of a multi-valued $\tilde{S}$). As discussed in more detail in section 3 this new 'degree of freedom' is intimately related to the existence of gauge fields. In contrast to $\tilde{S}$, the second dynamic variable ρ is a physical observable (in the statistical sense) and is treated as a single-valued function.

The necessary and sufficient condition for single-valuedness of a function $\tilde{S}(x,t)$ (in a subspace $\mathcal{G} \subseteq \mathcal{R}^4$) is that all second order derivatives of $\tilde{S}(x,t)$ with respect to x_k and t commute with each other (in $\mathcal{G}$) [see e.g. Kaempfer (1965)]. As a consequence, the order of two derivatives of $\tilde{S}$ with respect to anyone of the variables x_k, t must not be changed. We introduce the (single-valued) quantities

$$\tilde{S}_{[j,k]} = \left[\frac{\partial^2 \tilde{S}}{\partial x_j \partial x_k} - \frac{\partial^2 \tilde{S}}{\partial x_k \partial x_j} \right], \quad \tilde{S}_{[0,k]} = \left[\frac{\partial^2 \tilde{S}}{\partial t \partial x_k} - \frac{\partial^2 \tilde{S}}{\partial x_k \partial t} \right] \tag{6}$$

in order to describe the non-commuting derivatives in the following calculations.

The second of the assumptions listed above has been referred to in I as 'statistical conditions'. For the present three-dimensional theory these are obtained in the same way as in I,

by replacing the observables $x_k(t)$, $p_k(t)$ and the force field $F_k(x(t), p(t), t)$ of the type 1 theory (2) by averages $\overline{x_k}$, $\overline{p_k}$ and $\overline{F_k}$. This leads to the relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{x_k} = \frac{\overline{p_k}}{m} \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p_k} = \overline{F_k(x, p, t)}, \tag{8}$$

where the averages are given by the following integrals over the random variables x_k, p_k (which should be clearly distinguished from the type I observables $x_k(t)$, $p_k(t)$ which will not be used any more):

$$\overline{x_k} = \int_{-\infty}^{\infty} \mathrm{d}^3x\,\rho(x,t)\,x_k \tag{9}$$

$$\overline{p_k} = \int_{-\infty}^{\infty} \mathrm{d}^3p\,w(p,t)\,p_k \tag{10}$$

$$\overline{F_k(x, p, t)} = \int_{-\infty}^{\infty} \mathrm{d}^3x\,\mathrm{d}^3p\,W(x, p, t)F_k(x, p, t). \tag{11}$$

The time-dependent probability densities W, ρ, w should be positive semidefinite and normalized to unity, i.e. they should fulfill the conditions

$$\int_{-\infty}^{\infty} \mathrm{d}^3x\,\rho(x,t) = \int_{-\infty}^{\infty} \mathrm{d}^3p\,w(p,t) = \int_{-\infty}^{\infty} \mathrm{d}^3x\,\mathrm{d}^3p\,W(x, p, t) = 1 \tag{12}$$

The densities ρ and w may be derived from the fundamental probability density W by means of the relations

$$\rho(x,t) = \int_{-\infty}^{\infty} \mathrm{d}^3p\,W(x, p, t); \qquad w(p,t) = \int_{-\infty}^{\infty} \mathrm{d}^3x\,W(x, p, t). \tag{13}$$

The present construction of the statistical conditions (7) and (8) from the type 1 theory (2) shows two differences as compared to the treatment in I. The first is that we allow now for a $p-$dependent external force. This leads to a more complicated probability density $W(x, p, t)$ as compared to the two decoupled densities $\rho(x,t)$ and $w(p,t)$ of I. The second difference, which is in fact related to the first, is the use of a multi-valued $\tilde{S}(x,t)$.

Note, that the $p-$dependent probability densities $w(p,t)$ and $W(x, p, t)$ have been introduced in the above relations in a purely formal way. We defined an expectation value $\overline{p_k}$ [via Eq. (7)] and assumed [in Eq. (10)] that a random variable p_k and a corresponding probability density $w(p,t)$ exist. But the validity of this assumption is not guaranteed . There is no compelling conceptual basis for the existence of these quantities in a pure configuration-space theory. If they exist, they must be defined with the help of additional considerations (see section 6 of I). The deeper reason for this problem is that the concept of measurement of momentum (which is proportional to the time derivative of position) is ill-defined in a theory whose observables are defined in terms of a large number of experiments at *one and the same* instant of time (measurement of a derivative requires measurements at different times). Fortunately, these considerations, which have been discussed in more detail in I, play not a prominent role [apart from the choice of $W(x, p, t)$ discussed in section 4], for the derivation of Schrödinger's equation reported in the present paper[3].

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[3] These considerations seem relevant for attempts to define phase-space densities, e.g. of the Wigner type, in QT

Using the continuity equation (3) and the statistical conditions (7) and (8) the present generalization of the integral equation Eq. (24) of I may be derived. The steps leading to this result are very similar to the corresponding steps in I and may be skipped. The essential difference to the one-dimensional treatment is - apart from the number of space dimensions - the non-commutativity of the second order derivatives of $\tilde{S}(x,t)$ leading to non-vanishing quantities $\tilde{S}_{[j,k]}$, $\tilde{S}_{[0,k]}$ defined in Eq. (6). The result takes the form

$$
\begin{aligned}
&- \int_{-\infty}^{\infty} \mathrm{d}^3 x \frac{\partial \rho}{\partial x_k} \left[\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \sum_j \left(\frac{\partial \tilde{S}}{\partial x_j} \right)^2 + V \right] \\
&+ \int_{-\infty}^{\infty} \mathrm{d}^3 x \rho \left[\frac{1}{m} \frac{\partial \tilde{S}}{\partial x_j} \tilde{S}_{[j,k]} + \tilde{S}_{[0,k]} \right] = \overline{F_k^{(e)}(x,\,p,\,t)}
\end{aligned}
\tag{14}
$$

In the course of the calculation leading to (14) it has been assumed that the macroscopic force $F_k(x,\,p,\,t)$ entering the second statistical condition (8) may be written as a sum of two contributions, $F_k^{(m)}(x,t)$ and $F_k^{(e)}(x,\,p,\,t)$,

$$
F_k(x,\,p,\,t) = F_k^{(m)}(x,t) + F_k^{(e)}(x,\,p,\,t),
\tag{15}
$$

where $F_k^{(m)}(x,t)$ takes the form of a negative gradient of a scalar function $V(x,t)$ (mechanical potential) and $F_k^{(e)}(x,\,p,\,t)$ is the remaining $p-$dependent part.

Comparing Eq. (14) with the corresponding formula obtained in I [see Eq. (24) of I] we see that two new terms appear now, the expectation value of the $p-$dependent force on the r.h.s., and the second term on the l.h.s. of Eq. (14). The latter is a direct consequence of our assumption of a multi-valued variable $\tilde{S}$. In section 4 it will be shown that for vanishing multi-valuedness Eq. (14) has to agree with the three-dimensional generalization of the corresponding result [Eq. (24) of I] obtained in I. This means that the $p-$dependent term on the r.h.s. has to vanish too in this limit and indicates a relation between multi-valuedness of $\tilde{S}$ and $p-$dependence of the external force.

3. Gauge coupling as a consequence of a multi-valued phase

In this section we study the consequences of the multi-valuedness [London (1927), Weyl (1929), Dirac (1931)] of the quantity $\tilde{S}(x,t)$ in the present theory. We assume that $\tilde{S}(x,t)$ may be written as a sum of a single-valued part $S(x,t)$ and a multi-valued part $\tilde{N}$. Then, given that (5) holds, the derivatives of $\tilde{S}(x,t)$ may be written in the form

$$
\frac{\partial \tilde{S}}{\partial t} = \frac{\partial S}{\partial t} + e\Phi, \qquad \frac{\partial \tilde{S}}{\partial x_k} = \frac{\partial S}{\partial x_k} - \frac{e}{c} A_k,
\tag{16}
$$

where the four functions Φ and A_k are proportional to the derivatives of $\tilde{N}$ with respect to t and x_k respectively (Note the change in sign of Φ and A_k in comparison to Klein (2009); this is due to the fact that the multi-valued phase is now denoted by $\tilde{S}$). The physical motivations for introducing the pre-factors e and c in Eq. (16) have been extensively discussed elsewhere, see Kaempfer (1965), Klein (2009), in an electrodynamical context. In agreement with Eq. (16), $\tilde{S}$ may be written in the form [Kaempfer (1965), Klein (2009)]

$$
\tilde{S}(x,t;\mathcal{C}) = S(x,t) - \frac{e}{c} \int_{x_0,t_0;\mathcal{C}}^{x,t} \left[\mathrm{d}x_k' A_k(x',t') - c\mathrm{d}t' \phi(x',t') \right],
\tag{17}
$$

as a path-integral performed along an arbitrary path $\mathcal{C}$ in four-dimensional space; the multi-valuedness of $\tilde{S}$ simply means that it depends not only on x, t but also on the path $\mathcal{C}$ connecting the points x_0, t_0 and x, t.

The quantity $\tilde{S}$ cannot be a physical observable because of its multi-valuedness. The fundamental physical quantities to be determined by our (future) theory are the four derivatives of $\tilde{S}$ which will be rewritten here as two observable fields $-\tilde{E}(x, t), \tilde{p}_k(x, t)$,

$$-\tilde{E}(x, t) = \frac{\partial S(x,t)}{\partial t} + e\Phi(x,t), \tag{18}$$

$$\tilde{p}_k(x, t) = \frac{\partial S(x,t)}{\partial x_k} - \frac{e}{c} A_k(x,t), \tag{19}$$

with dimensions of energy and momentum respectively.

We encounter a somewhat unusual situation in Eqs. (18), (19): On the one hand the left hand sides are observables of our theory, on the other hand we cannot solve our (future) differential equations for these quantities because of the peculiar multi-valued structure of $\tilde{S}$. We have to use instead the decompositions as given by the right hand sides of (18) and (19). The latter eight terms (the four derivatives of S and the four scalar functions Φ and A_k) are single-valued (in the mathematical sense) but need not be unique because only the left hand sides are uniquely determined by the physical situation. We tentatively assume that the fields Φ and A_k are 'given' quantities in the sense that they represent an external influence (of 'external forces') on the considered statistical situation. An actual calculation has to be performed in such a way that fixed fields Φ and A_k are chosen and then the differential equations are solved for S (and ρ). However, as mentioned already, what is actually uniquely determined by the physical situation is the *sum* of the two terms on the right hand sides of (18) and (19). Consequently, a different set of fixed fields Φ' and A_k' may lead to a physically equivalent, but mathematically different, solution S' in such a way that the sum of the new terms [on the right hand sides of (18) and (19)] is the *same* as the sum of the old terms. We assume here, that the formalism restores the values of the physically relevant terms. This implies that the relation between the old and new terms is given by

$$S'(x,t) = S(x,t) + \varphi(x,t) \tag{20}$$

$$\Phi'(x,t) = \Phi(x,t) - \frac{1}{e}\frac{\partial \varphi(x,t)}{\partial t} \tag{21}$$

$$A_k'(x,t) = A_k(x,t) + \frac{c}{e}\frac{\partial \varphi(x,t)}{\partial x_k}, \tag{22}$$

where $\varphi(x,t)$ is an arbitrary, single-valued function of x_k, t. Consequently, all 'theories' (differential equations for S and ρ defined by the assumptions listed in section 2) should be form-invariant under the transformations (20)-(22). These invariance transformations, predicted here from general considerations, are (using an arbitrary function $\chi = c\varphi/e$ instead of φ) denoted as 'gauge transformations of the second kind'.

The fields $\Phi(x,t)$ and $A_k(x,t)$ describe an external influence but their numerical value is undefined; their value at x, t may be changed according to (21) and (22) without changing their physical effect. Thus, these fields *cannot play a local role* in space and time like forces and fields in classical mechanics and electrodynamics. What, then, is the physical meaning of these fields ? An explanation which seems obvious in the present context is the following: They describe the *statistical effect* of an external influence on the considered system (ensemble of

identically prepared individual particles). The statistical effect of a force field on an ensemble may obviously *differ* from the local effect of the same force field on individual particles; thus the very existence of fields Φ and A_k different from $\vec{E}$ and $\vec{B}$ is no surprise. The second common problem with the interpretation of the 'potentials' Φ and A_k is their non-uniqueness. It is hard to understand that a quantity ruling the behavior of individual particles should not be uniquely defined. In contrast, this non-uniqueness is much easier to accept if Φ and A_k rule the behavior of ensembles instead of individual particles. We have no problem to accept the fact that a function that represents a global (integral) effect may have many different local realizations.

It seems that this interpretation of the potentials Φ and A_k is highly relevant for the interpretation of the effect found by Aharonov & Bohm (1959). If QT is interpreted as a theory about individual particles, the Aharonov-Bohm effects imply that a charged particle may be influenced in a nonlocal way by electromagnetic fields in inaccessible regions. This paradoxical prediction, which is however in strict agreement with QT, led even to a discussion about the reality of these effects, see Bocchieri & Loinger (1978), Roy (1980), Klein (1981), Peshkin & Tonomura (1989). A statistical interpretation of the potentials has apparently never been suggested, neither in the vast literature about the Aharonov-Bohm effect nor in papers promoting the statistical interpretation of QT; most physicists discuss this nonlocal 'paradox' from the point of view of 'the wave function of a single electron'. Further comments on this point may be found in section 11.

The expectation value $\overline{F_k^{(e)}}(x, p, t)$ on the right hand side of (14) is to be calculated using local, macroscopic forces whose functional form is still unknown. Both the potentials and these local forces represent an external influence, and it is reasonable to assume that the (nonlocal) potentials are the statistical representatives of the local forces on the r.h.s. of Eq. (14). The latter have to be determined by the potentials but must be uniquely defined at each space-time point. The gauge-invariant fields

$$E_k = -\frac{1}{c}\frac{\partial A_k}{\partial t} - \frac{\partial \Phi}{\partial x_k}, \qquad B_k = \epsilon_{kij}\frac{\partial A_j}{\partial x_i}, \tag{23}$$

fulfill these requirements. As a consequence of the defining relations (23) they obey automatically the homogeneous Maxwell equations.

In a next step we rewrite the second term on the l.h.s. of Eq. (14). The commutator terms (6) take the form

$$\tilde{S}_{[0,k]} = -e\left(\frac{1}{c}\frac{\partial A_k}{\partial t} + \frac{\partial \Phi}{\partial x_k}\right), \qquad \tilde{S}_{[j,k]} = \frac{e}{c}\left(\frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j}\right). \tag{24}$$

As a consequence, they may be expressed in terms of the local fields (23), which have been introduced above for reasons of gauge-invariance. Using (24), (23) and the relation (19) for the momentum field, Eq. (14) takes the form

$$-\int_{-\infty}^{\infty} d^3 x \frac{\partial \rho}{\partial x_k}\left[\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m}\sum_j\left(\frac{\partial \tilde{S}}{\partial x_j}\right)^2 + V\right],$$
$$+\int_{-\infty}^{\infty} d^3 x \rho\left[\frac{e}{c}\epsilon_{kij}\tilde{v}_i B_j + e E_k\right] = \overline{F_k^{(e)}}(x, p, t) \tag{25}$$

with a velocity field defined by $\tilde{v}_i = \tilde{p}_i/m$. Thus, the new terms on the l.h.s. of (25) - stemming from the multi-valuedness of $\tilde{S}$ - take the form of an expectation value (with $\mathcal{R}^3$ as sample

space) of the Lorentz force field

$$\vec{F}_L(x,t) = e\vec{E}(x,t) + \frac{e}{c}\vec{v}(x,t) \times \vec{B}(x,t), \tag{26}$$

if the particle velocity is identified with the velocity field $\vec{v}(x,t)$.

The above steps imply a relation between potentials and local fields. From the present statistical (nonlocal) point of view the potentials are more fundamental than the local fields. In contrast, considered from the point of view of macroscopic physics, the local fields are the physical quantities of primary importance and the potentials may (or may not) be introduced for mathematical convenience.

4. A constraint for forces in statistical theories

Let us discuss now the nature of the macroscopic forces $F_k^{(e)}(x, p, t)$ entering the expectation value on the r.h.s. of Eq. (25). In our type I parent theory, classical mechanics, there are no constraints for the possible functional form of $F_k^{(e)}(x, p, t)$. However, this need not be true in the present statistical framework. As a matter of fact, the way the mechanical potential $V(x,t)$ entered the differential equation for S (in the previous work I) indicates already that such constraints do actually exist. Let us recall that in I we tacitly restricted the class of forces to those derivable from a potential $V(x,t)$. If we eliminate this restriction and admit arbitrary forces, with components $F_k(x,t)$, we obtain instead of the above relation (25) the simpler relation [Eq. (24) of I, generalized to three dimensions and arbitrary forces of the form $F_k(x,t)$]

$$-\int_{-\infty}^{\infty} \mathrm{d}^3x \, \frac{\partial \rho}{\partial x_k} \left[\frac{1}{2m} \sum_j \left(\frac{\partial S}{\partial x_j} \right)^2 + \frac{\partial S}{\partial t} \right] = \int_{-\infty}^{\infty} \mathrm{d}x \rho F_k(x,t). \tag{27}$$

This is a rather complicated integro-differential equation for our variables $\rho(x,t)$ and $S(x,t)$. We assume now, using mathematical simplicity as a guideline, that Eq. (27) can be written in the common form of a local differential equation. This assumption is of course not evident; in principle the laws of physics could be integro-differential equations or delay differential equations or take an even more complicated mathematical form. Nevertheless, this assumption seems rather weak considering the fact that all fundamental laws of physics take this 'simple' form. Thus, we postulate that Eq. (27) is equivalent to a differential equation

$$\frac{1}{2m} \sum_j \left(\frac{\partial S}{\partial x_j} \right)^2 + \frac{\partial S}{\partial t} + T = 0, \tag{28}$$

where the unknown term T describes the influence of the force F_k but may also contain other contributions. Let us write

$$T = -L_0 + V, \tag{29}$$

where L_0 does not depend on F_k, while V depends on it and vanishes for $F_k \to 0$. Inserting (28) and (29) in (27) yields

$$\int \mathrm{d}^3x \, \frac{\partial \rho}{\partial x_k} \, (-L_0 + V) = \int \mathrm{d}^3x \, \rho F_k(x,t). \tag{30}$$

For $F_k \to 0$ Eq. (30) leads to the relation

$$\int \mathrm{d}^3 x \, \frac{\partial \rho}{\partial x_k} L_0 = 0, \tag{31}$$

which remains true for finite forces because L_0 does not depend on F_k. Finally, performing a partial integration, we see that a relation

$$F_k = -\frac{\partial V}{\partial x_k} + s_k, \quad \int_{-\infty}^{\infty} \mathrm{d}^3 x \, \rho s_k = 0, \tag{32}$$

exists between F_k and V, with a vanishing expectation value of the (statistically irrelevant) functions s_k. This example shows that the restriction to gradient fields, made above and in I, is actually not necessary. We may *admit* force fields which are arbitrary functions of x and t; the statistical conditions (which play now the role of a 'statistical constraint') eliminate automatically all forces that cannot be written after statistical averaging as gradient fields.

This is very interesting and indicates the possibility that the present statistical assumptions leading to Schrödinger's equation may also be responsible, at least partly, for the structure of the real existing (gauge) interactions of nature.

Does this statistical constraint also work in the present $p-$dependent case ? We assume that the force in (25) is a standard random variable with the configuration space as sample space (see the discussion in section 4 of I) and that the variable p in $F_k^{(e)}(x, p, t)$ may consequently be replaced by the field $\tilde{p}(x, t)$ [see (19)]. Then, the expectation value on the r.h.s. of (25) takes the form

$$\overline{F_k^{(e)}(x, p, t)} = \int_{-\infty}^{\infty} \mathrm{d}^3 x \rho(x, t) H_k\left(x, \frac{\partial \tilde{S}(x, t)}{\partial x}, t\right). \tag{33}$$

The second term on the l.h.s. of (25) has the *same* form. Therefore, the latter may be eliminated by writing

$$H_k\left(x, \frac{\partial \tilde{S}}{\partial x}, t\right) = \frac{e}{c} \epsilon_{kij} \frac{1}{m} \frac{\partial \tilde{S}}{\partial x_i} B_j + eE_k + h_k\left(x, \frac{\partial \tilde{S}}{\partial x}, t\right), \tag{34}$$

with $h_k(x, p, t)$ as our new unknown functions. They obey the simpler relations

$$-\int_{-\infty}^{\infty} \mathrm{d}^3 x \frac{\partial \rho}{\partial x_k} \left[\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \sum_j \left(\frac{\partial \tilde{S}}{\partial x_j} \right)^2 + V \right] = \int_{-\infty}^{\infty} \mathrm{d}^3 x \rho h_k\left(x, \frac{\partial \tilde{S}}{\partial x}, t\right). \tag{35}$$

On a first look this condition for the allowed forces looks similar to the $p-$independent case [see (27)]. But the dependence of h_k on x, t cannot be considered as 'given' (externally controlled), as in the $p-$independent case, because it contains now the unknown x, t-dependence of the derivatives of $\tilde{S}$. We may nevertheless try to incorporate the r.h.s by adding a term $\tilde{T}$ to the bracket which depends on the derivatives of the multivalued quantity $\tilde{S}$. This leads to the condition

$$h_k\left(x, \frac{\partial \tilde{S}}{\partial x}, t\right) = -\frac{\partial \tilde{T}(x, \frac{\partial \tilde{S}}{\partial x}, t)}{\partial x_k} + s_k, \quad \int_{-\infty}^{\infty} \mathrm{d}^3 x \, \rho s_k = 0. \tag{36}$$

But this relation cannot be fulfilled for nontrivial $h_k, \tilde{T}$ because the derivatives of $\tilde{S}$ cannot be subject to further constraints beyond those given by the differential equation; on top of that the derivatives with regard to x on the r.h.s. create higher order derivatives of $\tilde{S}$ which are

not present at the l.h.s. of Eq. (36). The only possibility to fulfill this relation is for constant $\frac{\partial \tilde{S}}{\partial x}$, a special case which has in fact already be taken into account by adding the mechanical potential V. We conclude that the statistical constraint leads to $h_k = \tilde{T} = 0$ and that the statistical condition (35) takes the form

$$-\int \mathrm{d}^3 x \frac{\partial \rho}{\partial x_k} \left[\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \sum_j \left(\frac{\partial \tilde{S}}{\partial x_j} \right)^2 + V \right] = 0. \tag{37}$$

Thus, only a mechanical potential and the four electrodynamic potentials are compatible with the statistical constraint and will consequently - assuming that the present statistical approach reflects a fundamental principle of nature - be realized in nature. As is well known all existing interactions follow (sometimes in a generalized form) the gauge coupling scheme derived above. The statistical conditions imply not only Schrödinger's equation but also the form of the (gauge) coupling to external influences and the form of the corresponding local force, the Lorentz force,

$$\vec{F}_L = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}, \tag{38}$$

if the particle velocity $\vec{v}$ is identified with the velocity field $\vec{\tilde{v}}(x,t)$.

In the present derivation the usual order of proceeding is just inverted. In the conventional deterministic treatment the form of the local forces (Lorentz force), as taken from experiment, is used as a starting point. The potentials are introduced afterwards, in the course of a transition to a different formal framework (Lagrange formalism). In the present approach the fundamental assumptions are the statistical conditions. Then, taking into account an existing mathematical freedom (multi-valuedness of a variable) leads to the introduction of potentials. From these, the shape of the macroscopic (Lorentz) force can be derived, using the validity of the statistical conditions as a constraint.

5. Fisher information as the hallmark of quantum theory

The remaining nontrivial task is the derivation of a local differential equation for S and ρ from the integral equation (37). As our essential constraint we will use, besides general principles of simplicity (like homogeneity and isotropy of space) the principle of maximal disorder, as realized by the requirement of minimal Fisher information. Using the abbreviation

$$\bar{L}(x,t) = \frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \left(\frac{\partial \tilde{S}(x,t)}{\partial x} \right)^2 + V(x,t), \tag{39}$$

the general solution of (37) may be written in the form

$$\frac{\partial \rho}{\partial x_k} \bar{L}(x,t) = G_k(x,t), \tag{40}$$

where the three functions $G_k(x,t)$ have to vanish upon integration over $\mathcal{R}^3$ and are otherwise arbitrary. If we restrict ourselves to an isotropic law, we may write

$$G_k(x,t) = \frac{\partial \rho}{\partial x_k} L_0. \tag{41}$$

Then, our problem is to find a function L_0 which fulfills the differential equation

$$\bar{L}(x,t) - L_0 = 0, \tag{42}$$

and condition (31). The method used in I for a one-dimensional situation, to determine L_0 from the requirement of minimal Fisher information, remains essentially unchanged in the present three-dimensional case. The reader is referred to the detailed explanations reported in I.

In I it has been shown that this principle of maximal disorder leads to an anomalous variational problem and to the following conditions for our unknown function L_0:

$$\bar{L}(x,t) - L_0 \left(\rho, \frac{\partial \rho}{\partial x}, \frac{\partial^2 \rho}{\partial x \partial x} \right) = 0 \tag{43}$$

$$\delta \int \mathrm{d}^3 x \rho \left[\bar{L}(x,t) - L_0 \left(\rho, \frac{\partial \rho}{\partial x}, \frac{\partial^2 \rho}{\partial x \partial x} \right) \right] = 0, \tag{44}$$

where L_0 contains only derivatives of ρ up to second order and does not explicitly depend on x, t. If Eq. (43) is taken into account, the Euler-Lagrange equations of the variational problem (44) lead to the following differential equation

$$- \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \beta}{\partial \left(\frac{\partial^2 \rho}{\partial x_k \partial x_i} \right)} + \frac{\partial}{\partial x_k} \frac{\partial \beta}{\partial \left(\frac{\partial \rho}{\partial x_k} \right)} - \frac{\partial \beta}{\partial \rho} + \frac{\beta}{\rho} = 0 \tag{45}$$

for the variable $\beta = \rho L_0$. Eq. (45) is a straightforward generalization of the corresponding one-dimensional relation [equation (68) of I] to three spatial dimensions.

Besides (45) a further (consistency) condition exists, which leads to a simplification of the problem. The function L_0 may depend on second order derivatives of ρ but this dependence must be of a special form not leading to any terms in the Euler-Lagrange equations [according to (43) our final differential equation for S and ρ must not contain higher than second order derivatives of ρ]. Consequently, the first term in Eq. (45) (as well as the sum of the remaining terms) has to vanish separately and (45) can be replaced by the two equations

$$\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \beta}{\partial \left(\frac{\partial^2 \rho}{\partial x_k \partial x_i} \right)} = 0 \tag{46}$$

$$\frac{\partial}{\partial x_k} \frac{\partial \beta}{\partial \left(\frac{\partial \rho}{\partial x_k} \right)} - \frac{\partial \beta}{\partial \rho} + \frac{\beta}{\rho} = 0. \tag{47}$$

In I a new derivation of Fisher's functional has been obtained, using the general solution of the one-dimensional version of (45), as well as the so-called composition law. In the present three-dimensional situation we set ourselves a less ambitious aim. We know that Fisher's functional describes the maximal amount of disorder. If we are able to find a solution of (46), (47) that agrees with this functional (besides 'null-terms' giving no contribution to the Euler-Lagrange equations) then we will accept it as our correct solution. It is easy to see that this solution is given by

$$L_0 = B_0 \left[-\frac{1}{2\rho^2} \sum_j \left(\frac{\partial \rho}{\partial x_j} \right)^2 + \frac{1}{\rho} \sum_j \frac{\partial^2 \rho}{\partial x_j^2} \right], \tag{48}$$

where B_0 is an arbitrary constant. Eq. (48) presents again the three-dimensional (and isotropic) generalization of the one-dimensional result obtained in I. By means of the identity

$$\frac{\partial}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_k} = \frac{\partial \sqrt{\rho}}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_k} \frac{\partial \sqrt{\rho}}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_i}, \tag{49}$$

it is easily verified that the solution (48) obeys also condition (31). Using the decomposition (16) and renaming B according to $B = \hbar^2/4m$, the continuity equation (3) and the second differential equation (43) respectively, take the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} \frac{\rho}{m} \left(\frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right) = 0, \tag{50}$$

$$\frac{\partial S}{\partial t} + e\phi + \frac{1}{2m} \sum_k \left(\frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right)^2 + V - \frac{\hbar^2}{2m} \frac{\triangle \sqrt{\rho}}{\sqrt{\rho}} = 0. \tag{51}$$

The function S occurring in (50), (51) is single-valued but not unique (not gauge-invariant). If now the complex-valued variable

$$\psi = \sqrt{\rho} e^{i\frac{S}{\hbar}}, \tag{52}$$

is introduced, the two equations (50), (51) may be written in compact form as real and imaginary parts of the linear differential equation

$$\left(\frac{\hbar}{i} \frac{\partial}{\partial t} + e\phi \right) \psi + \frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial \vec{x}} - \frac{e}{c} \vec{A} \right)^2 \psi + V\psi = 0, \tag{53}$$

which completes our derivation of Schrödinger's equation in the presence of a gauge field. Eq. (53) is in manifest gauge-invariant form. The gauge-invariant derivatives of $\tilde{S}$ with respect to t and $\vec{x}$ correspond to the two brackets in (53). In particular, the canonical momentum $\partial S/\partial \vec{x}$ corresponds to the momentum operator proportional to $\partial/\partial \vec{x}$. Very frequently, Eq. (53) is written in the form

$$-\frac{\hbar}{i} \frac{\partial}{\partial t} \psi = H\psi,, \tag{54}$$

with the Hamilton operator

$$H = \frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial \vec{x}} - \frac{e}{c} \vec{A} \right)^2 + V + e\phi, \tag{55}$$

Our final result, Eqs. (54), (55), agrees with the result of the conventional quantization procedure. In its simplest form, the latter starts from the classical relation $H(x, p) = E$, where $H(x, p)$ is the Hamiltonian of a classical particle in a conservative force field, and E is its energy. To perform a "canonical quantization" means to replace p and E by differential expressions according to (1) and let then act both sides of the equation $H(x, p) = E$ on states ψ of a function space. The 'black magic' involved in this process has been eliminated, or at least dramatically reduced, in the present approach, where Eqs. (54), (55) have been derived from a set of assumptions which can all be interpreted in physical terms.

The Hamiltonian (55) depends on the potentials Φ and $\vec{A}$ and is consequently a non-unique (not gauge-invariant) mathematical object. The same is true for the time-development operator $U(H)$ which is an operator function of H, see e.g. Kobe & Yang (1985). This non-uniqueness is a problem if $U(H)$ is interpreted as a quantity ruling the time-evolution of a single particle. It is no problem from the point of view of the SI where H and $U(H)$ are primarily convenient mathematical objects which occur in a natural way if the time-dependence of statistically relevant (uniquely defined) quantities, like expectation values and transition probabilities, is to be calculated.

6. Spin as a statistical degree of freedom

Spin is generally believed to be a phenomenon of quantum-theoretic origin. For a long period of time, following Dirac's derivation of his relativistic equation, it was also believed to be essentially of relativistic origin. This has changed since the work of Schiller (1962b), Levy-Leblond (1967), Arunsalam (1970), Gould (1995), Reginatto (1998b) and others, who showed that spin may be derived entirely in the framework of non-relativistic QT without using any relativistic concepts. Thus, a new derivation of non-relativistic QT like the present one should also include a derivation of the phenomenon of spin. This will be done in this and the next two sections.

A simple idea to extend the present theory is to assume that sometimes - under certain external conditions to be identified later - a situation occurs where the behavior of our statistical ensemble of particles cannot longer be described by ρ, S alone but requires, e.g., the double number of field variables; let us denote these by ρ_1, S_1, ρ_2, S_2 (we restrict ourselves here to spin one-half). The relations defining this generalized theory should be formulated in such a way that the previous relations are obtained in the appropriate limits. One could say that we undertake an attempt to introduce a new (discrete) degree of freedom for the ensemble. If we are able to derive a non-trivial set of differential equations - with coupling between ρ_1, S_1 and ρ_2, S_2 - then such a degree of freedom could exist in nature.

Using these guidelines, the basic equations of the generalized theory can be easily formulated. The probability density and probability current take the form $\rho = \rho_1 + \rho_2$ and $\vec{j} = \vec{j}_1 + \vec{j}_2$, with $\vec{j}_i$ ($i = 1, 2$) defined in terms of ρ_i, S_i exactly as before (see section 2). Then, the continuity equation is given by

$$\frac{\partial(\rho_1 + \rho_2)}{\partial t} + \frac{\partial}{\partial x_l}\left(\frac{\rho_1}{m}\frac{\partial \tilde{S}_1}{\partial x_l} + \frac{\rho_2}{m}\frac{\partial \tilde{S}_2}{\partial x_l}\right) = 0, \tag{56}$$

where we took the possibility of multi-valuedness of the "phases" already into account, as indicated by the notation $\tilde{S}_i$. The statistical conditions are given by the two relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{x_k} = \frac{\overline{p_k}}{m} \tag{57}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p_k} = \overline{F_k^{(T)}(x, p, t)}, \tag{58}$$

which are similar to the relations used previously (in section 2 and in I), and by an additional equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{s_k} = \overline{F_k^{(R)}(x, p, t)}, \tag{59}$$

which is required as a consequence of our larger number of dynamic variables. Eq. (59) is best explained later; it is written down here for completeness. The forces $F_k^{(T)}(x, p, t)$ and $F_k^{(R)}(x, p, t)$ on the r.h.s. of (58) and (59) are again subject to the "statistical constraint", which has been defined in section 3. The expectation values are defined as in (9)-(11).

Performing mathematical manipulations similar to the ones reported in section 2, the l.h.s. of Eq. (58) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p_k} = \int \mathrm{d}^3x \left[\frac{\partial \rho_1}{\partial t}\frac{\partial \tilde{S}_1}{\partial x_k} + \frac{\partial \rho_2}{\partial t}\frac{\partial \tilde{S}_2}{\partial x_k}\right.$$
$$\left. - \frac{\partial \rho_1}{\partial x_k}\frac{\partial \tilde{S}_1}{\partial t} - \frac{\partial \rho_2}{\partial x_k}\frac{\partial \tilde{S}_2}{\partial t} + \rho_1 \tilde{S}_{[0,k]}^{(1)} + \rho_2 \tilde{S}_{[0,k]}^{(2)}\right], \tag{60}$$

where the quantities $\tilde{S}^{(i)}_{[j,k]}$, $i = 1, 2$ are defined as above [see Eq. (6)] but with $\tilde{S}$ replaced by $\tilde{S}_i$. Let us write now $\tilde{S}$ in analogy to section 2 in the form $\tilde{S}_i = S_i + \tilde{N}_i$, as a sum of a single-valued part S_i and a multi-valued part $\tilde{N}_i$. If $\tilde{N}_1$ and $\tilde{N}_2$ are to represent an external influence, they must be identical and a single multi-valued part $\tilde{N} = \tilde{N}_1 = \tilde{N}_2$ may be used instead. The derivatives of $\tilde{N}$ with respect to t and x_k must be single-valued and we may write

$$\frac{\partial \tilde{S}_i}{\partial t} = \frac{\partial S_i}{\partial t} + e\Phi, \qquad \frac{\partial \tilde{S}_i}{\partial x_k} = \frac{\partial S_i}{\partial x_k} - \frac{e}{c} A_k, \tag{61}$$

using the same familiar electrodynamic notation as in section 2. In this way we arrive at eight single-valued functions to describe the external conditions and the dynamical state of our system, namely Φ, A_k and ρ_i, S_i.

In a next step we replace ρ_i, S_i by new dynamic variables ρ, S, ϑ, φ defined by

$$\begin{aligned}
\rho_1 &= \rho \cos^2 \frac{\vartheta}{2}, & S_1 &= S + \frac{\hbar}{2}\varphi, \\
\rho_2 &= \rho \sin^2 \frac{\vartheta}{2}, & S_2 &= S - \frac{\hbar}{2}\varphi.
\end{aligned} \tag{62}$$

A transformation similar to Eq. (62) has been introduced by Takabayasi (1955) in his reformulation of Pauli's equation. Obviously, the variables S, ρ describe 'center of mass' properties (which are common to both states 1 and 2) while ϑ, φ describe relative (internal) properties of the system.

The dynamical variables S, ρ and ϑ, φ are not decoupled from each other. It turns out (see below) that the influence of ϑ, φ on S, ρ can be described in a (formally) similar way as the influence of an external electromagnetic field if a 'vector potential' $\vec{A}^{(s)}$ and a 'scalar potential' $\phi^{(s)}$, defined by

$$A_l^{(s)} = -\frac{\hbar c}{2e} \cos \vartheta \frac{\partial \varphi}{\partial x_l}, \qquad \phi^{(s)} = \frac{\hbar}{2e} \cos \vartheta \frac{\partial \varphi}{\partial t}, \tag{63}$$

are introduced. Denoting these fields as 'potentials', we should bear in mind that they are not externally controlled but defined in terms of the internal dynamical variables. Using the abbreviations

$$\hat{A}_l = A_l + A_l^{(s)}, \qquad \hat{\phi} = \phi + \phi^{(s)}, \tag{64}$$

the second statistical condition (58) can be written in the following compact form

$$\begin{aligned}
-\int &d^3x \frac{\partial \rho}{\partial x_l} \left[\left(\frac{\partial S}{\partial t} + e\hat{\phi} \right) + \frac{1}{2m} \sum_j \left(\frac{\partial S}{\partial x_j} - \frac{e}{c} \hat{A}_j \right)^2 \right] \\
&+ \int d^3x \rho \left[-\frac{e}{c} v_j \left(\frac{\partial \hat{A}_l}{\partial x_j} - \frac{\partial \hat{A}_j}{\partial x_l} \right) - \frac{e}{c} \frac{\partial \hat{A}_l}{\partial t} - e \frac{\partial \hat{\phi}}{\partial x_l} \right] \\
&= \overline{F_l^{(T)}(x, p, t)} = \int d^3x \rho F_l^{(T)}(x, p, t),
\end{aligned} \tag{65}$$

which shows a formal similarity to the spinless case [see (14) and (24)]. The components of the velocity field in (65) are given by

$$v_j = \frac{1}{m} \left(\frac{\partial S}{\partial x_j} - \frac{e}{c} \hat{A}_j \right). \tag{66}$$

If now fields E_l, B_l and $E_l^{(s)}$, $B_l^{(s)}$ are introduced by relations analogous to (23), the second line of (65) may be written in the form

$$\int \mathrm{d}^3x \rho \left[\left(e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}\right)_l + \left(e\vec{E^{(s)}} + \frac{e}{c}\vec{v} \times \vec{B^{(s)}}\right)_l \right],$$ (67)

which shows that both types of fields, the external fields as well as the internal fields due to ϑ, φ, enter the theory in the same way, namely in the form of a Lorentz force.

The first, externally controlled Lorentz force in (67) may be eliminated in exactly the same manner as in section 3 by writing

$$\overline{F_l^{(T)}}(x, p, t) = \int \mathrm{d}^3x \rho \left(e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}\right)_l + \int \mathrm{d}^3x \rho F_l^{(I)}(x, p, t).$$ (68)

This means that one of the forces acting on the system as a whole is again given by a Lorentz force; there may be other nontrivial forces $F^{(I)}$ which are still to be determined. The second 'internal' Lorentz force in (67) can, of course, not be eliminated in this way. In order to proceed, the third statistical condition (59) must be implemented. To do that it is useful to rewrite Eq. (65) in the form

$$-\int \mathrm{d}^3x \frac{\partial \rho}{\partial x_l} \left[\left(\frac{\partial S}{\partial t} + e\hat{\phi}\right) + \frac{1}{2m}\sum_j \left(\frac{\partial S}{\partial x_j} - \frac{e}{c}\hat{A}_j\right)^2 \right]$$
$$+ \int \mathrm{d}^3x \frac{\hbar}{2}\rho \sin\vartheta \left(\frac{\partial \vartheta}{\partial x_l}\left[\frac{\partial \varphi}{\partial t} + v_j\frac{\partial \varphi}{\partial x_j}\right] - \frac{\partial \varphi}{\partial x_l}\left[\frac{\partial \vartheta}{\partial t} + v_j\frac{\partial \vartheta}{\partial x_j}\right]\right)$$ (69)
$$= \overline{F_l^{(I)}}(x, p, t) = \int \mathrm{d}^3x \rho F_l^{(I)}(x, p, t),$$

using (67), (68) and the definition (63) of the fields $A_l^{(s)}$ and $\phi^{(s)}$.

We interpret the fields φ and ϑ as angles (with φ measured from the $y-$axis of our coordinate system) determining the direction of a vector

$$\vec{s} = \frac{\hbar}{2}\left(\sin\vartheta \sin\varphi \, \vec{e}_x + \sin\vartheta \cos\varphi \, \vec{e}_y + \cos\vartheta \, \vec{e}_z\right),$$ (70)

of constant length $\frac{\hbar}{2}$. As a consequence, $\dot{\vec{s}}$ and $\vec{s}$ are perpendicular to each other and the classical force $\vec{F}^{(R)}$ in Eq. (59) should be of the form $\vec{D} \times \vec{s}$, where $\vec{D}$ is an unknown field. In contrast to the 'external force', we are unable to determine the complete form of this 'internal' force from the statistical constraint [an alternative treatment will be reported in section 8] and set

$$\vec{F}^{(R)} = -\frac{e}{mc}\vec{B} \times \vec{s},$$ (71)

where $\vec{B}$ is the external 'magnetic field', as defined by Eq. (23), and the factor in front of $\vec{B}$ has been chosen to yield the correct $g-$factor of the electron.

The differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{s} = -\frac{e}{mc}\vec{B} \times \vec{s}$$ (72)

for particle variables $\vartheta(t)$, $\varphi(t)$ describes the rotational state of a classical magnetic dipole in a magnetic field, see Schiller (1962b). Recall that we do *not* require that (72) is fulfilled in the

present theory. The present variables are the fields $\vartheta(x,t)$, $\varphi(x,t)$ which may be thought of as describing a kind of 'rotational state' of the statistical ensemble as a whole, and have to fulfill the 'averaged version' (59) of (72).

Performing steps similar to the ones described in I (see also section 2), the third statistical condition (59) implies the following differential relations,

$$\dot{\varphi} + v_j \frac{\partial \varphi}{\partial x_j} = \frac{e}{mc} \frac{1}{\sin \vartheta} \left(B_z \sin \vartheta - B_y \cos \vartheta \cos \varphi - B_z \cos \vartheta \sin \varphi \right)$$
$$+ \frac{\cos \varphi}{\sin \vartheta} G_1 - \frac{\sin \varphi}{\sin \vartheta} G_2, \tag{73}$$

$$\dot{\vartheta} + v_j \frac{\partial \vartheta}{\partial x_j} = \frac{e}{mc} \left(B_x \cos \varphi - B_y \sin \varphi \right) - \frac{G_3}{\sin \vartheta}, \tag{74}$$

for the dynamic variables ϑ and φ. These equations contain three fields $G_i(x,t)$, $i = 1,2,3$ which have to obey the conditions

$$\int \mathrm{d}^3 x \rho \, G_i = 0, \qquad \vec{G} \vec{s} = 0, \tag{75}$$

and are otherwise arbitrary. The 'total derivatives' of φ and ϑ in (69) may now be eliminated with the help of (73),(74) and the second line of Eq. (69) takes the form

$$\int \mathrm{d}^3 x \frac{\partial \rho}{\partial x_l} \frac{e}{mc} s_j B_j + \int \mathrm{d}^3 x \rho \frac{e}{mc} s_j \frac{\partial}{\partial x_l} B_j$$
$$+ \int \mathrm{d}^3 x \rho \frac{\hbar}{2} \left(\cos \varphi \frac{\partial \vartheta}{\partial x_l} G_1 - \sin \varphi \frac{\partial \vartheta}{\partial x_l} G_2 + \frac{\partial \varphi}{\partial x_l} G_3 \right). \tag{76}$$

The second term in (76) presents an external macroscopic force. It may be eliminated from (69) by writing

$$\overline{F_l^{(I)}(x,\, p,\, t)} = \int \mathrm{d}^3 x \rho \left(- \mu_j \frac{\partial}{\partial x_l} B_j \right) + \overline{F_l^{(V)}(x,\, p,\, t)}, \tag{77}$$

where the magnetic moment of the electron $\mu_i = -(e/mc)s_i$ has been introduced. The first term on the r.h.s. of (77) is the expectation value of the well-known electrodynamical force exerted by an inhomogeneous magnetic field on the translational motion of a magnetic dipole; this classical force plays an important role in the standard interpretation of the quantum-mechanical Stern-Gerlach effect. It is satisfying that both translational forces, the Lorentz force as well as this dipole force, can be derived in the present approach. The remaining unknown force $\vec{F}^{(V)}$ in (77) leads (in the same way as in section 3) to a mechanical potential V, which will be omitted for brevity.

The integrand of the first term in (76) is linear in the derivative of ρ with respect to x_l. It may consequently be added to the first line of (69) which has the same structure. Therefore, it represents (see below) a contribution to the generalized Hamilton-Jacobi differential equation. The third term in (76) has the mathematical structure of a force term, but does not contain any externally controlled fields. Thus, it must also represent a contribution to the generalized Hamilton-Jacobi equation. This implies that this third term can be written as

$$\int \mathrm{d}^3 x \rho \frac{\hbar}{2} \left(\cos \varphi \frac{\partial \vartheta}{\partial x_l} G_1 - \sin \varphi \frac{\partial \vartheta}{\partial x_l} G_2 + \frac{\partial \varphi}{\partial x_l} G_3 \right) = \int \mathrm{d}^3 x \frac{\partial \rho}{\partial x_l} L_0', \tag{78}$$

where L_0' is an unknown field depending on G_1, G_2, G_3.

Collecting terms and restricting ourselves, as in section 5, to an isotropic law, the statistical condition (69) takes the form of a generalized Hamilton-Jacobi equation:

$$\bar{L} := \left(\frac{\partial S}{\partial t} + e\hat{\phi}\right) + \frac{1}{2m}\sum_j\left(\frac{\partial S}{\partial x_j} - \frac{e}{c}\hat{A}_j\right)^2 + \mu_i B_i = L_0. \tag{79}$$

The unknown function L_0 must contain L_0' but may also contain other terms, let us write $L_0 = L_0' + \Delta L_0$.

7. 'Missing' quantum spin terms from Fisher information

Let us summarize at this point what has been achieved so far. We have four coupled differential equations for our dynamic field variables ρ, S, ϑ, φ. The first of these is the continuity equation (56), which is given, in terms of the present variables, by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_l}\left[\frac{\rho}{m}\left(\frac{\partial S}{\partial x_j} - \frac{e}{c}\hat{A}_j\right)\right] = 0. \tag{80}$$

The three other differential equations, the evolution equations (73), (74) and the generalized Hamilton-Jacobi equation (79), do not yet possess a definite mathematical form. They contain four unknown functions G_i, L_0 which are constrained, but not determined, by (75), (78).

The simplest choice, from a formal point of view, is $G_i = L_0 = 0$. In this limit the present theory agrees with Schiller's field-theoretic (Hamilton-Jacobi) version, see Schiller (1962b), of the equations of motion of a classical dipole. This is a classical (statistical) theory despite the fact that it contains [see (63)] a number $\hbar$. But this classical theory is not realized in nature; at least not in the microscopic domain. The reason is that the simplest choice from a formal point of view is not the simplest choice from a physical point of view. The postulate of maximal simplicity (Ockham's razor) implies equal probabilities and the principle of maximal entropy in classical statistical physics. A similar principle which is able to 'explain' the nonexistence of classical physics (in the microscopic domain) is the principle of minimal Fisher information Frieden (2004). The relation between the two (classical and quantum-mechanical) principles has been discussed in detail in I.

The mathematical formulation of the principle of minimal Fisher information for the present problem requires a generalization, as compared to I, because we have now several fields with coupled time-evolution equations. As a consequence, the spatial integral (spatial average) over $\rho(\bar{L} - L_0)$ in the variational problem (44) should be replaced by a space-time integral, and the variation should be performed with respect to all four variables. The problem can be written in the form

$$\delta \int dt \int d^3x\, \rho\, (\bar{L} - L_0) = 0 \tag{81}$$

$$E_a = 0, \quad a = S, \rho, \vartheta, \varphi, \tag{82}$$

where $E_a = 0$ is a shorthand notation for the equations (80), (79), (74) (73). Eqs. (81), (82) require that the four Euler-Lagrange equations of the variational problem (81) agree with the differential equations (82). This imposes conditions for the unknown functions L_0, G_i. If the *solutions* of (81), (82) for L_0, G_i are inserted in the variational problem (81), the four

relations (82) become redundant and $\rho(\bar{L} - L_0)$ becomes the Lagrangian density of our problem. Thus, Eqs. (81) and (82) represent a method to construct a Lagrangian.

We assume a functional form $L_0(\chi_\alpha, \partial_k\chi_\alpha, \partial_k\partial_l\chi_\alpha)$, where $\chi_\alpha = \rho, \vartheta, \varphi$. This means L_0 does not possess an explicit x, t-dependence and does not depend on S (this would lead to a modification of the continuity equation). We further assume that L_0 does not depend on time-derivatives of χ_α (the basic structure of the time-evolution equations should not be affected) and on spatial derivatives higher than second order. These second order derivatives must be taken into account but should not give contributions to the variational equations (a more detailed discussion of the last point has been given in I).

The variation with respect to S reproduces the continuity equation which is unimportant for the determination of L_0, G_i. Performing the variation with respect to ρ, ϑ, φ and taking the corresponding conditions (79), (74) (73) into account leads to the following differential equations for L_0, $G_1 \cos\varphi - G_2 \sin\varphi$ and G_3,

$$-\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}\frac{\partial\rho L_0}{\partial\frac{\partial^2\rho}{\partial x_k\partial x_i}} + \frac{\partial}{\partial x_k}\frac{\partial\rho L_0}{\partial\frac{\partial\rho}{\partial x_k}} - \rho\frac{\partial L_0}{\partial\rho} = 0 \tag{83}$$

$$-\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}\frac{\partial\rho L_0}{\partial\frac{\partial^2\vartheta}{\partial x_k\partial x_i}} + \frac{\partial}{\partial x_k}\frac{\partial\rho L_0}{\partial\frac{\partial\vartheta}{\partial x_k}} - \frac{\partial\rho L_0}{\partial\vartheta} - \frac{\hbar\rho}{2}\left(G_1\cos\varphi - G_2\sin\varphi\right) = 0 \tag{84}$$

$$-\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}\frac{\partial\rho L_0}{\partial\frac{\partial^2\varphi}{\partial x_k\partial x_i}} + \frac{\partial}{\partial x_k}\frac{\partial\rho L_0}{\partial\frac{\partial\varphi}{\partial x_k}} - \frac{\partial\rho L_0}{\partial\varphi} - \frac{\hbar}{2}\rho G_3 = 0. \tag{85}$$

The variable S does not occur in (83)-(85) in agreement with our assumptions about the form of L_0. It is easy to see that a proper solution (with vanishing variational contributions from the second order derivatives) of (83)-(85) is given by

$$L_0 = \frac{\hbar^2}{2m}\left[\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial\vec{x}}\frac{\partial}{\partial\vec{x}}\sqrt{\rho} - \frac{1}{4}\sin^2\vartheta\left(\frac{\partial\varphi}{\partial\vec{x}}\right)^2 - \frac{1}{4}\left(\frac{\partial\vartheta}{\partial\vec{x}}\right)^2\right] \tag{86}$$

$$\hbar G_1\cos\varphi - \hbar G_2\sin\varphi = \frac{\hbar^2}{2m}\left[\frac{1}{2}\sin 2\vartheta\left(\frac{\partial\varphi}{\partial\vec{x}}\right)^2 - \frac{1}{\rho}\frac{\partial}{\partial\vec{x}}\rho\frac{\partial\vartheta}{\partial\vec{x}}\right] \tag{87}$$

$$\hbar G_3 = -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{\partial}{\partial\vec{x}}\left(\rho\sin\vartheta^2\frac{\partial\varphi}{\partial\vec{x}}\right). \tag{88}$$

A new adjustable parameter appears on the r.h.s of (86)-(88) which has been identified with $\hbar^2/2m$, where $\hbar$ is again Planck's constant. This second $\hbar$ is related to the quantum-mechanical principle of maximal disorder. It is in the present approach not related in any obvious way to the previous "classical" $\hbar$ which denotes the amplitude of a rotation; compare, however, the alternative derivation of spin in section 8.

The solutions for G_1, G_2 may be obtained with the help of the second condition ($\vec{G}\vec{s} = 0$) listed in Eq. (75). The result may be written in the form

$$\begin{aligned} G_1 &= \frac{\hbar}{2m}\frac{1}{\rho}\frac{\partial}{\partial\vec{x}}\rho\left(\frac{1}{2}\sin 2\vartheta\sin\varphi\frac{\partial\varphi}{\partial\vec{x}} - \cos\varphi\frac{\partial\vartheta}{\partial\vec{x}}\right) \\ G_2 &= \frac{\hbar}{2m}\frac{1}{\rho}\frac{\partial}{\partial\vec{x}}\rho\left(\frac{1}{2}\sin 2\vartheta\cos\varphi\frac{\partial\varphi}{\partial\vec{x}} + \sin\varphi\frac{\partial\vartheta}{\partial\vec{x}}\right). \end{aligned} \tag{89}$$

Eqs. (88) and (89) show that the first condition listed in (75) is also satisfied. The last condition is also fulfilled: L_0 can be written as $L_0' + \Delta L_0$, where

$$L_0' = -\frac{\hbar^2}{8m}\left[\sin^2\vartheta\left(\frac{\partial\varphi}{\partial\vec{x}}\right)^2 - \left(\frac{\partial\vartheta}{\partial\vec{x}}\right)^2\right], \quad \Delta L_0 = \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial\vec{x}}\frac{\partial}{\partial\vec{x}}\sqrt{\rho}, \tag{90}$$

and L_0' fulfills (78). We see that L_0' is a quantum-mechanical contribution to the rotational motion while ΔL_0 is related to the probability density of the ensemble (as could have been guessed considering the mathematical form of these terms). The last term is the same as in the spinless case [see (51)].

The remaining task is to show that the above solution for L_0 does indeed lead to a (appropriately generalized) Fisher functional. This can be done in several ways. The simplest is to use the following result due to Reginatto (1998b):

$$\int \mathrm{d}^3x\,(-\rho L_0) = \frac{\hbar^2}{8m}\sum_{j=1}^{3}\int \mathrm{d}^3x\sum_{k=1}^{3}\frac{1}{\rho^{(j)}}\left(\frac{\partial\rho^{(j)}}{\partial x_k}\right)^2, \tag{91}$$

$$\rho^{(1)} := \rho\sin^2\frac{\vartheta}{2}\cos^2\frac{\varphi}{2}, \quad \rho^{(2)} := \rho\sin^2\frac{\vartheta}{2}\sin^2\frac{\varphi}{2}, \quad \rho^{(3)} := \rho\cos^2\frac{\vartheta}{2}. \tag{92}$$

The functions $\rho^{(j)}$ represent the probability that a particle is at space-time point x, t and $\vec{s}$ points into direction j. Inserting (86) the validity of (91) may easily be verified. The r.h.s. of Eq. (91) shows that the averaged value of L_0 represents indeed a Fisher functional, which completes our calculation of the 'quantum terms' L_0, G_i.

Summarizing, our assumption, that under certain external conditions four state variables instead of two may be required, led to a nontrivial result, namely the four coupled differential equations (80), (79), (74), (73) with L_0, G_i given by (86), (89), (88). The external condition which stimulates this splitting is given by a gauge field; the most important case is a magnetic field $\vec{B}$ but other possibilities do exist (see below). These four differential equations are equivalent to the much simpler differential equation

$$\left(\frac{\hbar}{\iota}\frac{\partial}{\partial t} + e\phi\right)\hat{\psi} + \frac{1}{2m}\left(\frac{\hbar}{\iota}\frac{\partial}{\partial\vec{x}} - \frac{e}{c}\vec{A}\right)^2\hat{\psi} + \mu_B\vec{\sigma}\vec{B}\hat{\psi} = 0, \tag{93}$$

which is linear in the complex-valued two-component state variable $\hat{\psi}$ and is referred to as Pauli equation (the components of the vector $\vec{\sigma}$ are the three Pauli matrices and $\mu_B = -e\hbar/2mc$). To see the equivalence one writes, see Takabayasi (1955), Holland (1995),

$$\hat{\psi} = \sqrt{\rho}\,\mathrm{e}^{\frac{\iota}{\hbar}S}\begin{pmatrix}\cos\frac{\vartheta}{2}\mathrm{e}^{\iota\frac{\varphi}{2}} \\[2mm] \iota\sin\frac{\vartheta}{2}\mathrm{e}^{-\iota\frac{\varphi}{2}}\end{pmatrix}, \tag{94}$$

and evaluates the real and imaginary parts of the two scalar equations (93). This leads to the four differential equations (80), (79), (74) (73) and completes the present spin theory.

In terms of the real-valued functions $\rho, S, \vartheta, \varphi$ the quantum-mechanical solutions (86), (88), (89) for L_0, G_i look complicated in comparison to the classical solutions $L_0 = 0$, $G_i = 0$. In terms of the variable $\hat{\psi}$ the situation changes to the contrary: The quantum-mechanical equation becomes simple (linear) and the classical equation, which has been derived by Schiller (1962b), becomes complicated (nonlinear). The simplicity of the

underlying physical principle (principle of maximal disorder) leads to a simple mathematical representation of the final basic equation (if a complex-valued state function is introduced). One may also say that the linearity of the equations is a consequence of this principle of maximal disorder. This is the deeper reason why it has been possible, see Klein (2009), to derive Schrödinger's equation from a set of assumptions including linearity.

Besides the Pauli equation we found, as a second important result of our spin calculation, that the following local force is compatible with the statistical constraint:

$$\vec{F}^L + \vec{F}^I = e\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) - \vec{\mu} \cdot \frac{\partial}{\partial \vec{x}}\vec{B}. \tag{95}$$

Here, the velocity field $\tilde{\vec{v}}(x,t)$ and the magnetic moment field $\vec{\mu}(x,t) = -(e/mc)\vec{s}(x,t)$ have been replaced by corresponding particle quantities $\vec{v}(t)$ and $\vec{\mu}(t)$; the dot denotes the inner product between $\vec{\mu}$ and $\vec{B}$. The first force in (95), the Lorentz force, has been derived here from first principles without any additional assumptions. The same cannot be said about the second force which takes this particular form as a consequence of some additional assumptions concerning the form of the 'internal force' $\vec{F}^R$ [see (71)]. In particular, the field appearing in $\vec{F}^R$ was arbitrary as well as the proportionality constant (g-factor of the electron) and had to be adjusted by hand. It is well-known that in a relativistic treatment the spin term appears automatically if the potentials are introduced. Interestingly, this unity is not restricted to the relativistic regime. Following Arunsalam (1970) and Gould (1995) we report in the next section an alternative (non-relativistic) derivation of spin, which does not contain any arbitrary fields or constants - but is unable to yield the expression (95) for the macroscopic electromagnetic forces.

In the present treatment spin has been introduced as a property of an ensemble and not of individual particles. Similar views may be found in the literature, see Ohanian (1986). Of course, it is difficult to imagine the properties of an ensemble as being completely independent from the properties of the particles it is made from. The question whether or not a property 'spin' can be ascribed to single particles is a subtle one. Formally, we could assign a probability of being in a state i ($i = 1, 2$) to a particle just as we assign a probability for being at a position $\vec{x} \in R^3$. But contrary to position, no classical meaning - and no classical measuring device - can be associated with the discrete degree of freedom i. Experimentally, the measurement of the 'spin of a single electron' is - in contrast to the measurement of its position - a notoriously difficult task. Such experiments, and a number of other interesting questions related to spin, have been discussed by Morrison (2007).

8. Spin as a consequence of a multi-valued phase

As shown by Arunsalam (1970), Gould (1995), and others, spin in non-relativistic QT may be introduced in exactly the same manner as the electrodynamic potentials. In this section we shall apply a slightly modified version of their method and try to derive spin in an alternative way - which avoids the shortcoming mentioned in the last section.

Arunsalam (1970) and Gould (1995) introduce the potentials by applying the well-known minimal-coupling rule to the free Hamiltonian. In the present treatment this is achieved by making the quantity S multi-valued. The latter approach seems intuitively preferable considering the physical meaning of the corresponding classical quantity. Let us first review the essential steps [see Klein (2009) for more details] in the process of creating potentials in the *scalar* Schrödinger equation:

- Chose a free Schrödinger equation with single-valued state function.
- 'Turn on' the interaction by making the state function multi-valued (multiply it with a multi-valued phase factor)
- Shift the multi-valued phase factor to the left of all differential operators, creating new terms (potentials) in the differential equation.
- Skip the multi-valued phase. The final state function is again single-valued.

Let us adapt this method for the derivation of spin (considering spin one-half only). The first and most important step is the identification of the free Pauli equation. An obvious choice is

$$\left[\frac{\hbar}{\imath}\frac{\partial}{\partial t} + \frac{1}{2m}\left(\frac{\hbar}{\imath}\frac{\partial}{\partial \vec{x}}\right)^2 + V\right]\bar{\psi} = 0, \tag{96}$$

where $\bar{\psi}$ is a single-valued *two-component* state function; (96) is essentially a duplicate of Schrödinger's equation. We may of course add arbitrary vanishing terms to the expression in brackets. This seems trivial, but some of these terms may vanish *only* if applied to a single-valued $\bar{\psi}$ and may lead to non-vanishing contributions if applied later (in the second of the above steps) to a multi-valued state function $\bar{\psi}^{multi}$.

In order to investigate this possibility, let us rewrite Eq. (96) in the form

$$\left[\hat{p}_0 + \frac{1}{2m}\vec{\hat{p}}\vec{\hat{p}} + V\right]\bar{\psi} = 0, \tag{97}$$

where $\hat{p}_0$ is an abbreviation for the first term of (96) and the spatial derivatives are given by

$$\vec{\hat{p}} = \hat{p}_k\vec{e}_k, \qquad \hat{p}_k = \frac{\hbar}{\imath}\frac{\partial}{\partial x_k}. \tag{98}$$

All terms in the bracket in (97) are to be multiplied with a 2x2 unit-matrix E which has not be written down. Replace now the derivatives in (97) according to

$$\hat{p}_0 \Rightarrow \hat{p}_0 M_0, \qquad \vec{\hat{p}} \Rightarrow \vec{\hat{p}}_k M_k, \tag{99}$$

where M_0, M_k are hermitian 2x2 matrices with constant coefficients, which should be constructed in such a way that the new equation agrees with (97) for single-valued $\bar{\psi}$, i.e. assuming the validity of the condition

$$(\hat{p}_i\hat{p}_k - \hat{p}_k\hat{p}_i)\bar{\psi} = 0. \tag{100}$$

This leads to the condition

$$M_0^{-1}M_iM_k = E\delta_{ik} + T_{ik}, \tag{101}$$

where T_{ik} is a 2x2 matrix with two cartesian indices i, k, which obeys $T_{ik} = -T_{ki}$. A solution of (101) is given by $M_0 = \sigma_0$, $M_i = \sigma_i$, where σ_0, σ_i are the four Pauli matrices. In terms of this solution, Eq. (101) takes the form

$$\sigma_i\sigma_k = \sigma_0\delta_{ik} + \imath\varepsilon_{ikl}\sigma_l. \tag{102}$$

Thus, an alternative free Pauli-equation, besides (96) is given by

$$\left[\frac{\hbar}{\imath}\frac{\partial}{\partial t} + \frac{1}{2m}\left(\frac{\hbar}{\imath}\right)^2\sigma_i\frac{\partial}{\partial x_i}\sigma_k\frac{\partial}{\partial x_k} + V\right]\bar{\psi} = 0. \tag{103}$$

The quantity in the bracket is the generalized Hamiltonian constructed by Arunsalam (1970) and Gould (1995). In the present approach gauge fields are introduced by means of a multi-valued phase. This leads to the same formal consequences as the minimal coupling rule but allows us to conclude that the second free Pauli equation (103) is *more appropriate* than the first, Eq. (96), because it is more general with regard to the consequences of multi-valuedness. This greater generality is due to the presence of the second term on the r.h.s. of (102).

The second step is to turn on the multi-valuedness in Eq. (103), $\bar{\psi} \Rightarrow \bar{\psi}^{multi}$, by multiplying $\bar{\psi}$ with a multi-valued two-by-two matrix. This matrix must be chosen in such a way that the remaining steps listed above lead to Pauli's equation (93) in presence of an gauge field. Since in our case the final result (93) is known, this matrix may be found by performing the inverse process, i.e. performing a singular gauge transformation $\hat{\psi} = \Gamma \bar{\psi}^{multi}$ of Pauli's equation (93) from $\hat{\psi}$ to $\bar{\psi}^{multi}$, which *removes* all electrodynamic terms from (93) and creates Eq. (103). The final result for the matrix Γ is given by

$$\Gamma = E \exp\left\{ \imath \frac{e}{\hbar c} \int^{x,t} \left[\mathrm{d}x'_k A_k(x',t') - c\mathrm{d}t' \phi(x',t') \right] \right\}, \tag{104}$$

and agrees, apart from the unit matrix E, with the multi-valued factor introduced previously [see (17) and (52)] leading to the electrodynamic potentials. The inverse transition from (103) to (93), i.e. the creation of the potentials and the Zeeman term, can be performed by using the inverse of (104).

The Hamiltonian (103) derived by Arunsalam (1970) and Gould (1995) shows that spin can be described by means of the same abelian gauge theory that leads to the standard quantum mechanical gauge coupling terms; no new adjustable fields or parameters appear. The only requirement is that the appropriate free Pauli equation (103) is chosen as starting point. The theory of Dartora & Cabrera (2008), on the other hand, started from the alternative (from the present point of view inappropriate) free Pauli equation (96) and leads to the conclusion that spin must be described by a non-abelian gauge theory.

As far as our derivation of non-relativistic QT is concerned we have now two alternative, and in a sense complementary, possibilities to introduce spin. The essential step in the second (Arunsalam-Gould) method is the transition from (96) to the equivalent free Pauli equation (103). This step is a remarkable short-cut for the complicated calculations, performed in the last section, leading to the various terms required by the principle of minimal Fisher information. The Arunsalam-Gould method is unable to provide the shape (95) of the corresponding macroscopic forces but is very powerful insofar as no adjustable quantities are required. It will be used in the next section to perform the transition to an arbitrary number of particles.

9. Transition to N particles as final step to non-relativistic quantum theory

In this section the present derivation of non-relativistic QT is completed by deriving Schrödinger's equation for an arbitrary number N of particles or, more precisely, for statistical ensembles of identically prepared experimental arrangements involving N particles.

In order to generalize the results of sections 2 and 5, a convenient set of $n = 3N$ coordinates $q_1, ... q_n$ and masses $m_1, ... m_n$ is defined by

$$\begin{aligned} (q_1, q_2, q_3, ... q_{n-2}, q_{n-1}, q_n) &= (x_1, y_1, z_1, ..., x_N, y_N, z_N), \\ (m_1, m_2, m_3, ... m_{n-2}, m_{n-1}, m_n) &= (m_1, m_1, m_1, ..., m_N, m_N, m_N). \end{aligned} \tag{105}$$

The index $I = 1, ... N$ is used to distinguish particles, while indices $i, k, ..$ are used here to distinguish the $3N$ coordinates $q_1, ... q_n$. No new symbol has been introduced in (105) to distinguish the masses m_I and m_i since there is no danger of confusion in anyone of the formulas below. However, the indices of masses will be frequently written in the form $m_{(i)}$ in order to avoid ambiguities with regard to the summation convention. The symbol Q in arguments denotes dependence on all $q_1, ... q_n$. In order to generalize the results of section 8 a notation $x_{I,k}$, $\vec{x}_I$, and m_I (with $I = 1, ..., N$ and $k = 1, 2, 3$) for coordinates, positions, and masses will be more convenient.

The basic relations of section 2, generalized in an obvious way to N particles, take the form

$$\frac{\partial \rho(Q,t)}{\partial t} + \frac{\partial}{\partial q_k} \frac{\rho(Q,t)}{m_{(k)}} \frac{\partial S(Q,t)}{\partial q_k} = 0 \tag{106}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{q_k} = \frac{1}{m_{(k)}} \overline{p_k} \tag{107}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{p_k} = \overline{F_k(Q,t)} \tag{108}$$

$$\overline{q_k} = \int \mathrm{d}Q\, \rho(Q,t)\, q_k \tag{109}$$

Here, S is a single-valued variable; the multi-valuedness will be added later, following the method of section 8.

The following calculations may be performed in complete analogy to the corresponding steps of section 2. For the present $N-$dimensional problem, the vanishing of the surface integrals, occurring in the course of various partial integrations, requires that ρ vanishes exponentially in arbitrary directions of the configuration space. The final conclusion to be drawn from Eqs. (106)- (109) takes the form

$$\sum_{j=1}^n \frac{1}{2m_{(j)}} \left(\frac{\partial S}{\partial q_j} \right)^2 + \frac{\partial S}{\partial t} + V = L_0, \qquad \int \mathrm{d}Q\, \frac{\partial \rho}{\partial q_k} L_0 = 0, \tag{110}$$

The remaining problem is the determination of the unknown function L_0, whose form is constrained by the condition defined in Eq. (110).

L_0 can be determined using again the principle of minimal Fisher information, see I for details. Its implementation in the present framework takes the form

$$\delta \int \mathrm{d}t \int \mathrm{d}Q\, \rho\, (L - L_0) = 0 \tag{111}$$

$$E_a = 0, \quad a = S, \rho, \tag{112}$$

where $E_S = 0$, $E_\rho = 0$ are shorthand notations for the two basic equations (110) and (106). As before, Eqs. (111), (112) represent a method to construct a Lagrangian. After determination of L_0 the three relations listed in (111), (112) become redundant and (112) become the fundamental equations of the $N-$particle theory.

The following calculation can be performed in complete analogy to the case $N = 1$ reported in section 5. All relations remain valid if the upper summation limit 3 is replaced by $3N$. This is also true for the final result, which takes the form

$$L_0 = \frac{\hbar^2}{4\rho} \left[-\frac{1}{2\rho} \frac{1}{m_{(j)}} \frac{\partial \rho}{\partial q_j} \frac{\partial \rho}{\partial q_j} + \frac{1}{m_{(j)}} \frac{\partial^2 \rho}{\partial q_j \partial q_j} \right]. \tag{113}$$

If a complex-valued variable ψ, defined as in (52), is introduced, the two basic relations $E_a = 0$ may be condensed into the single differential equation,

$$\left[\frac{\hbar}{\iota} \frac{\partial}{\partial t} + \sum_{I=1}^{N} \frac{1}{2m_{(I)}} \left(\frac{\hbar}{\iota} \frac{\partial}{\partial x_{I,k}} \right) \left(\frac{\hbar}{\iota} \frac{\partial}{\partial x_{I,k}} \right) + V \right] \psi = 0, \tag{114}$$

which is referred to as $N-$particle Schrödinger equation, rewritten here in the more familiar form using particle indices. As is well-known, only approximate solutions of this partial differential equation of order $3N + 1$ exist for realistic systems. The inaccessible complexity of quantum-mechanical solutions for large N is not reflected in the abstract Hilbert space structure (which is sometimes believed to characterize the whole of QT) but plays probably a decisive role for a proper description of the mysterious relation between QT and the macroscopic world.

Let us now generalize the Arunsalam-Gould method, discussed in section 8, to an arbitrary number of particles. We assume, that the considered $N-$particle statistical ensemble responds in 2^N ways to the external electromagnetic field. This means we restrict ourselves again, like in section 6, 7 to spin one-half. Then, the state function may be written as $\psi(x_1, s_1; \dots x_I, s_I; \dots x_N, s_N)$ where $s_I = 1, 2$. In the first of the steps listed at the beginning of section 8, a differential equation, which is equivalent to Eq. (114) for single-valued ψ but may give non-vanishing contributions for multi-valued ψ, has to be constructed. The proper generalization of Eq. (103) to arbitrary N takes the form

$$\left[\frac{\hbar}{\iota} \frac{\partial}{\partial t} + \sum_{I=1}^{N} \frac{1}{2m_{(I)}} \left(\frac{\hbar}{\iota} \sigma_k^{(I)} \frac{\partial}{\partial x_{I,k}} \right) \left(\frac{\hbar}{\iota} \sigma_l^{(I)} \frac{\partial}{\partial x_{I,l}} \right) + V \right] \psi = 0, \tag{115}$$

where the Pauli matrices $\sigma_k^{(I)}$ operate by definition only on the two-dimensional subspace spanned by the variable s_I. In the second step we perform the replacement

$$\psi \Rightarrow \exp \left\{ -\frac{\iota}{\hbar} \sum_{I=1}^{N} \frac{e_I}{c} \sum_{k=1}^{3} \int^{\vec{x}_I, t} \left[dx'_{I,k} A_k(x'_I, t') - c dt' \phi(x'_I, t') \right] \right\} \psi, \tag{116}$$

using a multi-valued phase factor, which is an obvious generalization of Eq. (104). The remaining steps, in the listing of section 8, lead in a straightforward way to the final result

$$\begin{bmatrix} \frac{\hbar}{\iota} \frac{\partial}{\partial t} + \sum_{I=1}^{N} e_I \phi(x_I, t) + \sum_{I=1}^{N} \sum_{k=1}^{3} \frac{1}{2m_{(I)}} \left(\frac{\hbar}{\iota} \frac{\partial}{\partial x_{I,k}} - \frac{e_I}{c} A_k(x_I, t) \right)^2 \\ + \sum_{I=1}^{N} \mu_B^{(I)} \sigma_k^{(I)} B_k(x_I, t) + V(x_1, \dots, x_N, t) \end{bmatrix} \psi = 0 \tag{117}$$

where $\mu_B^{(I)} = -\hbar e_I / 2m_I c$ and $\vec{B} = \text{rot} \vec{A}$. The mechanical potential $V(x_1, \dots, x_N, t)$ describes a general many-body force but contains, of course, the usual sum of two-body potentials as a special case. Eq. (117) is the $N-$body version of Pauli's equation and completes - in the sense discussed at the very beginning of this paper - the present derivation of non-relativistic QT.

10. The classical limit of quantum theory is a statistical theory

The classical limit of Schrödinger's equation plays an important role for two topics discussed in the next section, namely the interpretation of QT and the particular significance of potentials in QT; to study these questions it is sufficient to consider a single-particle ensemble described by a single state function. This 'classical limit theory' is given by the two differential equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} \frac{\rho}{m} \left(\frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right) = 0, \tag{118}$$

$$\frac{\partial S}{\partial t} + e\phi + \frac{1}{2m} \sum_k \left(\frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right)^2 + V = 0, \tag{119}$$

which are obtained from Eqs. (50) and (51) by performing the limit $\hbar \to 0$. The quantum mechanical theory (50) and (51) and the classical theory (118) and (119) show fundamentally the same mathematical structure; both are initial value problems for the variables S and ρ obeying two partial differential equations. The difference is the absence of the last term on the l.h.s. of (51) in the corresponding classical equation (119). This leads to a *decoupling* of S and ρ in (119); the identity of the classical object described by S is no longer affected by statistical aspects described by ρ.

The field theory (118), (119) for the two 'not decoupled' fields S and ρ is obviously very different from classical mechanics which is formulated in terms of trajectories. The fact that one of these equations, namely (119), agrees with the Hamilton-Jacobi equation, does not change the situation since the presence of the continuity equation (118) cannot be neglected. On top of that, even if it could be neglected, Eq. (119) would still be totally different from classical mechanics: In order to construct particle trajectories from the partial differential equation (119) for the field $S(x,t)$, a number of clearly defined mathematical manipulations, which are part of the classical theory of canonical transformations, see Greiner (1989), must be performed. The crucial point is that the latter theory is *not* part of QT and cannot be added 'by hand' in the limit $\hbar \to 0$. Thus, (118), (119) is, like QT, an *indeterministic* theory predicting not values of single event observables but only probabilities, which must be verified by ensemble measurements.

Given that we found a solution $S(x,t)$, $\rho(x,t)$ of (118), (119) for given initial values, we may ask which experimental predictions can be made with the help of these quantities. Using the fields $\vec{p}(x,t)$, $\tilde{E}(x,t)$ defined by Eqs. (19), (18), the Hamilton-Jacobi equation (119) takes the form

$$\frac{\vec{p}^2(x,t)}{2m} + V(x,t) = \tilde{E}(x,t), \tag{120}$$

The l.h.s. of (120) depends on the field $\vec{p}$ in the same way as a classical particle Hamiltonian on the (gauge-invariant) kinetic momentum $\vec{p}$. We conclude that the field $\vec{p}(x,t)$ describes a mapping from space-time points to particle momenta: If a particle (in an external electromagnetic field) is found at time t at the point x, then its kinetic momentum is given by $\vec{p}(x,t)$. This is not a deterministic prediction since we can not predict if a single particle will be or will not be at time t at point x; the present theory gives only a probability $\rho(x,t)$ for such an event. Combining our findings about $\vec{p}(x,t)$ and x we conclude that the experimental prediction which can be made with the help of $S(x,t)$, $\rho(x,t)$ is given by the following phase

space probability density:

$$w(x, p, t) = \rho(x, t)\delta^{(3)}(p - \frac{\partial \tilde{S}(x, t)}{\partial x}).$$ (121)

Eq. (121) confirms our claim that the classical limit theory is a statistical theory. The one-dimensional version of (121) has been obtained before by means of a slightly different method in I. The deterministic element [realized by the delta-function shaped probability in (121)] contained in the classical statistical theory (118), (119) is *absent* in QT, see I.

Eqs. (118), (119) constitute the mathematically well-defined limit $\hbar \to 0$ of Schrödinger's equation. Insofar as there is general agreement with regard to two points, namely that (i) 'non-classicality' (whatever this may mean precisely) is expressed by a nonzero $\hbar$, and that (ii) Schrödinger's equation is the most important relation of quantum theory, one would also expect general agreement with regard to a further point, namely that Eqs. (118), (119) present essentially (for a three-dimensional configuration space) the *classical limit* of quantum mechanics. But this is, strangely enough, not the case. With a few exceptions, see Van Vleck (1928), Schiller (1962a), Ballentine (1994), Shirai (1998), Klein (2009), most works (too many to be quoted) take it for granted that the classical limit of quantum theory is classical mechanics. The objective of papers like Rowe (1991), Werner & Wolff (1995), Landau (1996), Allori & Zanghi (2009) devoted to "..the classical limit of quantum mechanics.." is very often not the problem: *"what is* the classical limit of quantum mechanics ?" but rather: "*how to bridge the gap* between quantum mechanics and classical mechanics ?". Thus, the fact that classical mechanics is the classical limit of quantum mechanics is considered as *evident* and any facts not compatible with it - like Eqs. (118), (119) - are denied.

What, then, is the reason for this widespread denial of reality ? One of the main reasons is the principle of reductionism which still rules the thinking of most physicists today. The reductionistic ideal is a hierarchy of physical theories; better theories have an enlarged domain of validity and contain 'inferior' theories as special cases. This principle which has been extremely successful in the past *dictates* that classical mechanics is a special case of quantum theory. Successful as this idea might have been during a long period of time it is not necessarily universally true; quantum mechanics and classical mechanics describe different domains of reality, both may be true in their own domains of validity. Many phenomena in nature indicate that the principle of reductionism (alone) is insufficient to describe reality, see Laughlin & Pines (2000). Releasing ourselves from the metaphysical principle of reductionism, we accept that the classical limit of quantum mechanics for a three-dimensional configuration space is the statistical theory defined by Eqs. (118), (119). It is clear that this theory is not realized in nature (with the same physical meaning of the variables) because $\hbar$ is different from zero. But this is a different question and does not affect the conclusion.

11. Extended discussion

In this paper it has been shown, continuing the work of I, that the basic differential equation of non-relativistic QT may be derived from a number of clearly defined assumptions of a statistical nature. Although this does not exclude the possibility of other derivations, we consider this success as a strong argument in favor of the statistical interpretation of QT.

This result explains also, at least partly, the success of the canonical quantization rules (1). Strictly speaking, these rules have only be derived for a particular (though very important) special case, the Hamiltonian. However, one can expect that (1) can be verified for all

meaningful physical observables[4]. On the other hand, it cannot be expected that the rules (1) hold for arbitrary functions of x, p; each case has to be investigated separately. Thus, the breakdown of (1), as expressed by Groenewold's theorem, is no surprise.

The fundamental Ehrenfest-like relations of the present theory establish [like the formal rules (1)] a *correspondence* between particle mechanics and QT. Today, philosophical questions concerning, in particular, the 'reality' of particles play an important role in the thinking of some physicists. So: 'What is this theory about.. ?' While the present author is no expert in this field, the concept of *indeterminism*, as advocated by the philosopher Popper (1982), seems to provide an appropriate philosophical basis for the present work.

The present method to introduce gauge fields by means of a multi-valued dynamic variable ('phase function') has been invented many years ago but leads, in the context of the present statistical theory, nevertheless to several new results. In particular, it has been shown in section 3, that only the Lorentz force can exist as fundamental macroscopic force if the statistical assumptions of section 2 are valid. It is the only force (in the absence of spin effects, see the remarks below) that can be incorporated in a 'standard' differential equation for the dynamical variables ρ, S. The corresponding terms in the statistical field equations, *representing* the Lorentz force, are given by the familiar gauge (minimal) coupling terms containing the potentials. The important fact that all forces in nature follow this 'principle of minimal coupling' is commonly explained as a consequence of local gauge invariance. The present treatment offers an alternative explanation.

Let us use the following symbolic notation to represent the relation between the local force and the terms representing its action in a statistical context:

$$\Phi, \vec{A} \Rightarrow e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}. \tag{122}$$

The fields $\vec{E}$ and $\vec{B}$ are uniquely defined in terms of the potentials ϕ and $\vec{A}$ [see (23)] while the inverse is not true. Roughly speaking, the local fields are 'derivatives' of the potentials - and the potentials are 'integrals' of the local field; this mathematical relation reflects the physical role of the potentials ϕ and $\vec{A}$ as statistical representatives of the the local fields $\vec{E}$ and $\vec{B}$, as well as their non-uniqueness. It might seem that the logical chain displayed in (122) is already realized in the classical treatment of a particle-field system, where potentials have to be introduced in order to construct a Lagrangian, see e.g. Landau & Lifshitz (1967). However, in this case, the form of the local force is not derived but postulated. The present treatment 'explains' the form of the Lagrangian - as a consequence of the basic assumptions listed in section 2.

The generalization of the present theory to spin, reported in sections 6 and 7, leads to a correspondence similar to Eq. (122), namely

$$\vec{\mu}\vec{B} \rightarrow \vec{\mu} \cdot \frac{\partial}{\partial \vec{x}}\vec{B}. \tag{123}$$

The term linear in $\vec{B}$, on the l.h.s. of (123), plays the role of a 'potential' for the local force on the r.h.s. The points discussed after Eq. (122) apply here as well [As a matter of fact we consider $\vec{B}$ as a unique physical quantity; it would not be unique if it would be defined in terms of the tensor on the r.h.s. of (123)]. We see here a certain analogy between gauge and spin interaction terms. Unfortunately, the derivation of the spin force on the r.h.s. of (123) requires - in contrast to the Lorentz force - additional assumptions (see the remarks in sections 7, 8).

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[4] As indicated by preliminary calculations of the angular momentum relation corresponding to Eq. (8)

Our notation for potentials ϕ, $\vec{A}$, fields $\vec{E}$, $\vec{B}$, and parameters e, c suggests that these quantities are electrodynamical in nature. However, this is not necessarily true. By definition, the fields $\vec{E}$, $\vec{B}$ obey four equations (the homogeneous Maxwell equations), which means that additional conditions are required in order to determine these six fields. The most familiar possibility is, of course, the second pair of Maxwell's equations. A second possible realization for the fields $\vec{E}$, $\vec{B}$ is given by the *inertial* forces acting on a mass m in an arbitrarily accelerated reference frame, see Hughes (1992). The inertial gauge field may also lead to a spin response of the ensemble; such experiments have been proposed by Mashhoon & Kaiser (2006). It is remarkable that the present theory establishes a (admittedly somewhat vague) link between the two extremely separated physical fields of inertia and QT.

It is generally assumed that the electrodynamic potentials have a particular significance in QT which they do not have in classical physics. Let us analyze this statement in detail. The first part of the statement, concerning the significance of the potentials, is of course true. The second part, asserting that in classical physics all external influences *can* be described solely in terms of field strengths, is wrong. More precisely, it is true for classical mechanics but not for classical physics in general. A counterexample - a theory belonging to classical physics but with potentials playing an indispensable role - is provided by the classical limit (118),(119) of Schrödinger's equation. In this field theory the potentials play an indispensable role because (in contrast to particle theories, like the canonical equations) no further derivatives of the Hamiltonian, which could restore the fields, are to be performed. This means that the significance of the potentials is not restricted to quantum theory but rather holds for the whole class of *statistical* theories discussed above, which contains both quantum theory and its classical limit theory as special cases. This result is in agreement with the statistical interpretation of potentials proposed in section 3.

The precise characterization of the role of the potentials is of particular importance for the interpretation of the Aharonov-Bohm effect. The 'typical quantum-mechanical features' observed in these phase shift experiments should be identified by comparing the quantum mechanical results not with classical mechanics but with the predictions of the classical statistical theory (118), (119). The predictions of two statistical theories, both of which use potentials to describe the influence of the external field, have to be compared.

The limiting behavior of Schrödinger's equation as $\hbar \to 0$, discussed in section 10, is very important for the proper interpretation of QT. The erroneous belief (wish) that this limit can (must) be identified with classical mechanics is closely related to the erroneous belief that QT is able to describe the dynamics of individual particles. In this respect QT is obviously an *incomplete* theory, as has been pointed out many times before, during the last eighty years, see e.g. Einstein (1949), Margenau (1935) , Ballentine (1970), Held (2008). Unfortunately, this erroneous opinion is historically grown and firmly established in our thinking as shown by the ubiquitous use of phrases like 'the wave function of the electron'. But it is clear that an erroneous identification of the domain of validity of a physical theory will automatically create all kinds of mysteries and unsolvable problems - and this is exactly what happens. Above, we have identified one of the (more subtle) problems of this kind, concerning the role of potentials in QT, but many more could be found. Generalizing the above argumentation concerning potentials, we claim that characteristic features of QT cannot be identified by comparison with classical mechanics. Instead, quantum theory should be compared with its classical limit, which is in the present $3D$-case given by (118), (119) - we note in this context that several 'typical' quantum phenomena have been explained by Kirkpatrick (2003) in terms of classical probability theory. One has to compare the solutions of the classical, nonlinear

equations (118), (119) with those of the quantum mechanical, linear equations, (50), (51), in order to find out which 'typical quantum-mechanical features' are already given by statistical (nonlocal) correlations of the classical limit theory and which features are really quantum-theoretical in nature - related to the nonzero value of $\hbar$.

12. Summary

In the present paper it has been shown that the method reported in I, for the derivation of Schrödingers's equation, can be generalized in such a way that essentially all aspects of non-relativistic QT are taken into account. The success of this derivation from statistical origins is interpreted as an argument in favor of the SI. The treatment of gauge fields and spin in the present statistical framework led to several remarkable new insights. We understand now why potentials (and not local fields) occur in the field equations of QT. The non-uniqueness of the potentials and the related concept of gauge invariance is not a mystery any more. Spin is derived as a kind of two-valuedness of a statistical ensemble. The local forces associated with the gauge potentials, the Lorentz force and the force experienced by a particle with magnetic moment, can also be derived. Apart from some open questions in the area of non-relativistic physics, a major problem for future research is a relativistic generalization of the present theory.

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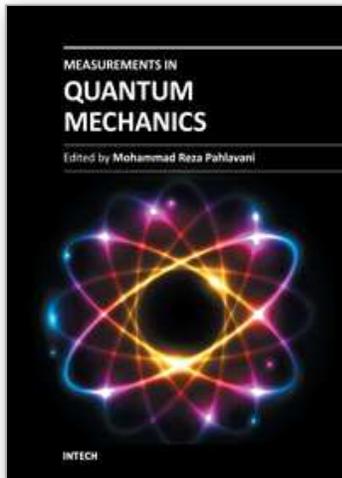
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