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Optical Fibers in Phase Space: A Theoretical Framework

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1. Introduction

Since their discovery, optical fibers have received increasing attention due to its important technological applications (Bottacchi, 2002; Culshaw, 1997; Harmon, 2001; Herrmann, 1973; Keyl, 2002; Lauterborn et al., 1997; Prasad & Williams, 1991; Shimizu et al., 1997; Way, 1998; Young, 2000). A fiber is an optical waveguide in which the propagation of an optical wave is confined to two dimensions (the cross section dimensions). The dimension of the confinement must be comparable to the wavelength of the light which one would like to confine (Kogelnik, 1979).

A fiber is formed by a region with a refractive index larger than that of the surrounding media; this condition assures the total internal reflections at the interfaces required for the beam propagation. If an integer number of wavelengths have traveled between two consecutive reflections, a standing wave pattern will be developed, producing a constructive interference which allows high electromagnetic power to be transmitted along the fiber. A change in the fiber geometry, modifies the reflection angle and consequently the total number of reflections: rays traveling thicker fibers require more reflections to travel along the same length in the propagation direction. This implies that the effective velocity of light in thicker fibers must be slower; this effect also produces a change in the phase of the output signal (Arnaud, 1976). These phenomena are cumbersome to explain by standard optical methods (Kogelnik, 1979; Torchigin & Torchigin, 2003) and are easily predicted by the use of the Wigner Distribution Function (Reyes et al., 1999). One of the main advantages of the phase space approach to solve optical problems is the simplicity of the mathematical calculations compared with the traditional treatment (Bottacchi, 2002). This is due to the fact that in phase space representation, the relevant properties of the system can be obtained by simple matrices products.

Quantum mechanically, the development of nonlinear optics allowed the generation and manipulation of new quantum states of light, going from the simplest and common one, the so-called coherent states (Glauber, 1963), to squeezed states (Walls, 1983; Yuen, 1976), Fock states (Lvovsky et al., 2001) or entangled states (Ou et al., 1992). A full quantum theoretical analysis of the three-photon states is contained in the Wigner function (Leonhardt, 2001) that has proven to be very helpful to visualize in the phase space (the amplitude $\mathbf{q}$ and phase $\mathbf{p}$ quadratures) quantum mechanical system defined by its density matrix. This has already been the case for some quantum states of light such as the coherent state, the squeezed

vacuum or the bright squeezed state (Breitenbach et al., 1995; 1997; Smithey et al., 1993), whose Wigner function has been experimentally reconstructed using homodyne quantum tomography, a technique that allows the measurement of the marginal probability distribution that expresses the quadrature amplitude distribution. The use of optical fiber for quantum squeezing has considerable technological advantages, such as generating squeezing directly at the communications wavelength and the use of existing transmission technology (Corney et al., 2008).

More generally, the Wigner function contains the full information about the quantum states (Wigner, 1932) and their moments allow to differentiate between paraxial regime, wave-like regime and chaotic behavior (Rivera et al., 1997). More particularly, it allows us to establish the quantum correlations between the different generated modes in the case of twin photons or photon triplets (Benchekh et al., 2007). The Wigner function is a positive definite function in the phase space only for classical states with Gaussian marginal probability distributions (Rivera & Castano, 2010). However, it can be negative in some circumstances for particular quantum states of light. These negativities are the signature of highly nonclassical behaviour of a quantum state (Rivera & Castano, 2010) as it has been observed for a quantum state of light prepared in a single-photon Fock state (Lvovsky et al., 2001). These quantum negativities are also present in the case of complete degenerate three-photon states obtained by third order optical parametric fluorescence or amplification, and also for aberrated optical systems.

Historically, the so-called *Wigner Distribution Function* (Wigner, 1932) has been of central importance as an alternative description of Quantum Mechanics (Kim & Noz, 1991). However, these phase-space mathematical tool has found exciting applications in a wide range of the physical sciences and even engineering ranging from statistical mechanics (Green, 1951; Mori et al., 1962) to optics (Perinova et al., 1998; Schleich, 2001; Wolf, 2004). Moreover, it has become the basis of an entire discipline: time-frequency representation of wave phenomena (Allen & Mills, 2004; Boashash, 2003; Cohen, 1995; Grochenig, 2000). There exist several reviews of the quantum phase-space distribution functions, in particular of the Wigner distribution function. A concise but authorative review of the quantum distribution functions is that by Wigner (Wigner, 1971). A good mathematical treatment of the quantum distribution functions and related operator algebra is given in the book by (Louisell, 1973). Some extensive reviews of the quantum distribution functions are given by (Balazs & Jennings, 1984; Berry, 1977; Filinov et al., 2008; Groot & Suttorp, 1972; Hillery et al., 1984; Lee, 1995; OConnell, 1983; Takabayasi, 1954). Applications of the Wigner distribution function to Optics are reviewed by (Dragoman, 1997; Dodonov, 2002; Mack & Schleich, 2003; Zalevsky & Mendlovic, 1997), and for the particular case of fibers on the works (Bao & Chen, 2011; Benabid & Roberts, 2011; Benchekh et al., 2007; Corney et al., 2008; Leonhardt, 2001; Rivera & Castano, 2010a).

As Wigner functions, the *Lie Algebra*, due to its mathematical simplicity to solve differential equations by numerical integration, has become an important aid for the solution of different problems in classical and quantum mechanics (Bakhturin, 2003; Frank & van Isacker, 1994; Hamermesh, 1962; Jacobson, 1979). A Lie treatment of geometrical optics and aberrations has been developed by (Dragt & Finn, 1976), and it is a new approach to fiber optics (Reyes et al., 1999; Reyes & Castano, 2000) that simplifies the traditional solution of optical problems (Born & Wolf, 1999) to the determination of the corresponding Symplectic Map associated to the optical system, thus reducing the problem to simple matrices products. The Gaussian Symplectic map helps to find the Wigner distribution function of the probability density of an optical fiber, and from it, it is possible to obtain all the physical information required to analyze the fiber (Rivera & Castano, 2010a).

This chapter presents a brief review of the phase-space analysis applied to fiber Optics, using the Wigner Distribution Function. The hope is that it will show the beauty, elegance and usefulness of this mathematical construction. The rest of the chapter is organized as follows. Section 2 gives a short review of phase space representation using the Wigner Distribution Function which describes some of its important properties and its physical interpretation. Section 3 presents the description of the Maxwell equations under paraxial approach (considering parallel rays close to the optical axis of the system) that describe the light propagation in a fiber by a parabolic type equation that is completely equivalent to the quantum system Schrödinger equation for a bidimensional potential-well time-dependent. Section 4 shows an example, analyzing a gaussian beam propagation through a fiber.

2. Phase space representation

The standard formulation of quantum mechanics either in the Schrödinger (Schrodinger, 1946) or in Heisenberg pictures (Heisenberg, 1930) may create an impression that quantum and classical dynamics are completely different (Dirac, 1935). However, there are representations in which quantum dynamics seems to resemble classical statistical mechanics, and where the state of a quantum system is represented by the quasiprobability distribution in phase space of the corresponding classical system (Kim & Noz, 1991). Of course, there are at least two important differences (Hillery et al., 1984):

1. Quasiprobability distributions may take negative values (unlike the true probability distributions).

2. The classical distribution can be localized at a point in phase space, whereas the quantum distribution must always be spread in a finite phase volume, in agreement with uncertainty relations.

Among different quasiprobability distributions Cohen (1995), the Wigner Distribution Function, introduced by Wigner in 1932[1] (Wigner, 1932), is the only one for which the quantum evolution law coincides with the classical one for the case of linear dynamics (Moyal, 1949). The Wigner distribution function is a real valued quasiprobability distribution containing all information available about the system. Its popularity stems from its characteristics (Wigner, 1932):

• It has a close connection to the marginal probability distributions characterizing the probabilities of the outcomes of von Neumann measurements of the system.

• It lends itself to a visualization of quantum states, and some of their properties.

• It is a versatile calculation tool for normally ordered operators.

With the use of this distribution function, it is straightforward to cast quantum mechanics in a form which resembles the classical theory of statistical averages over the classical phase space, with the Wigner distribution function playing a role analogous to a probability function

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[1] Wigner's original motivation for introducing it, was to be able to calculate the quantum correction to the second virial coefficient of a gas, which indicates how it deviates from the ideal gas law (Wigner, 1932). Classically, to ca1culate the second virial coefficient one needs a joint distribution of position and momentum. So Wigner devised the simplest joint distribution that gave, as marginals, the quantum mechanical distributions of position and momentum. The quantum mechanics came in the distribution, but the distribution was used in the classical manner. It was a hybrid method. Also, Wigner was motivated in part by the work of Kirkwood (Kirkwood, 1933) who had previously calculated this quantity but Wigner improved it.

(Fairlie, 1964). Consequently, the Wigner distribution function has been used extensively to study the classical limit of quantum mechanical systems (Kim & Noz, 1991; Mayer & Band, 1947; Moyal, 1949).

There is in principle an infinite variety of quantum phase-space distribution functions corresponding to an infinite number of possible ordering rules of two noncommuting operators and their linear combinations (Lee, 1995). The general class of this distributions is given by (Cahill & Glauber, 1969; Cohen, 1966; Kakazu et al., 2007). Distribution functions in general have different properties and are associated with various dynamical equations, so they may be described most conveniently by distribution functions having different characteristics (Hillery et al., 1984). Other distribution functions that have been considered in the past include those of Kirkwood (Kirkwood, 1933), Margenau-Hill (Johansen & Luis, 2004; Margenau & Hill, 1961; Terletsky, 1937)[2], Husimi (Husimi, 1940), Q-functions (Husimi, 1940; Kano, 1965; Smith, 2006), Page (Page, 1952), Glauber-Sudarshan (Glauber, 1963; Sudarshan, 1963), Rihaczek (Rihaczek, 1968), and Choi-Williams (Choi & Williams, 1989). Another very used function is the ambiguity function (Bastiaans, 1980; Marks & Hall, 1979; Woodward, 1963).

Phase space representation, in particular through the so-called *Wigner Distribution Function*, has proven to be a very effective tool applied in many branches of physics (Cohen, 1995; Kim & Wigner, 1987; Kim & Noz, 1991; Mecklenbrauker et al., 1997; Moyal, 1949; Stewart et al., 2002; Wigner, 1932), and more specifically in fiber optics (Dragoman & Meunier, 1998; Kominis & Hizanidis, 2002; Reyes et al., 1999a; Rivera & Castano, 2010a; Sheppard & Larkin, 2000; Voss et al., 1999). The Wigner distribution function was invented by Wigner (Wigner, 1932) to study the quantum corrections to the classical behavior of certain statistical systems described by the Boltzmann formula. For the evaluation of the Wigner function are various implementations (Bala & Prabhu, 1989; Easton et al., 84; Eilouti & Khadra, 1989; Flandrin et al., 1984; Frank et al., 2000; Gupta & Asakura, 1986; Lohmann, 1980; Lopez et al., 2002; Maanen, 1985; Mateeva & Sharlandjiev, 1986; Rivera et al., 1997; Subotic & Saleh, 1984).

In this chapter the notation will be for the optical position coordinates $\mathbf{q}$ (that corresponds to the interaction of the ray with a $z = 0$ reference plane), and for the canonically conjugate momentum $\mathbf{p}$ (which describes the direction of the ray with respect to the normal at the point $\mathbf{q}$ that evolves over the system's optical axis, z) (Buchdahl, 1970).

Consider a particle in one dimension. Classically, the particle is described by a *phase space distribution* $P_{cl}(\mathbf{q}, \mathbf{p})$. The average of a function of position and momentum $A(\mathbf{q}, \mathbf{p})$ can then be expressed as

$$\langle A \rangle_{cl} = \int_{-\infty}^{\infty} d\mathbf{q} \int_{-\infty}^{\infty} d\mathbf{p} \, A(\mathbf{q}, \mathbf{p}) \, P_{cl}(\mathbf{q}, \mathbf{p}) \, . \tag{1}$$

A quantum mechanical particle is described by a density matrix $\hat{\rho}$, and the average of a function of the position and momentum operators $\hat{A}(\hat{q}, \hat{p})$ as

$$\langle A \rangle_{quant} = \text{Tr} \left(\hat{A} \, \hat{\rho} \right) . \tag{2}$$

It must be admitted that, given the classical expression $A(\mathbf{q}, \mathbf{p})$, the corresponding self adjoin operator $\hat{A}$ is not uniquely defined. The use of a *quasiprobability phase space distribution*

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[2] Kirkwood attempted to extend the classical theory to the quantum case and devised the distribution commonly called the Rihaczek or Margenau-Hill distribution to do that. Many years later, Margenau and Hill derived the distribution that bears their name. The importance of the Margenau-Hill work is not the distribution but the derivation. They were also the first to consider joint distributions involving spin.

$P_Q(\mathbf{q}, \mathbf{p})$, however, does give such a definition by expressing the quantum mechanical average as

$$\langle A \rangle_{quant} = \int_{-\infty}^{\infty} d\mathbf{q} \int_{-\infty}^{\infty} d\mathbf{p} \, A(\mathbf{q}, \mathbf{p}) \, P_Q(\mathbf{q}, \mathbf{p}) \, , \tag{3}$$

where the function $A(\mathbf{q}, \mathbf{p})$ can be derived from the operator $\hat{A}(\hat{q}, \hat{p})$ by a well defined correspondence rule. This allows one to cast quantum mechanical results into a form in which they resemble classical ones. This is a reformulation of Schrödinger quantum mechanics which describes states by functions in configuration space (Kim & Noz, 1991).

In the case where P_Q in (3) is chosen to be the Wigner Distribution function (Wigner, 1932), then the correspondence between $A(\mathbf{q}, \mathbf{p})$ and $\hat{A}$ is that proposed by Weyl (Weyl, 1927), as was first demonstrated by Moyal (Moyal, 1949).

The requirement given by Eq. (3) let us to define a function in the $6N$ dimensional $\mathbf{q}, \mathbf{p}$ phase space, called the *Wigner Distribution Function* in terms of the density matrix, ρ as:

$$\mathcal{W}_\rho(\mathbf{q}, \mathbf{p}; t) \equiv \left(\frac{1}{\pi\hbar}\right)^{3N} \int_{-\infty}^{\infty} d\mathbf{r} \, \exp\left(\frac{2i}{\hbar} \mathbf{p} \cdot \mathbf{r}\right) \rho(\mathbf{q} - \mathbf{r}, \mathbf{q} + \mathbf{r}; t) \, , \tag{4}$$

Because for pure states described by a wavefunction Ψ, the density matrix is given by (vonNeumann, 1927)

$$\rho(\mathbf{q}, \mathbf{q}') = \Psi^*(\mathbf{q}') \, \Psi(\mathbf{q}) \, , \tag{5}$$

the expression (4) for pure states can be rewritten in coordinate representation as:

$$\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\mathbf{r} \, \Psi^*\left(\mathbf{q} - \frac{1}{2}\mathbf{r}; z\right) e^{-i[\mathbf{p} \cdot \mathbf{r}]/\hbar} \, \Psi\left(\mathbf{q} + \frac{1}{2}f\mathbf{r}; z\right) \tag{6}$$

or taking the Fourier transform (Goodman, 1968) in momentum representation as

$$\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\mathbf{r} \, \tilde{\Psi}\left(\mathbf{p} + \frac{1}{2}\mathbf{r}; z\right) e^{-i[\mathbf{q} \cdot \mathbf{r}]/\hbar} \, \tilde{\Psi}^*\left(\mathbf{p} - \frac{1}{2}\mathbf{r}; z\right) \, , \tag{7}$$

where $\hbar$ is the Planck constant divided by 2π, $\tilde{\Psi}$ denotes the Fourier transform of Ψ and the asterisk represents the complex conjugate. In geometric optics, $\hbar$ corresponds to the wavelength λ of the beam (Wolf, 2004).

In Wigner phase-space representation everything we have said for the coordinate domain holds for the momentum domain because the Wigner distribution is basically identical in form in both domains (compare equations 6 and 7). The complete symmetry between $\mathbf{q}$ and $\mathbf{p}$ in the former definitions of the Wigner function (equations 6 and 7), indicates that space and momentum have equal weight in this description (Moyal, 1949). Due to this, the Wigner distribution function can be thought as the expected value of the parity operator around $(\mathbf{q}, \mathbf{p})$ in the phase space (Royer, 1997); i.e. the Wigner function is proportional to the overlap of $\Psi(\mathbf{q}, z)$ with its specular image around $(\mathbf{q}, \mathbf{p})$, that is a measure of "how much centered" is $\Psi(\mathbf{q}, z)$. Note that the Wigner distribution function is a 4-dimensional phase space distribution function, where two dimensions correspond to real space and the other two to momentum space. The Wigner distribution function is a real function that can take either positive and negative values, however, only for a Gaussian the Wigner distribution function is positive everywhere (Hudson, 1974; Soto & Claverie, 1983); therefore, one cannot interpret it as a classical probability function in phase space (Lee, 1995). Tthe value of $\mathcal{W}_\Psi$ mirrors closely the intuitive objects in the model, that in the case of quantum optics may be the coherent states

of the radiation field (Glauber, 1965), and in monochromatic paraxial wave optics, they are often beams with Gaussian position and inclination distributions (Hillery et al., 1984; Rivera & Castano, 2010).

From all phase space representations, the Wigner Distribution Function can be uniquely distinguished (among shift-invariant joint distributions) by imposing a requirement of correct marginals with respect to arbitrary directions in the time-frequency plane, thus connecting the Wigner distribution with the fractional Fourier transform (Atakishiyev et al., 1999). It also contains all the information of the system and it can be proved that it contains the hologram of the signal (Wolf & Rivera, 1997).

For numerical calculations it is very useful to note that the Wigner distribution function is the Fourier transform of the kernel (Wigner, 1932):

$$\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z) = \Psi\left(\mathbf{q} + \frac{1}{2}\,\mathbf{r}\right)\,\Psi^*\left(\mathbf{q} - \frac{1}{2}\,\mathbf{r}\right). \tag{8}$$

Because $\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z)$ is Hermitian $[\mathcal{W}_\Psi(\mathbf{q}, \mathbf{r}) = \mathcal{W}_\Psi^*(\mathbf{q}, -\mathbf{r})]$, the Wigner distribution function is real (Moyal, 1949).

When we integrate $\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p})$ over $\mathbf{p}$, we obtain the probability distribution in $\mathbf{q}$, while if we integrate $\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p})$ over $\mathbf{q}$, we obtain the probability distribution in $\mathbf{p}$, (Moyal, 1949). Then, to recover either the image $|\Psi(\mathbf{q}; z)|^2$ (light intensity on the two-dimensional screen of coordinate $\mathbf{q}$ at the optical axis position z) or the diffraction pattern $|\tilde{\Psi}(\mathbf{p}; z)|^2$, it is necessary to make a simple projection of the Wigner distribution function (Wigner, 1932):

$$|\Psi(\mathbf{q}; z)|^2 = \int_{-\infty}^{\infty} d\mathbf{p}\,\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z), \tag{9}$$

$$|\tilde{\Psi}(\mathbf{p}; z)|^2 = \int_{-\infty}^{\infty} d\mathbf{q}\,\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z). \tag{10}$$

If the signal or image of interest is nonstationary, the Wigner distribution function gives the local spectrum centered at $\mathbf{p}$ as a function of location (Bartelt et al., 1980). Thus, the total energy of $\Psi(\mathbf{q}, z)$ can be obtained from integration of $\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z)$ over the entire phase space (Hillery et al., 1984).

Moreover $|\mathcal{W}_\Psi(\mathbf{q}, \mathbf{p}; z)| \leq (2\pi\hbar)^{-1}$.

Another interesting property (Schempp, 1986) is that the Wigner distribution function has the same extension and is band-limited as the function $\Psi(\mathbf{q}, z)$.

The Wigner distribution function is the expectation value of the parity operator about the phase-space point $\mathbf{q}, \mathbf{p}$ (Royer, 1997). To show this, let us first rewrite

$$\mathcal{W}(\mathbf{q}, \mathbf{p}) = \left(\frac{1}{\pi\hbar}\right)^{3N} \langle\Psi|\hat{\Pi}_{\mathbf{q}, \mathbf{p}}|\Psi\rangle, \tag{11}$$

where the operator $\hat{\Pi}_{\mathbf{q}, \mathbf{p}}$ has the following three equivalent expressions:

$$\begin{aligned}
\hat{\Pi}_{\mathbf{q}, \mathbf{p}} &= \int_{-\infty}^{\infty} dr\, e^{2ipr/\hbar}\,|q - r\rangle\langle q + r|, \\
&= \int_{-\infty}^{\infty} dk\, e^{-2ikq/\hbar}\,|p + k\rangle\langle p - k|, \\
&= \left(\frac{1}{\pi\hbar}\right)^{3N} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dr\, e^{i[k(\hat{R} - q) + r(\hat{P} - p)]/\hbar}.
\end{aligned} \tag{12}$$

Let us now consider the special case $q = 0, p = 0$, and denote $\hat{\Pi}_{q=0,p=0} = \hat{\Pi}$; we have

$$
\hat{\Pi} = \int_{-\infty}^{\infty} \mathrm{d}q \, | -q \rangle \, \langle q | \, ,
$$

$$
= \int_{-\infty}^{\infty} \mathrm{d}p \, | p \rangle \, \langle -p | \, ,
$$

$$
= \left(\frac{1}{\pi \hbar} \right)^{3N} \int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}y \, e^{i[k\hat{R} + y\hat{P}]/\hbar} \, . \tag{13}
$$

From (13) it is immediately apparent that $\hat{\Pi}$ is the parity operator (about the origin): it changes $\Psi(\mathbf{q})$ into $\Psi(-\mathbf{q})$ and $\tilde{\Psi}(\mathbf{p})$ into $\tilde{\Psi}(-\mathbf{p})$, or equivalently

$$
\hat{\Pi} \hat{R} \hat{\Pi} = -\hat{R} \, , \qquad \hat{\Pi} \hat{P} \hat{\Pi} = -\hat{P} \, , \tag{14}
$$

moreover,

$$
\hat{\Pi}^{-1} = \hat{\Pi} \, . \tag{15}
$$

We now observe that $\hat{\Pi}_{\mathbf{q},\mathbf{p}}$ may be obtained from $\hat{\Pi}$ by a unitary transformation

$$
\hat{\Pi}_{\mathbf{q},\mathbf{p}} = \hat{\mathcal{D}}(\mathbf{q}, \mathbf{p}) \, \hat{\Pi} \, \hat{\mathcal{D}}(\mathbf{q}, \mathbf{p})^{-1} \, ; \tag{16}
$$

here

$$
\hat{\mathcal{D}}(\mathbf{q}, \mathbf{p}) = e^{i(p\hat{R} - q\hat{P})/\hbar} \tag{17}
$$

is a phase-space displacement operator, introduced by Glauber (Glauber, 1963) in connection with a different, though related, type of phase-space representation of quantum mechanics, the coherent-state representation. We have the actions

$$
\hat{\mathcal{D}}(\mathbf{q}, \mathbf{p})^{-1} \, \hat{R} \, \hat{\mathcal{D}}(\mathbf{q}, \mathbf{p}) = \hat{R} + q \, , \tag{18}
$$

$$
\hat{\mathcal{D}}(\mathbf{q}, \mathbf{p})^{-1} \, \hat{P} \, \hat{\mathcal{D}}(\mathbf{q}, \mathbf{p}) = \hat{P} + p \, , \tag{19}
$$

$$
\hat{\mathcal{D}}(\mathbf{q}, \mathbf{p})^{-1} \, F(\hat{R}, \hat{P}) \, \hat{\mathcal{D}}(\mathbf{q}, \mathbf{p}) = F(\hat{R} + q, \hat{P} + p) \, . \tag{20}
$$

From this follows directly

$$
\hat{\Pi}_{\mathbf{q},\mathbf{p}} (\hat{R} - q) \hat{\Pi}_{\mathbf{q},\mathbf{p}} = -(\hat{R} - q) \, , \tag{21}
$$

$$
\hat{\Pi}_{\mathbf{q},\mathbf{p}} (\hat{P} - p) \hat{\Pi}_{\mathbf{q},\mathbf{p}} = -(\hat{P} - p) \, , \tag{22}
$$

that is, $\hat{\Pi}_{\mathbf{q},\mathbf{p}}$ reflects about the phase-space point $\mathbf{q}, \mathbf{p}$ and is thus the parity operator about that point. Note that

$$
(\hat{\Pi}_{\mathbf{q},\mathbf{p}})^2 = 1 \, . \tag{23}
$$

The Wigner function, is thus $\left(\frac{1}{\pi \hbar} \right)^{3N}$ times the expectation value of the parity operator about $\mathbf{q}, \mathbf{p}$. Alternatively, $\mathcal{W}(\mathbf{q}, \mathbf{p})$ is proportional to the overlap of Ψ with its mirror image about $\mathbf{q}, \mathbf{p}$, which is clearly a measure of how much Ψ is "centered" about $\mathbf{q}, \mathbf{p}$.

3. Light propagation on a fiber

Propagation of light in a fiber is governed by Maxwell equations (Born & Wolf, 1999). Consider a monocromatic light beam of frequency ω propagating through a fiber of refractive index n, described by the wavefunction $\Psi(x, y, z)$. It can be shown that this beam obeys the Helmholtz equation (Born & Wolf, 1999):

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{\omega^2}{c^2} n^2 \Psi = 0 \, , \tag{24}$$

where $n = n(x, y, z)$ is the refractive index of the fiber.

Under paraxial approach, the beam is almost parallel and close to the optical axis of the system, z, then $n = n(0, 0, z)$, and Ψ vary slowly with z allowing to neglect second order derivatives in the z direction. This considerations let to write equation (24) as the parabolic type equation (Leontovich & Fock, 1946)

$$\frac{i}{k} \frac{\partial \Psi}{\partial \tau} = \frac{1}{2k^2} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) + \left(n_0^2 - n^2 \right) \Psi \, , \tag{25}$$

where n_0 is the vacuum refractive index, and

$$\tau = - \int_0^z \frac{1}{n_0(z')} dz' \, . \tag{26}$$

Equation (25) shows that the light beam propagation in the paraxial approximation is described by a Schrödinger equation where the wavelenght $\lambda = \frac{1}{k}$ plays the role of the Planck constant and instead of time appears z. The potential well is given by the refractive index $n_0^2 - n^2(x, y, z)$. This treatment translate the problem of solving the Helmholtz equation (24) to solve the Schrödinger equation for a system with two degrees of freedom (x, y) in a time-dependent (z) potential well.

This Schrödinger equation (25) is valid for any wave that follows the Helmholtz equation under the paraxial approach (for a detailed description check (Arnaud, 1976; Manko, 1986; Marcuse, 1972). The validity of this approximation can be verified using the moments of the Wigner distribution function of the solution as shown in (Rivera et al., 1997). To solve the problem in fiber optics it can be applied the formalism of symplectic groups through coherent state representation of quantum mechanics (Manko & Wolf, 1985).

In general, the output Wigner function of an optical system is related to the input Wigner through (Castano et al., 1982; Gutierrez & Castano, 1992):

$$\mathcal{W}_{\Psi_{out}}(\mathbf{q}, \mathbf{p}; z) = \mathcal{W}_{\Psi_{in}}(a\,\mathbf{q} + b\,\mathbf{p}, \, c\,\mathbf{q} + d\,\mathbf{p}; \, z) \, , \tag{27}$$

where a, b, c and d are parameters which depend on the specific system under study.

As an example, the free space propagator is given by

$$\mathcal{W}_{\Psi_{out}}(\mathbf{q}, \mathbf{p}; z) = \mathcal{W}_{\Psi_{in}} \left(\mathbf{q} - \frac{z}{k}\mathbf{p}, \mathbf{p}; z \right) \, , \tag{28}$$

for a lens of focal length f, we have

$$\mathcal{W}_{\Psi_{out}}(\mathbf{q}, \mathbf{p}; z) = \mathcal{W}_{\Psi_{in}} \left(\mathbf{q}, \frac{1}{f}\mathbf{q} + \mathbf{p}; z \right) \, , \tag{29}$$

and to obtain a Fourier transform we use:

$$\mathcal{W}_{\Psi_{out}}(\mathbf{q}, \mathbf{p}; z) = \mathcal{W}_{\Psi_{in}} \left(-\mathbf{p}, \mathbf{q}; z \right) \, . \tag{30}$$

4. Gaussian beam propagation in optical fibers

To model optical fibers it is common to consider gaussian beams that travel freely through space (Rivera & Castano, 2010). Gaussians are also ubiquitous in quantum mechanics, where they are intimately related to the harmonic oscillator (Gitterman, 2003; Moshinsky, 1996; Sako & Diercksen, 2003), to the coherent (Gori et al., 2003; Grewal, 2002; Grosshans et al., 2003; Lauterborn et al., 1993; Lesurf et al., 1993) and squeezed states formalism (Agarwal & Ponomarenko, 2003; Dodonov, 2002; Kim et al., 2002; Sohma & Hirota, 2003). In Quantum Optics, Gaussian beams are fundamental to test and to compare wave optical models and systems (Berry, 1994; Oraevsky, 1998; Rivera et al., 1997).

Using Fermat minimal action principle, it can be proved that the system is governed by the optical Hamiltonian (Rivera et al., 1995):

$$H = -\sqrt{n^2 - p^2} \, . \tag{31}$$

This Hamiltonian generates a ray path, i.e. a unidimensional group of canonical transformations of the points of the optical phase space. In a three-dimensional optical medium we denote the two screen coordinates (perpendicular to the optical axis) by $\mathbf{q} = (x.y)$ and the optical axis coordinate as z.

When the canonical transformation has a nonlinear part, it is possible to identify this nonlinearity as the effect of aberrations as is studied in (Rivera et al., 1997; Rivera & Castano, 2010). An alternative approach (called coherent states for Lie groups) uses the continuous representations in quantum mechanics as a particular case of arbitrary Lie groups and can be used in fiber optics for analyzing nonquadratic media under the action of Hamiltonians that are the linear form of the Lie group representation with z dependent coefficients (Klauder, 1964). In geometric optics (paraxial approach), momentum is $|\mathbf{p}| = n \sin \theta$, where n denotes the refractive index and θ is the angle between the ray and the optical axis (Wolf, 2004).

A Gaussian function Γ associated to the one-dimensional real coordinate x is defined as (Simon, 2002)

$$\Gamma(x) = M \, \exp \left[-\frac{(x - x_0)^2}{2w_0} + ip_0 x \right] \, , \tag{32}$$

where $M = \left(\frac{w_1}{\pi |w_0|^2} \right)^{1/4}$, x_0, p_0 are real numbers, and $w_0 = w_1 + iw_2$, $w_1 > 0$ is a complex number. The dimension of w_0 is $[x^2]$, the one of x_0 is $[x]$, and that of p_0 is $[x^{-1}]$. The pre-exponential factor M guarantees the normalization condition

$$\langle \Gamma \mid \Gamma \rangle = \int_{-\infty}^{\infty} \mathrm{d}x \, \Gamma^*(x) \, \Gamma(x) = 1 \, . \tag{33}$$

This Gaussian is centered at x_0 and has a complex width $\sqrt{2w_0}$. The value at its maximum is M. If $p_0 \neq 0$ or $w_2 \neq 0$, this Gaussian shows oscillations.

The Fourier transform of the Gaussian Γ (Equation 32) provides the momentum representation of the beam (Goodman, 1968):

$$\tilde{\Gamma}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ipx} \, \Gamma(x) = \left(\frac{w_1}{\pi} \right)^{\frac{1}{4}} \exp \left[-\frac{w_0 (p - p_0)^2}{2} - ix_0 (p - p_0) \right] \, . \tag{34}$$

Interestingly, it is another Gaussian, centered in p_0, with width $\sqrt{2/w_0}$ and it oscillates for $x_0 \neq 0$.

In phase space, the Gaussian Γ, is represented by its Wigner distribution function (Rivera & Castano, 2010):

$$\mathcal{W}_\Gamma(x,p) = 2\exp\left\{-\frac{(x-x_0)^2}{w_1} - \frac{|w_0|^2}{w_1}(p-p_0)^2 + \frac{2w_2}{w_1}(x-x_0)(p-p_0)\right\}, \qquad (35)$$

that is a two-dimensional Gaussian; coordinate centered at x_0 with width $\sqrt{w_1}$, momentum center p_0 with width $|w_0|/\sqrt{w_1}$, and tilted by $\arctan(2w_2/w_1)$.

A *Vacuum Coherent State* (Dodonov, 2002) is a Gaussian function with $x_0 = p_0 = 0$ and $w_0 = 1$. It has the important property of being the only state described by the same function in both coordinate and momentum representation (Dodonov & Manko, 2000). A generalized *coherent state* is described by a Gaussian function with $w_0 = 1$, but x_0 and p_0 arbitrary. The state for which $w_0 \neq 1$ is called a *squeezed state* (Dodonov, 2002).

To evaluate the Wigner distribution function of a sectioned fiber we assume that an optical fiber is a cylinder of radius a and infinite length. We will consider a Gaussian traveling through a fiber with constant refractive index n until a break generated by a section with different refractive index m encapsulated by two parabolic surfaces. The Symplectic Map for this system is given by the product of the initial propagator (before the break), first refraction, propagation between refraction surface, second refraction (after the break), and last propagation. It can be shown that the Symplectic Map of this system is (Reyes & Castano, 2000)

$$M_{total} = \dots e^{:F_4:}\, e^{:F_2:}, \qquad (36)$$

where

$$e^{:F_2:} = e^{-\frac{a}{2n}\sin(2k\pi z):\vec{\mathbf{p}}^2:}e^{\alpha(n-m):\vec{\mathbf{q}}^2:}e^{-\frac{\gamma}{2m}:\vec{\mathbf{p}}^2:}$$
$$\times e^{-\frac{\gamma}{2}:\vec{\mathbf{p}}^2:}e^{-\alpha(n-m):\vec{\mathbf{q}}^2:}e^{-\frac{a}{2n}\sin(2k\pi z'):\vec{\mathbf{p}}^2:},$$

$$e^{:F_4:} = e^{\mathcal{A}:(\vec{\mathbf{p}}^2)^2:+\mathcal{B}:\vec{\mathbf{p}}^2(\vec{\mathbf{p}}\cdot\vec{\mathbf{q}}):+\mathcal{C}:(\vec{\mathbf{p}}\cdot\vec{\mathbf{q}})^2:+\mathcal{D}:\vec{\mathbf{p}}^2\vec{\mathbf{q}}^2:}$$
$$\times e^{\mathcal{E}:\vec{\mathbf{q}}^2(\vec{\mathbf{p}}\cdot\vec{\mathbf{q}}):+\mathcal{F}:(\vec{\mathbf{q}}^2)^2:}.$$

Here, z and z' give the propagation before and after the break, respectively.

In Eq. (36), the exponential $e^{:F_2:}$ is the Gaussian term, while $e^{:F_4:}$ corresponds to the aberration term. This method simplifies the optical problem of obtaining the image of an optical system to the determination of the corresponding Symplectic Map associated to the system, thus reducing the problem to simple matrices products.

In order to calculate the Wigner distribution function we need to use the following correspondence

$$:\vec{\mathbf{p}}: \longrightarrow \vec{\mathbf{p}}$$
$$:\vec{\mathbf{q}}: \longrightarrow \vec{\mathbf{q}}$$

to make the Symplectic map, and consider the convolution between a point source and M_{total}. The point source is defined by the Dirac Delta

$$F(\vec{\mathbf{p}}',\vec{\mathbf{q}}') = \delta(\vec{\mathbf{q}}' - \vec{\mathbf{q}}, \vec{\mathbf{p}}' - \vec{\mathbf{p}}). \qquad (37)$$

The convolution between F and M_{total} (up to fourth order) is

$$F * M_{total} = F(\vec{\mathbf{p}}',\vec{\mathbf{q}}') * e^{F_2}e^{F_4}\dots$$
$$\simeq [F(\vec{\mathbf{p}}',\vec{\mathbf{q}}') * e^{F_2}](1+F_4)F(\vec{\mathbf{p}}',\vec{\mathbf{q}}') * e^{F_2}$$
$$+[F(\vec{\mathbf{p}}',\vec{\mathbf{q}}') * e^{F_2}]F_4. \qquad (38)$$

The aberration of the system respect to the Gaussian ray (second term of the last equation) is

$$F * e^{F_2} = F(\vec{\mathbf{p}'}, \vec{\mathbf{q}'}) * e^{F_2} = C_1 \frac{\pi}{\sqrt{\alpha_5}} e^{-\left(\frac{\alpha_1 \vec{\mathbf{p}}'^2 - \alpha_2 \vec{\mathbf{p}'} \cdot \vec{\mathbf{p}} + \alpha_3 \vec{\mathbf{p}}^2}{\alpha_4}\right)} ,$$

where

$$\alpha_1 = \frac{a}{2n} \sin(2k\pi z')(\beta + \frac{\gamma}{2}),$$

$$\alpha_2 = \beta \frac{a}{2n} \sin(2k\pi z'),$$

$$\alpha_3 = \beta(\frac{\gamma}{2} + \frac{a}{2n} \sin(2k\pi z')),$$

$$\alpha_4 = \beta + \frac{\gamma}{2} + \frac{a}{2n} \sin(2k\pi z'),$$

$$\alpha_5 = (\frac{\gamma}{2n} + \frac{a}{2n} \sin(2k\pi z))(\beta + \frac{\gamma}{2n} + \frac{a}{2n} \sin(2k\pi z')).$$

Without perturbation ($\gamma \to 0$), the aberration yield

$$F * e^{F_2} = C_1 \frac{\pi}{\sqrt{\frac{a}{2n} \sin(2k\pi z)}} ,$$

that corresponds to the convolution between $e^{-\frac{a}{2n} sin(2k\pi z)\vec{\mathbf{p}}^2}$ and a point source.

Now we can calculate the Wigner distribution function in the image plane substituting in Equation (6) the Symplectic Map of this system, Eqs. (36) and (38):

$$W(\vec{\mathbf{q}'}, \vec{\mathbf{p}'}) = \int_{-\infty}^{\infty} d\vec{\mathbf{r}}\, e^{-i\vec{\mathbf{p}'} \cdot \vec{\mathbf{r}}}$$

$$\times \left\{ F * e^{F_2}(\vec{\mathbf{q}'} + \frac{\vec{\mathbf{r}}}{2}) + [F * e^{F_2}]F_4(\vec{\mathbf{q}'} + \frac{\vec{\mathbf{r}}}{2}) \right\}$$

$$\times \left\{ F * e^{F_2}(\vec{\mathbf{q}'} - \frac{\vec{\mathbf{r}}}{2}) + [F * e^{F_2}]F_4(\vec{\mathbf{q}'} - \frac{\vec{\mathbf{r}}}{2}) \right\},$$

that can be rewritten up to fourth order as

$$W(\vec{\mathbf{q}'}, \vec{\mathbf{p}'}) =$$

$$C_1^2 \frac{2\pi^3}{\alpha_5} e^{i\pi} e^{-(2/\alpha_4)(\alpha_1 \vec{\mathbf{p}}'^2 - [2\alpha_1 + \alpha_2]\vec{\mathbf{p}''} \cdot \vec{\mathbf{p}} + [\alpha_1 + \alpha_2 + \alpha_3]\vec{\mathbf{p}}^2)}$$

$$\times \left\{ 1 + 2F_4\delta(\vec{\mathbf{p}'}) + \pi^2 \mathcal{F} \left(\frac{d^2}{dp_1'^2} + \frac{d^2}{dp_2'^2} \right)^2 + 8\pi^2 e^{i\pi} \right.$$

$$\times \left[\mathcal{C} \left(p_1' \frac{d}{dp_1'} + p_2' \frac{d}{dp_2'} \right)^2 + 2\mathcal{F} \left(q_1' \frac{d}{dp_1'} + q_2' \frac{d}{dp_2'} \right)^2 \right.$$

$$+ \mathcal{E} \left(p_1' q_1' \frac{d^2}{dp_1'^2} + p_2' q_2' \frac{d^2}{dp_2'^2} + (p_2' q_1' + p_1' q_2') \frac{d}{dp_1'} \frac{d}{dp_2'} \right)$$

$$\left. \left. + \frac{1}{2} [\mathcal{D}(\vec{\mathbf{p}'}) + \mathcal{E}(\vec{\mathbf{p}'}) \cdot \vec{\mathbf{q}'}] \left(\frac{d^2}{dp_1'^2} + \frac{d^2}{dp_2'^2} \right) \right] \right\} \delta(\vec{\mathbf{p}'}),$$

$$(39)$$

with $\vec{\mathbf{p}}' = \vec{\mathbf{p}}'' - \vec{\mathbf{p}}$. From this equation is clear that after the break, the Wigner distribution function is a Gaussian with center in $(2\alpha_1 + \alpha_2\vec{\mathbf{p}})/(2\alpha_1)$. The Wigner function found have a general phase of 2π, except in the term $(4\pi^2/3)\mathcal{F}\{\frac{d^2}{dp_1'^2} + \frac{d^2}{dp_2'^2}\}^2\delta(\vec{\mathbf{p}}')$ where the phase is π. Thus the initial Gaussian is modified by a corrective term, the F_4 polynomial (the exponent of the Aberration Lie Operator).

The limiting case without break, $\gamma \to 0$, has a Wigner distribution function given by

$$W(\vec{\mathbf{q}}', \vec{\mathbf{p}}') = \frac{C_1^2 e^{i\pi}}{\frac{a}{2n}sin(2k\pi z)}\delta(\vec{\mathbf{p}}')\left(1 + 2F_4 + \ldots\right) \tag{40}$$

5. Conclusions

As shown in this chapter, phase space approach (through the Wigner distribution function) simplifies the calculation and helps in the description of optical fibers. This is due to the fact that in phase space representation, the relevant properties of the system can be obtained by simple matrices products. Quasiprobability distribution functions are useful not only as calculation tools but can also provide insights into the connections between geometric opticsl and wave optics due to the fact that they allow one to express wave optics averages in a form which is very similar to that for classical averages. In this sense it serves to validate paraxial approximation.

6. References

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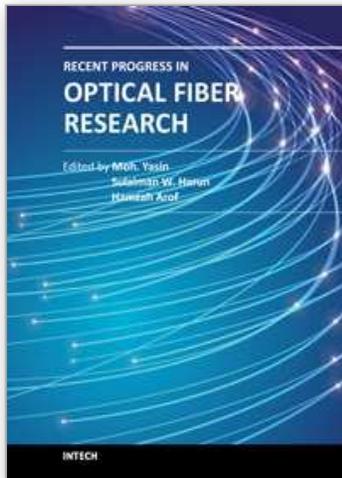
Recent Progress in Optical Fiber Research
Edited by Dr Moh. Yasin

This book presents a comprehensive account of the recent progress in optical fiber research. It consists of four sections with 20 chapters covering the topics of nonlinear and polarisation effects in optical fibers, photonic crystal fibers and new applications for optical fibers. Section 1 reviews nonlinear effects in optical fibers in terms of theoretical analysis, experiments and applications. Section 2 presents polarization mode dispersion, chromatic dispersion and polarization dependent losses in optical fibers, fiber birefringence effects and spun fibers. Section 3 and 4 cover the topics of photonic crystal fibers and a new trend of optical fiber applications. Edited by three scientists with wide knowledge and experience in the field of fiber optics and photonics, the book brings together leading academics and practitioners in a comprehensive and incisive treatment of the subject. This is an essential point of reference for researchers working and teaching in optical fiber technologies, and for industrial users who need to be aware of current developments in optical fiber research areas.

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