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Robust Control Design of Uncertain Discrete-Time Systems with Delays
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1. Introduction
When we consider control problems of physical systems, we often see time-delay in the process of control algorithms and the transmission of information. Time-delay often appear in many practical systems and mathematical formulations such as electrical system, mechanical system, biological system, and transportation system. Hence, a system with time-delay is a natural representation for them, and its analysis and synthesis are of theoretical and practical importance. In the past decades, research on continuous-time delay systems has been active. Difficulty that arises in continuous time-delay system is that it is infinite dimensional and a corresponding controller can be a memory feedback. This class of controllers may minimize a certain performance index, but it is difficult to implement it to practical systems due to a memory feedback. To overcome such a difficulty, a memoryless controller is used for time-delay systems. To this end, we define

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On the other hand, research on discrete-time delay systems has not attracted as much attention as that of continuous-time delay systems. In addition, most results have focused on state feedback stabilization of discrete-time systems with time-varying delays. Only a few results on observer design of discrete-time systems with time-varying delays have appeared in the literature(for example, (9)). The results in (3), (12), (14), (18) considered discrete-time systems with time-invariant delays. Gao and Chen (4), Hara and Yoneyama (5), (6) gave robust stability conditions. Fridman and Shaked (1) solved a guaranteed cost control problem. Fridman and Shaked (2), Yoneyama (17), Zhang and Han (19) considered the $H_{\infty}$ disturbance attenuation. They have given sufficient conditions via LMIs for corresponding control problems. Nonetheless, their conditions still show the conservatism. Hara and Yoneyama (5) and Yoneyama (17) gave least conservative conditions but their conditions require many LMI slack variables, which in turn require a large amount of computations. Furthermore, to authors’ best knowledge, few results on robust observer design problem for uncertain discrete-time systems with time-varying delays have given in the literature.

In this paper, we consider the stabilization for a nominal discrete-time system with time-varying delay and robust stabilization for uncertain system counterpart. The system under consideration has time-varying delays in state, control input and output measurement. First, we obtain a stability condition for a nominal time-delay system. To this end, we define
a Lyapunov function and use Leibniz-Newton formula and free weighting matrix method. These methods are known to reduce the conservatism in our stability condition, which are expressed as linear matrix inequality. Based on such a stability condition, a state feedback design method is proposed. Then, we extend our stabilization result to robust stabilization for uncertain discrete-time systems with time-varying delay. Next, we consider observer design and robust observer design. Similar to a stability condition, we obtain a condition such that the error system, which comes from the original system and its observer, is asymptotically stable. Using a stability condition of the error system, we proposed an observer design method. Furthermore, we give a robust observer design method for an uncertain time-delay system. Finally, we give some numerical examples to illustrate our results and to compare with existing results.

2. Time-delay systems

Consider the following discrete-time system with a time-varying delay and uncertainties in the state and control input.

\[ x(k + 1) = (A + \Delta A)x(k) + (A_d + \Delta A_d)x(k - d_k) + (B + \Delta B)u(k) + (B_d + \Delta B_d)u(k - d_k) \]

where \( x(k) \in \mathbb{R}^m \) is the state and \( u(k) \in \mathbb{R}^m \) is the control. \( A, A_d, B \) and \( B_d \) are system matrices with appropriate dimensions. \( d_k \) is a time-varying delay and satisfies \( 0 \leq d_m \leq d_k \leq d_M \) and \( d_{k+1} \leq d_k \) where \( d_m \) and \( d_M \) are known constants. Uncertain matrices are of the form

\[ [\Delta A \ \Delta A_d \ \Delta B \ \Delta B_d] = HF(k) \begin{bmatrix} E & E_d & E_1 & E_b \end{bmatrix} \]

where \( F(k) \in \mathbb{R}^{m \times j} \) is an unknown time-varying matrix satisfying \( F^T(k)F(k) \leq I \) and \( H, E, E_d, E_1 \) and \( E_b \) are constant matrices of appropriate dimensions.

**Definition 2.1.** The system (1) is said to be robustly stable if it is asymptotically stable for all admissible uncertainties (2).

When we discuss a nominal system, we consider the following system.

\[ x(k + 1) = Ax(k) + A_dx(k - d_k) + Bu(k) + B_du(k - d_k). \]

Our problem is to find a control law which makes the system (1) or (3) robustly stable. Let us now consider the following memoryless feedback:

\[ u(k) = Kx(k) \]

where \( K \) is a control gain to be determined. Applying the control (4) to the system (1), we have the closed-loop system

\[ x(k + 1) = ((A + \Delta A) + (B + \Delta B)K)x(k) + ((A_d + \Delta A_d) + (B_d + \Delta B_d)K)x(k - d_k). \]

For the nominal case, we have

\[ x(k + 1) = (A + BK)x(k) + (A_d + B_dK)x(k - d_k). \]

In the following section, we consider the robust stability of the closed-loop system (5) and the stability of the closed-loop system (6).

The following lemma is useful to prove our results.

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Lemma 2.2. \(((13))\) Given matrices $Q = Q^T$, $H$, $E$ and $R = R^T > 0$ with appropriate dimensions.

$$Q + HF(k)E + \ldots A_d + B_dK 00$$

for all $F(k)$ satisfying $F^T(k)F(k) \leq R$ if and only if there exists a scalar $\epsilon > 0$ such that

$$Q + \frac{1}{\epsilon}HH^T + \epsilon E^TRE < 0.$$ 

3. Stability analysis

This section analyzes the stability and robust stability of discrete-time delay systems. Section 3.1 gives a stability condition for nominal systems and Section 3.2 extends the stability result to a case of robust stability.

3.1 Stability for nominal systems

Stability conditions for discrete-time delay system (6) are given in the following theorem.

**Theorem 3.1.** Given integers $d_m$ and $d_M$, and control gain $K$. Then, the time-delay system (6) is asymptotically stable if there exist matrices $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $S > 0$, $M > 0$,

$$L = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 & L_5 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 \end{bmatrix}$$

satisfying

$$\Phi = \Phi_1 + \Xi'_L + \Xi'_L^T + \Xi_N + \Xi_N^T + \Xi'_T + \Xi_T^T \frac{\sqrt{M}}{S} Z^T < 0$$ \hspace{1cm} (7)

where

$$\Phi_1 = \begin{bmatrix} P_1 & 0 & 0 & 0 & 0 \\ 0 & P_{22} & 0 & 0 & 0 \\ 0 & 0 & -Q_1 - M & 0 & 0 \\ 0 & 0 & 0 & -P_2 & P_2 - Q_2 \\ 0 & 0 & 0 & 0 & -P_2 \\ -P_2 & P_2 - Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_2 \\ -P_2 & P_2 - Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} L \\ L \end{bmatrix} + N,$$

$$\Xi'_L = \begin{bmatrix} L - L & 0 & -L \end{bmatrix},$$

$$\Xi_N = \begin{bmatrix} 0 & N - N \end{bmatrix},$$

$$\Xi'_T = \begin{bmatrix} T - T(A + BK) - T(A_d + B_dK) & 0 & 0 \end{bmatrix}.$$
Proof: First, we note from Leibniz-Newton formula that
\[ 2\xi^T(k)\Delta x(k) - 2\xi^T(k)x(k) + 2\xi^T(k)e(k) = 0, \]  
(8)
\[ 2\xi^T(k)N[x(k) - x(k - d_k) + \sum_{i=k-d_k}^{k-1} e(i)] = 0 \]  
(9)
where \( e(k) = x(k+1) - x(k) \) and
\[ \xi^T(k) = [x^T(k+1) x^T(k) x^T(k-d_k) e^T(k) e^T(k-d_k)]. \]
It is also true that
\[ 2\xi^T(k)T[x(k+1) - (A+BK)x(k) - (A_d + B_dK)x(k-d_k) = 0. \]  
(10)
Now, we consider a Lyapunov function
\[ V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) \]
where
\[ V_1(k) = x^T(k)P_1x(k) + \sum_{i=k-d_1}^{k-1} e^T(i)P_2 \sum_{i=k-d_1}^{k-1} e(i), \]
\[ V_2(k) = \sum_{i=k-d_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-d_1}^{k-1} e^T(i)Q_2e(i), \]
\[ V_3(k) = \sum_{j=-d_2}^{-1} \sum_{i=k+j}^{k-1} e^T(j)Se(i), \]
\[ V_4(k) = \sum_{j=-d_3}^{-d_4} \sum_{i=k+j}^{k-1} x^T(i)Mx(i), \]
and \( P_1, P_2, Q_1, Q_2, S, M \) are positive definite matrices to be determined. Then, we calculate the difference \( \Delta V = V(k+1) - V(k) \) and add the left-hand-side of equations (8)-(10).
Since \( \Delta V_1(k) \), \( i = 1, \cdots, 4 \) are calculated as follows;
\[ \Delta V_1(k) = x^T(k+1)P_1x(k+1) + \sum_{i=k+1-d_{1+1}}^{k} e^T(i)P_2 \sum_{i=k+1-d_{1+1}}^{k} e(i) \]
\[ -x^T(k)P_1x(k) - \sum_{i=k-d_1}^{k-1} e^T(i)P_2 \sum_{i=k-d_1}^{k-1} e(i) \]
\[ \leq x^T(k+1)P_1x(k+1) - x^T(k)P_1x(k) + e^T(k)P_2e(k) \]
\[ -2e^T(k)P_2e(k-d_k) + 2e^T(k)P_2 \sum_{i=k-d_k}^{k-1} e(i) \]
\[ + e^T(k-d_k)P_2e(k-d_k) - 2e^T(k-d_k)P_2 \sum_{i=k-d_k}^{k-1} e(i) \],
\[ \Delta V_2(k) = \sum_{i=k+1-d_i+1}^{k} x^T(i)Q_1x(i) + \sum_{i=k+1-d_i+1}^{k} e^T(i)Q_2e(i) \]
\[ - \sum_{i=k-d}^{k-1} x^T(i)Q_1x(i) - \sum_{i=k-d}^{k-1} e^T(i)Q_2e(i) \]
\[ \leq x^T(k)Q_1x(k) + e^T(k)Q_2e(k) - x^T(k - d_i)Q_1x(k - d_i) \]
\[ - e^T(k - d_i)Q_2e(k - d_i), \]
\[ \Delta V_3(k) = d_{k+1}e^T(k)Se(k) - \sum_{i=k-d_i+1}^{k-1} e^T(i)Se(i) \]
\[ \leq d_Me^T(k)Se(k) - \sum_{i=k-d_i}^{k-1} e^T(i)Se(i), \]
\[ \Delta V_4(k) = (d_{M} - d_{m} + 1)x^T(k)Mx(k) - \sum_{i=k-d_{m}+1}^{k-d_{m}} x^T(i)Mx(i) \]
\[ \leq (d_{M} - d_{m} + 1)x^T(k)Mx(k) - x^T(k - d_i)Mx(k - d_i), \]

we have

\[ \Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) \]
\[ \leq x^T(k)[\Phi_1 + \Xi_L + \Xi_L^T + \Xi_N + \Xi_N^T + \Xi_T + \Xi_T^T]\xi(k) + \sum_{i=k-d_i}^{k-1} e^T(k)ZS^{-1}Z^T\xi(k) \]
\[ - \sum_{i=k-d_i}^{k-1} e^T(i)S^{-1}(Z^T\xi(k) + Se(i)) \]
\[ \leq x^T(k)[\Phi_1 + \Xi_L + \Xi_L^T + \Xi_N + \Xi_N^T + \Xi_T + \Xi_T^T + d_MZS^{-1}Z^T]\xi(k) \]

If (7) is satisfied, by Schur complement formula, we have \( \Phi_1 + \Xi_L + \Xi_L^T + \Xi_N + \Xi_N^T + \Xi_T + \Xi_T^T + d_MZS^{-1}Z^T < 0 \). It follows that \( \Delta V(k) < 0 \) and hence the proof is completed.

**Remark 3.2.** We employ \( \sum_{i=k-d_i}^{k-1} (*) \) in our Lyapunov function instead of \( \sum_{i=k-d_{m}+1}^{k-d_{m}} (*) \). This gives a less conservative stability condition.

### 3.2 Robust stability for uncertain systems

By extending Theorem 3.1, we obtain a condition for robust stability of uncertain system (5).

**Theorem 3.3.** Given integers \( d_m \) and \( d_M \), and control gain \( K \). Then, the time-delay system (5) is robustly stable if there exist matrices \( P_1 > 0, P_2 > 0, Q_1 > 0, Q_2 > 0, S > 0, M > 0, \)

\[
L = \begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4 \\
L_5
\end{bmatrix}, \quad N = \begin{bmatrix}
N_1 \\
N_2 \\
N_3 \\
N_4 \\
N_5
\end{bmatrix}, \quad T = \begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5
\end{bmatrix}
\]

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and a scalar $\lambda > 0$ satisfying

$$
\Pi = \begin{bmatrix}
\Phi + \lambda \bar{E}^T \bar{H}^T & \bar{H}^T \\
\bar{H} & -\lambda I
\end{bmatrix} < 0,
$$

(11)

where $\Phi$ is given in Theorem 3.1, and

$$
\bar{H} = \begin{bmatrix}
-H^T T_1^T & -H^T T_2^T & -H^T T_3^T & -H^T T_4^T & -H^T T_5^T & 0
\end{bmatrix},
$$

and

$$
\bar{E} = \begin{bmatrix}
0 & E_1 K & E_d & E_b K & 0 & 0
\end{bmatrix}.
$$

**Proof:** Replacing $A$, $A_d$, $B$ and $B_d$ in (7) with $A + HF(k)E$, $A_d + HF(k)E_d$, $B + HF(k)E_1$ and $B + HF(k)E_b$, respectively, we obtain a robust stability condition for the system (5).

$$
\Phi + \bar{H}^T F(k) \bar{E} + \bar{E}^T F^T(k) \bar{H} < 0
$$

(12)

By Lemma 2.2, a necessary and sufficient condition that guarantees (12) is that there exists a scalar $\lambda > 0$ such that

$$
\Phi + \lambda \bar{E}^T \bar{H}^T + \frac{1}{\lambda} \bar{H}^T \bar{H} < 0
$$

(13)

Applying Schur complement formula, we can show that (13) is equivalent to (11).

## 4. State feedback stabilization

This section proposes a state feedback stabilization method for the uncertain discrete-time delay system (1). First, stabilization of nominal system is considered in Section 4.1. Then, robust stabilization is proposed in Section 4.2.

### 4.1 Stabilization

First, we consider stabilization for the nominal system (3). Our problem is to find a control gain $K$ such that the closed-loop system (6) is asymptotically stable. Unfortunately, Theorem 3.1 does not give LMI conditions to find $K$ asymptotically stabilizes the time-delay system (3) if there exist matrices $\tilde{P}_1 > 0$, $\tilde{P}_2 > 0$, $\tilde{Q}_1 > 0$, $\tilde{Q}_2 > 0$, $\tilde{S} > 0$, $\tilde{M} > 0$, $G$, $Y$.

$$
\tilde{L} = \begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4 \\
L_5
\end{bmatrix}, \quad \tilde{N} = \begin{bmatrix}
N_1 \\
N_2 \\
N_3 \\
N_4 \\
N_5
\end{bmatrix},
$$

satisfying

$$
\Psi = \begin{bmatrix}
\Psi_1 + \Theta_L + \Theta_L^T + \Theta_N + \Theta_N^T + \Theta_T + \Theta_T^T \sqrt{\tilde{M}} \tilde{Z} & \sqrt{\tilde{M}} \tilde{Z}^T \\
& -\tilde{S}
\end{bmatrix} < 0
$$

(14)
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where

\[
\begin{bmatrix}
P_1 & 0 & 0 & 0 & 0 \\
0 & \Psi_{22} & 0 & 0 & 0 \\
0 & 0 & -Q_1 + \hat{M} & 0 & 0 \\
0 & 0 & 0 & \Psi_{44} & -P_2 \\
0 & 0 & 0 & -\bar{P}_2 & -\bar{Q}_2 \\
\end{bmatrix},
\]

\[
\begin{aligned}
\Psi_{22} &= -\bar{P}_1 + \bar{Q}_1 + (d_M - d_m + 1)\bar{M}, \\
\Psi_{44} &= \bar{P}_2 + \bar{Q}_2 + d_M\bar{S}, \\
Z &= \begin{bmatrix} 0 & 0 & -\bar{P}_2 \\
\end{bmatrix} + \bar{N}, \\
\Theta_T &= \begin{bmatrix}
p_1Y^T - p_1(AY^T + BG) - p_1(A_dY^T + B_dG) & 0 & 0 \\
p_2Y^T - p_2(AY^T + BG) - p_2(A_dY^T + B_dG) & 0 & 0 \\
p_3Y^T - p_3(AY^T + BG) - p_3(A_dY^T + B_dG) & 0 & 0 \\
p_4Y^T - p_4(AY^T + BG) - p_4(A_dY^T + B_dG) & 0 & 0 \\
p_5Y^T - p_5(AY^T + BG) - p_5(A_dY^T + B_dG) & 0 & 0 \\
\end{bmatrix}.
\end{aligned}
\]

In this case, a controller gain in the controller (4) is given by

\[
K = GY^{-T}
\]

Proof: Let \( T_i = \rho_iY^{-1}, i = 1, \ldots, 5 \) where each \( \rho_i \) is given. We substitute them into (7). Then, we calculate \( \Psi = \Sigma\Phi\Sigma^T \) with \( \Sigma = \text{diag}(Y Y Y Y Y) \). Defining \( \bar{P}_i = YP_iY^T, \bar{Q}_i = YQ_iY^T, \) \( i = 1, 2 \), \( \bar{S} = YSY^T, \bar{M} = YMY^T, \bar{L} = YLY^T, \bar{N} = YNY^T \), we obtain \( \Theta < 0 \) in (14) where we let \( G = KY^T \). If the condition (14) hold, state feedback control gain matrix \( K \) is obviously given by (15).

Remark 4.2. Should \( Y \) be singular, let \( \bar{L}_1 = 0 \). In this case, it follows from (1, 1)-block of \( \Psi \) that \( \bar{P}_1 + \rho_1(Y + Y^T) < 0 \). Then, if (14) holds, \( Y \) must be nonsingular.

4.2 Robust stabilization

In a similar way to robust stability, we extend a stabilization result in the previous section to robust stabilization for uncertain discrete-time delay system (1).

Theorem 4.3. Given integers \( d_m \) and \( d_M \), and scalars \( \rho_i, i = 1, \ldots, 5 \). Then, the controller (4) robustly stabilizes the time-delay system (1) if there exist matrices \( \bar{P}_1 > 0, \bar{P}_2 > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{S} > 0, \bar{M} > 0, G, Y \)

\[
\begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4 \\
L_5 \\
\end{bmatrix}, \quad \bar{N} = \begin{bmatrix} \bar{N}_1 \\
\bar{N}_2 \\
\bar{N}_3 \\
\bar{N}_4 \\
\bar{N}_5 \\
\end{bmatrix},
\]
and a scalar $\lambda > 0$ satisfying
\[ \Lambda = \begin{bmatrix} \Psi + \lambda \hat{H}^T \hat{H} & \hat{E}^T \\ \hat{E} & -\lambda I \end{bmatrix} < 0, \] (16)
where
\[ \hat{H} = \begin{bmatrix} -\rho_1 H^T - \rho_2 H^T - \rho_3 H^T - \rho_4 H^T - \rho_5 H^T \end{bmatrix}, \]
and
\[ \hat{E} = \begin{bmatrix} 0 EY^T + E_1 G E_d Y^T + E_3 G 0 0 \end{bmatrix}. \]
In this case, a controller gain in the controller (4) is given by (15).

**Proof:** Replacing $A$, $A_d$, $B$ and $B_d$ in (14) with $A + HF(k)E$, $A_d + HF(k)E_d$, $B + HF(k)E_1$ and $B + HF(k)E_3$, respectively, we obtain robust stability conditions for the system (1):
\[ \Psi + \hat{H}^T F(k) \hat{E} + \hat{E}^T F^T(k) \hat{H} < 0 \] (17)
By Lemma 2.2, a necessary and sufficient condition that guarantees (17) is that there exists a scalar $\lambda > 0$ such that
\[ \Psi + \lambda \hat{H}^T \hat{H} + \frac{1}{\lambda} \hat{E}^T \hat{E} < 0 \] (18)
Applying Schur complement formula, we can show that (18) is equivalent to (16).

5. State estimation
All the information on the state variables of the system is not always available in a physical situation. In this case, we need to estimate the values of the state variables from all the available information on the output and input. In the following, we make analysis of the existence of observers. Section 5.1 analyzes the observer of a nominal system, and Section 5.2 considers the robust observer analysis of an uncertain system.

5.1 Observer analysis
Using the results in the previous sections, we consider an observer design for the system (1), which estimates the state variables of the system using measurement outputs.
\[ x(k + 1) = (A + \Delta A)x(k) + (A_d + \Delta A_d)x(k - d_k), \] (19)
\[ y(k) = (C + \Delta C)x(k) + (C_d + \Delta C_d)x(k - d_k) \] (20)
where uncertain matrices are of the form:
\[ \begin{bmatrix} \Delta A & \Delta A_d \\ \Delta C & \Delta C_d \end{bmatrix} = \begin{bmatrix} H \\ H_2 \end{bmatrix} F(k) \begin{bmatrix} E \\ E_d \end{bmatrix} \]
where $F(k) \in \mathbb{R}^{l \times l}$ is an unknown time-varying matrix satisfying $F^T(k)F(k) \leq I$ and $H$, $H_2$, $E$ and $E_d$ are constant matrices of appropriate dimensions.
We consider the following system to estimate the state variables:
\[ \hat{x}(k + 1) = A\hat{x}(k) + \bar{K}(y(k) - C\hat{x}(k)) \] (21)

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where $\hat{x}$ is the estimated state and $\bar{K}$ is an observer gain to be determined. It follows from (19), (20) and (21) that

$$x_c(k+1) = (\bar{A} + HF(k)\bar{E})x_c(k) + (\bar{A}_d + HF(k)\bar{E}_d)x_c(k-d_k).$$

(22)

where $x_c^T = [\hat{x}^T \ e^T]^T$, $e(k) = x(k) - \hat{x}(k)$ and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A - \bar{K}C \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & A_d - \bar{K}C_d \end{bmatrix},$$

$$\bar{H} = \begin{bmatrix} H \\ H - \bar{K}\bar{H}_2 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} E & 0 \\ 0 & \bar{E}_d \end{bmatrix}.$$}

We shall find conditions for (22) to be robustly stable. In this case, the system (21) becomes an observer for the system (19) and (20).

For nominal case, we have

$$x_c(k+1) = \bar{A}x_c(k) + \bar{A}_dx_c(k-d_k).$$

(23)

We first consider the asymptotic stability of the system (23). The following theorem gives conditions for the system (23) to be asymptotically stable.

**Theorem 5.1.** Given integers $d_m$ and $d_M$, and observer gain $\bar{K}$. Then, the system (23) is asymptotically stable if there exist matrices $0 < \bar{P}_1 \in \mathbb{R}^{2n \times 2n}$, $0 < \bar{Q}_1 \in \mathbb{R}^{2n \times 2n}$, $0 < \bar{Q}_2 \in \mathbb{R}^{2n \times 2n}$, $0 < \bar{S} \in \mathbb{R}^{2n \times 2n}$, $0 < \bar{M} \in \mathbb{R}^{2n \times 2n}$, $L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix} \in \mathbb{R}^{10n \times 2n}$, $\bar{N} = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \\ \bar{N}_4 \\ \bar{N}_5 \end{bmatrix} \in \mathbb{R}^{10n \times 2n}$, $\bar{T} = \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \\ \bar{T}_4 \\ \bar{T}_5 \end{bmatrix} \in \mathbb{R}^{10n \times 2n}$ satisfying

$$\Phi = \Phi_1 + \Sigma_L + \Sigma_L^T + \Sigma_N + \Sigma_N^T + \Sigma_T + \Sigma_T^T \sqrt{d_M Z} - S < 0$$

(24)

where

$$\Phi_1 = \begin{bmatrix} \bar{P}_1 & 0 & 0 & 0 & 0 \\ 0 & \bar{P}_{22} & 0 & 0 & 0 \\ 0 & 0 & -\bar{Q}_1 - \bar{M} & 0 & 0 \\ 0 & 0 & 0 & \bar{P}_{44} & -\bar{P}_2 \\ 0 & 0 & 0 & -\bar{P}_2 & \bar{P}_2 - \bar{Q}_2 \end{bmatrix},$$

$$\Phi_2 = -\bar{P}_H + \bar{Q}_1 + (d_M - d_m + 1)\bar{M},$$

$$\Phi_4 = \bar{P}_2 + \bar{Q}_2 + d_M\bar{S},$$

$$\hat{Z} = \begin{bmatrix} 0 \\ 0 \\ -\bar{P}_2 \\ \bar{P}_2 \end{bmatrix} + \bar{N},$$

$$\hat{\Sigma}_L = \begin{bmatrix} L & -L & 0 & -L \end{bmatrix},$$

$$\hat{\Sigma}_N = \begin{bmatrix} 0 & \bar{N} - \bar{N} \end{bmatrix},$$

$$\hat{\Sigma}_T = \begin{bmatrix} \bar{T} - \bar{T} \bar{A} & -\bar{T} \bar{A}_d & 0 & 0 \end{bmatrix}.$$
5.2 Robust observer analysis

Now, we extend the result for the uncertain system (23).

**Theorem 5.2.** Given integers \(d_m\) and \(d_M\), and observer gain \(K\). Then, the system (22) is robustly stable if there exist matrices \(0 < P_1 \in \mathbb{R}^{2n \times 2n}, 0 < P_2 \in \mathbb{R}^{2n \times 2n}, 0 < Q_1 \in \mathbb{R}^{2n \times 2n}, 0 < Q_2 \in \mathbb{R}^{2n \times 2n}, 0 < \tilde{S} \in \mathbb{R}^{2n \times 2n}, 0 < \tilde{M} \in \mathbb{R}^{2n \times 2n}\),

\[
\begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4 \\
L_5
\end{bmatrix} \in \mathbb{R}^{10n \times 2n}, \quad
\begin{bmatrix}
\tilde{N}_1 \\
\tilde{N}_2 \\
\tilde{N}_3 \\
\tilde{N}_4 \\
\tilde{N}_5
\end{bmatrix} \in \mathbb{R}^{10n \times 2n}, \quad
\tilde{T} = \begin{bmatrix}
\hat{T}_1 \\
\hat{T}_2 \\
\hat{T}_3 \\
\hat{T}_4 \\
\hat{T}_5
\end{bmatrix} \in \mathbb{R}^{10n \times 2n}
\]

and a scalar \(\lambda > 0\) satisfying

\[
\tilde{\Pi} = \begin{bmatrix}
\Phi + \lambda \hat{E}^T \hat{E}^T \\
\hat{H}
\end{bmatrix} < 0
\]

where \(\Phi\) is given in Theorem 5.1, and

\[
\hat{H} = \begin{bmatrix}
-\hat{A}^T T_1 & -\hat{A}^T T_2 & -\hat{A}^T T_3 & -\hat{A}^T T_4 & -\hat{A}^T T_5 & 0 \\
-\hat{E}^T T_1 & -\hat{E}^T T_2 & -\hat{E}^T T_3 & -\hat{E}^T T_4 & -\hat{E}^T T_5 & 0
\end{bmatrix},
\]

\[
\hat{E} = \begin{bmatrix}
0 & \hat{E}_d & 0 & 0 & 0
\end{bmatrix}.
\]

**Proof:** Replacing \(\hat{A}\) and \(\hat{A}_d\) in (24) with \(\hat{A} + \hat{H} \Gamma(k) \hat{E}\) and \(\hat{A}_d + \hat{H} \Gamma(k) \hat{E}_d\), respectively, and following similar lines of proof of Theorem 3.3, we have the desired result.

6. Observer design

This section gives observer design methods for discrete-time delay systems. Section 6.1 provides an observer design method for a nominal delay system, and Section 6.2 proposes for an uncertain delay system.

6.1 Nominal observer

Similar to Theorem 3.1, Theorem 5.1 does not give a design method of finding an observer gain \(K\). Hence, we obtain another theorem below.

**Theorem 6.1.** Given integers \(d_m\) and \(d_M\), and scalars \(\rho_i\) and \(\hat{\rho}_i\), \(i = 1, \ldots, 5\). Then, (21) becomes an observer for the system (19) and (20) with \(\Delta \Lambda = \Delta A_i = 0, \Delta C = \Delta C_i = 0\) if there exist matrices

\[
\begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4 \\
L_5
\end{bmatrix} \in \mathbb{R}^{10n \times 2n}, \quad
\begin{bmatrix}
\tilde{N}_1 \\
\tilde{N}_2 \\
\tilde{N}_3 \\
\tilde{N}_4 \\
\tilde{N}_5
\end{bmatrix} \in \mathbb{R}^{10n \times 2n}, \quad
T = \begin{bmatrix}
\hat{T}_1 \\
\hat{T}_2 \\
\hat{T}_3 \\
\hat{T}_4 \\
\hat{T}_5
\end{bmatrix} \in \mathbb{R}^{S_{In} \times n}, \quad
\tilde{T} = \begin{bmatrix}
\hat{T}_1 \\
\hat{T}_2 \\
\hat{T}_3 \\
\hat{T}_4 \\
\hat{T}_5
\end{bmatrix} \in \mathbb{R}^{S_{In} \times n}
\]
satisfying
\[
\Psi = \begin{bmatrix} \Psi_1 + \hat{\Theta}_L^T + \hat{\Theta}_N^T + \hat{\Theta}_T^T \sqrt{\frac{d_M Z^T}{-S}} \end{bmatrix} < 0
\] (25)

where
\[
\Psi_1 = \begin{bmatrix} \rho_1 0 0 0 0 \\ 0 \Psi_{22} 0 0 0 \\ 0 0 -\hat{Q}_1 - \hat{M} 0 0 \\ 0 0 0 \Psi_{44} - \rho_2 \\ 0 0 0 0 -\hat{P}_2 \hat{P}_2 - \hat{Q}_2 \end{bmatrix},
\]
\[
\Psi_{22} = -\hat{P}_1 + \hat{Q}_1 + (d_M - d_m + 1) \hat{M},
\]
\[
\Psi_{44} = \hat{P}_2 + \hat{Q}_2 + d_M \hat{S},
\]
\[
\hat{Z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \hat{N},
\]
\[
\hat{\Theta}_L = \begin{bmatrix} L - L 0 - L \end{bmatrix},
\]
\[
\hat{\Theta}_N = \begin{bmatrix} 0 \hat{N} - \hat{N} 0 0 \end{bmatrix},
\]
\[
\hat{\Theta}_T = \begin{bmatrix} T_1 \rho_1 Y - T_1 A - \rho_1 (YA - \hat{G} C) - T_1 A d - \rho_1 (YA_d - \hat{G} C_d) 0 0 0 0 0 \\ T_2 \rho_1 Y - T_2 A - \rho_2 (YA - \hat{G} C) - T_2 A d - \rho_2 (YA_d - \hat{G} C_d) 0 0 0 0 0 \\ T_3 \rho_2 Y - T_3 A - \rho_3 (YA - \hat{G} C) - T_3 A d - \rho_3 (YA_d - \hat{G} C_d) 0 0 0 0 0 \\ T_4 \rho_3 Y - T_4 A - \rho_4 (YA - \hat{G} C) - T_4 A d - \rho_4 (YA_d - \hat{G} C_d) 0 0 0 0 0 \\ T_5 \rho_4 Y - T_5 A - \rho_5 (YA - \hat{G} C) - T_5 A d - \rho_5 (YA_d - \hat{G} C_d) 0 0 0 0 0 \\ T_6 \rho_5 Y - T_6 A - \rho_5 (YA - \hat{G} C) - T_6 A d - \rho_5 (YA_d - \hat{G} C_d) 0 0 0 0 0 \end{bmatrix}.
\]

In this case, an observer gain in the observer (21) is given by
\[
K = Y^{-1} \hat{G}.
\] (26)

**Proof:** Proof is similar to that of Theorem 4.1. Let
\[
T_i = \begin{bmatrix} T_i \rho_i Y \\ T_i \rho_i \hat{Y} \end{bmatrix}, \ i = 1, \ldots, 5
\]
where \(\rho_i\) and \(\hat{\rho}_i\), \(i = 1, \ldots, 5\) are given. We substitute them into (24). Defining \(\hat{G} = Y \hat{K}\), we obtain \(\Psi < 0\) in (25). If the condition (25) hold, observer gain matrix \(\hat{K}\) is obviously given by (26).

### 6.2 Robust observer

Extending Theorem 4.1, we have the following theorem, which proposes a robust observer design for an uncertain delay system.
Theorem 6.2. Given integers \( d_m \) and \( d_M \), and scalars \( \rho_i, \hat{\rho}_i, i = 1, \ldots, 5 \). Then, (21) becomes an observer for the system (19) and (20) if there exist matrices \( 0 < \hat{P}_1 \in \mathbb{R}^{2n \times 2n}, 0 < \hat{P}_2 \in \mathbb{R}^{2n \times 2n}, 0 < \hat{Q}_1 \in \mathbb{R}^{2n \times 2n}, 0 < \hat{Q}_2 \in \mathbb{R}^{2n \times 2n}, 0 < \hat{S} \in \mathbb{R}^{2n \times 2n}, 0 < \hat{M} \in \mathbb{R}^{2n \times 2n}, G \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{n \times n} \):

\[
\hat{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix} \in \mathbb{R}^{10n \times 2n}, \quad \hat{N} = \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \\ \hat{N}_3 \\ \hat{N}_4 \\ \hat{N}_5 \end{bmatrix} \in \mathbb{R}^{10n \times 2n}, \quad T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \in \mathbb{R}^{5n \times n}, \quad \hat{T} = \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \end{bmatrix} \in \mathbb{R}^{5n \times n},
\]

and a scalar \( \lambda > 0 \) satisfying

\[
\hat{\Lambda} = \begin{bmatrix} \Psi + \lambda E^T E & \hat{H}^T \\ \hat{H} & -\lambda I \end{bmatrix} < 0 \quad (27)
\]

where \( \Psi \) is given in Theorem 6.1, and

\[
\hat{H} = -\begin{bmatrix} H^T T_1^T + \rho_1 (YH - \hat{G}H_2)^T H^T T_2^T + \hat{\rho}_1 (YH - \hat{G}H_2)^T H^T T_3^T + \rho_2 (YH - \hat{G}H_2)^T \\ H^T T_4^T + \hat{\rho}_2 (YH - \hat{G}H_2)^T H^T T_5^T + \rho_4 (YH - \hat{G}H_2)^T H^T T_2^T + \hat{\rho}_4 (YH - \hat{G}H_2)^T \\ H^T T_3^T + \rho_3 (YH - \hat{G}H_2)^T H^T T_4^T + \hat{\rho}_3 (YH - \hat{G}H_2)^T H^T T_5^T + \rho_5 (YH - \hat{G}H_2)^T \end{bmatrix},
\]

\[
\hat{E} = \begin{bmatrix} 0 & 0 & E_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

In this case, an observer gain in the observer (21) is given by (26).

Proof: Replacing \( A \) and \( A_d \) in (27) with \( A + HF(k)E \) and \( A_d + HF(k)E_d \), respectively, and following similar lines of proof of Theorem 4.3, we have the desired result.

7. Examples

In this section, the following examples are provided to illustrate the proposed results. First example shows stabilization and robust stabilization. Second one gives observer design and robust observer design.

Example 7.1. Consider the following discrete-time delay system:

\[
x(k+1) = \begin{bmatrix} 1.1 + \alpha & 0 \\ 0 & 0.97 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x(k-d_k) + \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.8 \end{bmatrix} u(k) + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} u(k-d_k)
\]

where \( \alpha \) satisfies \( |\alpha| \leq \bar{\alpha} \) for \( \bar{\alpha} \) is an upper bound of \( a(k) \). First, we consider the stabilization for a nominal time-delay system with \( a(k) = 0 \) by Theorem 4.1. Table 1 shows control gains for different time-invariant delay \( d_k \), while Table 2 gives control gains for different time-varying delay \( d_k \).

Next, we consider the robust stabilization for the uncertain time-delay system with \( a(k) \neq 0 \). In this case, system matrices can be represented in the form of (11) with matrices given by

\[
A = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_d = E_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad H = \begin{bmatrix} \bar{\alpha} \\ 0 \end{bmatrix}, \quad F(k) = \frac{a(k)}{\bar{\alpha}}.
\]

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For time-invariant delay \( d_k \), Theorem 4.3 gives control gains for different \( \bar{\alpha} \) in Table 3. Table 4 provides the result for time-varying delay \( d_k \).

**Example 7.2.** Consider the following discrete-time delay system:

\[
egin{align*}
x(k + 1) &= \begin{bmatrix} 0.85 + 0.1\bar{\alpha} & 0 \\ 0 & 0.97 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x(k - d_k), \\
y(k) &= \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} x(k - d_k),
\end{align*}
\]

where \( |\bar{\alpha}| \leq \bar{\alpha} \) for \( \bar{\alpha} \) is an upper bound of \( \alpha(k) \). We first consider the observer design for a nominal time-delay system with \( \alpha(k) = 0 \) by Theorem 6.1. Table 5 shows observer gains for different time-invariant delay \( d_k \), while Table 6 gives observer gains for different time-varying delay \( d_k \). In the following observer design, all \( \rho \)’s are set to be zero for simplicity.

Next, we consider the robust observer design for the uncertain time-delay system with \( \alpha(k) \neq 0 \). In this case, system matrices can be represented in the form of (1) with matrices given by

\[
A = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \quad E = \begin{bmatrix} \bar{\alpha} & 0 \end{bmatrix}, \quad E_d = E_1 = \begin{bmatrix} 0 & 0 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 0.5 & 0.2 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad F(k) = \frac{\alpha(k)}{\bar{\alpha}}.
\]
For time-invariant delay $d_k$, Theorem 6.2 gives observer gains for different $\bar{\alpha}$ in Table 7. Table 8 provides observer gains for time-varying delay $d_k$ by the same theorem.

### 8. Conclusions

In this paper, we proposed stabilization and robust stabilization method for discrete-time systems with time-varying delay. Our conditions were obtained by introducing new Lyapunov function and using Leibniz-Newton formula and free weighting matrix method.
Similarly, we also gave observer design and robust observer design methods. Numerical examples were given to illustrate our proposed design method.

### 9. References


Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.

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