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## Sliding Mode Control of Second Order Dynamic System with State Constraints

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### 1. Introduction

In recent years much of the research in the area of control theory focused on the design of discontinuous feedback which switches the structure of the system according to the evolution of its state vector. This control idea may be illustrated by the following example.

*Example 1.* Let us consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 + u_i \quad i = 1, 2, \end{aligned} \quad (1)$$

where  $x_1(t)$  and  $x_2(t)$  denote the system state variables, with the following two feedback control laws

$$u_1 = f_1(x_1, x_2) = -x_2 - x_1 \quad (2)$$

$$u_2 = f_2(x_1, x_2) = -x_2 - 4x_1 \quad (3)$$

The performance of system (1) controlled according to (2) is shown in Fig. 1, and Fig. 2 presents the behaviour of the same system with feedback control (3). It can be clearly seen from those two figures that each of the feedback control laws (2) and (3) ensures the system stability only in the sense of Lyapunov.

However, if the following switching strategy is applied

$$i = \begin{cases} 1 & \text{for } \min\{x_1, x_2\} < 0 \\ 2 & \text{for } \min\{x_1, x_2\} \geq 0 \end{cases} \quad (4)$$

then the system becomes asymptotically stable. This is illustrated in Fig. 3. Moreover, it is worth to point out that system (1) with the same feedback control laws may exhibit completely different behaviour (and even become unstable). For example, if the switching strategy (4) is modified as

$$i = \begin{cases} 1 & \text{for } \min\{x_1, x_2\} \geq 0 \\ 2 & \text{for } \min\{x_1, x_2\} < 0 \end{cases} \quad (5)$$

then the system output increases to infinity. The system dynamic behaviour, in this situation, is illustrated in Fig. 4.

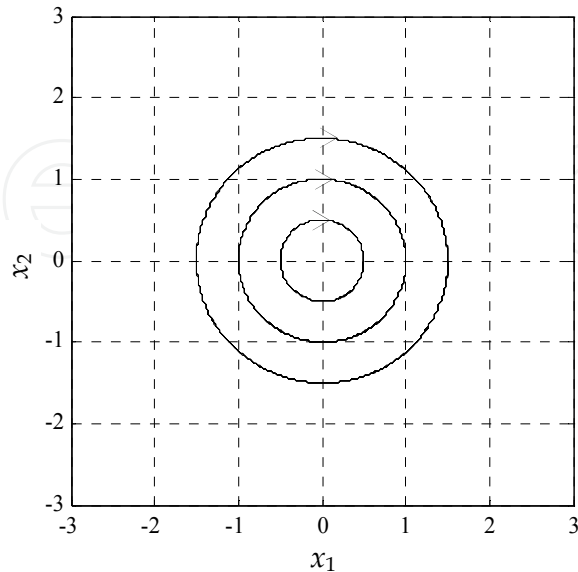


Fig. 1. Phase portrait of system (1) with controller (2).

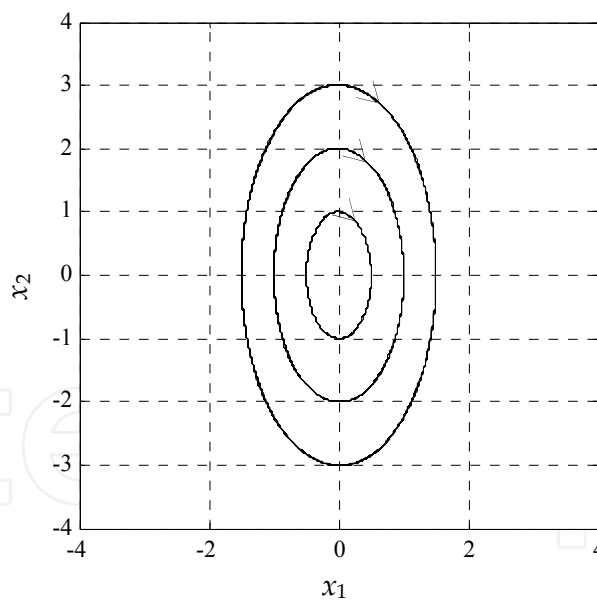


Fig. 2. Phase portrait of system (1) with controller (3).

This example presents the concept of variable structure control (VSC) and stresses that the system dynamics in VSC is determined not only by the applied feedback controllers but also, to a large extent, by the adopted switching strategy. VSC is inherently a nonlinear technique and as such, it offers a variety of advantages which cannot be achieved using conventional linear controllers. Our next example shows one of those favourable features – namely it demonstrates that VSC may enable finite time error convergence.

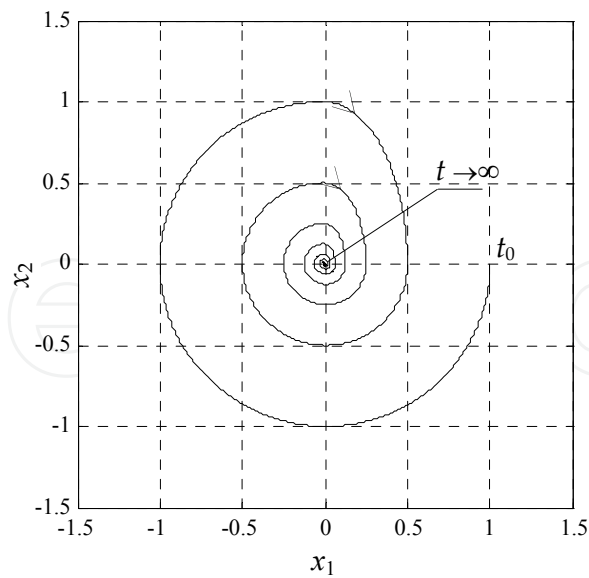


Fig. 3. Phase portrait of system (1) when switching strategy (4) is applied.

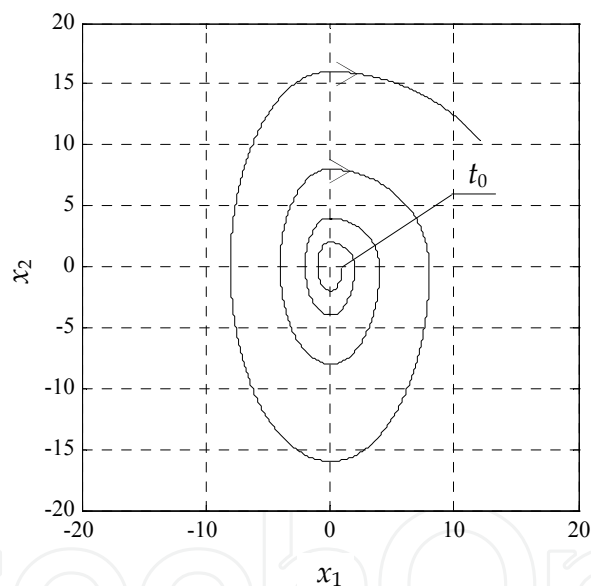


Fig. 4. Phase portrait of system (1) when switching strategy (5) is applied.

**Example 2.** In this example, again we consider system (1), however now we apply the following controller

$$u = -x_2 - a \operatorname{sgn}(x_1) - b \operatorname{sgn}(x_2) \quad (6)$$

where  $a > b > 0$ . Closer analysis of the behaviour of system (1) with control law (6) demonstrates that, in this example, the system error converges to zero in finite time which can be expressed as

$$T = \frac{a}{b} \sqrt{2x_{10}} \left( \frac{1}{\sqrt{a-b}} + \frac{1}{\sqrt{a+b}} \right) \quad (7)$$

where  $x_{10}$  and  $x_{20} = 0$  represent initial conditions of system (1). Even though the error converges to zero in finite time, the number of oscillations in the system tends to infinity, with the period of the oscillations decreasing to zero. This is illustrated in Figs. 5 and 6. In the simulation example presented in the figures, the following parameters are used  $a = 7$ ,  $b = 3$ ,  $x_{10} = 20$  and  $x_{20} = 0$ . Consequently, the system error is nullified at the time instant  $T = 12.045$  and remains equal to zero for any time greater than  $T$ . Clearly these favourable properties are achieved using finite control signal. This controller, due to the way the phase trajectory – shown in Fig. 5 – is drawn, is usually called “twisting controller”. It also serves as a good, simple example of the second order sliding mode controllers.

The two examples presented up to now demonstrate the principal properties of VSC systems. However, the main advantage of the systems is obtained when the controlled plant exhibits the sliding motion (DeCarlo et al., 1988; Hung et al., 1993; Slotine & Li, 1991; Utkin, 1977). The idea of sliding mode control (SMC) is to employ different feedback controllers acting on the opposite sides of a predetermined surface in the system state space. Each of those controllers pushes the system representative point (RP) towards the surface, so that the RP approaches the surface, and once it hits the surface for the first time it stays on it ever after. The resulting motion of the system is restricted to the surface, which graphically can be interpreted as “sliding” of the system RP along the surface. This idea is illustrated by the following example.

**Example 3.** Let us consider another second order plant

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= b \cos(m x_1) + u \quad |b| < 1, \end{aligned} \quad (8)$$

where  $b$  and  $m$  are possibly unknown constants. We select the following line in the state space

$$s = x_2 + c x_1 = 0 \quad (9)$$

( $c = \text{const.}$ ) and apply the controller

$$u = -c x_2 - \text{sgn}(s) \quad (10)$$

In this equation  $\text{sgn}(\cdot)$  function represents the sign of its argument, i.e.  $\text{sgn}(s < 0) = -1$  and  $\text{sgn}(s > 0) = +1$ . With this controller the system representative point moves towards line (9) always when it does not belong to the line. Then, once it hits the line, the controller switches the plant input (in the ideal case) with infinite frequency. Therefore, line (9) is called the switching line. Furthermore, since after reaching the line, the system RP slides along it, then the line is also called the sliding line. This example is illustrated in Fig. 7. The system parameters used in the presented simulation are  $c = 0.5$ ,  $b = 0.75$ ,  $m = 10$  and the simulation is performed for the following initial conditions  $x_{10} = 5$  and  $x_{20} = 1$ . Notice that when the plant remains in the sliding mode, its dynamics is completely determined by the switching line (or in general the switching hypersurface) parameters. This implies that neither model uncertainty nor matched external disturbance affects the plant dynamics (Draženović, 1969) which is a highly desirable system property. This property can also be justified geometrically, if one notices that in our example the slope of line (9) fully governs the plant motion in the sliding mode. Therefore, in SMC systems we usually make the distinction between two phases: the first one – called the reaching phase – lasts until the controlled

plant RP hits the switching surface, and the second one – the sliding phase – begins when the RP reaches the surface. In the latter phase the plant insensitivity to a class of modeling inaccuracies and external disturbances is ensured. Let us stress that the system robustness with respect to unmodeled dynamics, parameter uncertainty and external disturbances is guaranteed only in the sliding mode. Therefore, shortening or (if possible) even complete elimination of the reaching phase is an important and timely research issue (see for example Bartoszewicz & Nowacka-Leverton, 2009; Pan & Furuta, 2007; Sivert, 2004; Utkin & Shi, 1996) in the field of SMC.

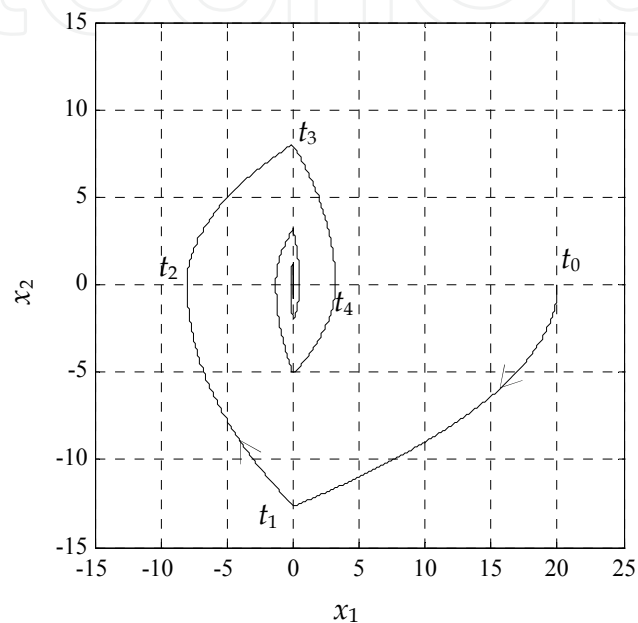


Fig. 5. Phase portrait of system (1) controlled according to (6).

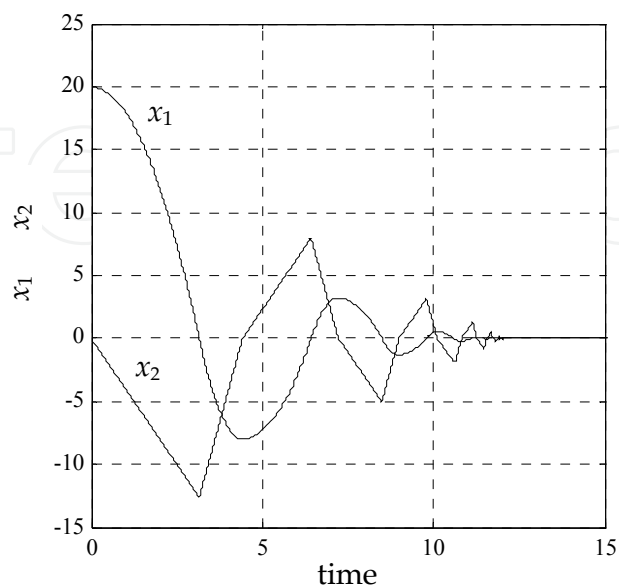


Fig. 6. State variables of system (1) controlled according to (6).

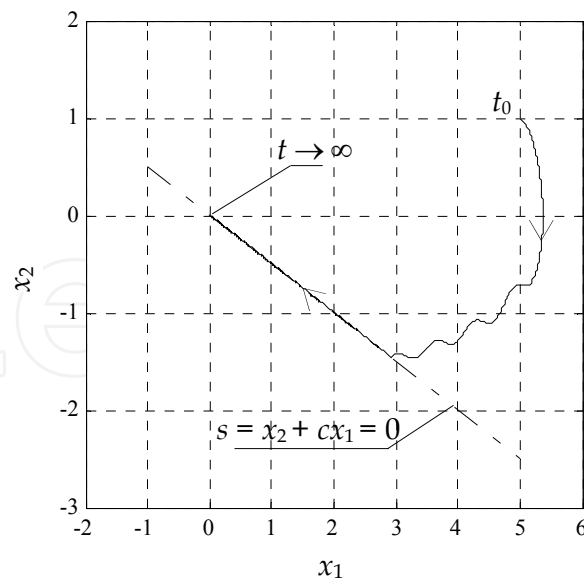


Fig. 7. Phase trajectory of system (1) controlled according to (10).

Another immediate consequence of the fact that in the sliding mode, the system RP is restricted to the switching hypersurface (which is a subset of the state space) is reduction of the system order. If the system of the order  $n$  has  $m$  independent inputs, then the sliding mode takes place on the intersection of  $m$  hypersurfaces and the reduced order of the system is equal to the difference  $n - m$ . To be more precise, in multi-input systems the sliding mode may take place either independently on each switching hypersurface or only on the intersection of the surfaces. In the first case the system RP approaches each surface at any time instant and once it hits any of the surfaces it stays on this surface ever after. In the latter case, however, the system RP does not necessarily approach each of the surfaces, but it always moves towards their intersection. In this case the system RP may hit a surface and move away from it (might possibly cross a switching surface), but once it reaches the intersection of all the surfaces, then the RP never leaves it.

One of the major tasks in the SMC system design is the selection of an appropriate control law. This can be achieved either by assuming a certain kind of the control law (usually motivated by some previous engineering experience) and proving that this control satisfies one of the so-called reaching conditions or by applying the reaching law approach. The reaching conditions (Edwards & Spurgeon, 1998) ensure stability of the sliding motion and therefore they are naturally derived using Lyapunov stability theory. On the other hand, if the reaching law approach is adopted for the purpose of a sliding mode controller construction (Bartoszewicz, 1998; Bartoszewicz, 1996; Gao et al, 1995; Golo & Milosavljević, 2000; Hung et al., 1993), then a totally different design philosophy is employed. In this case the desired evolution of the switching variable  $s$  is specified first, and then a control law ensuring that  $s$  changes according to the specification is determined.

Sliding mode controllers guarantee system insensitivity with respect to matched disturbance and model uncertainty (Draženović, 1969), and cause reduction of the plant order. Moreover, they are computationally efficient, and may be applied to a wide range of various, possibly nonlinear and time-varying plants. However, often they also exhibit a serious drawback which essentially hinders their practical applications. This drawback – high frequency oscillations which inevitably appear in any real system whose input is

supposed to switch infinitely fast – is usually called chattering. If system (8) exhibits any, even arbitrarily small, delay in the input channel, then control strategy (10), will cause oscillations whose frequency and amplitude depend on the delay. With the decreasing of the delay time, the frequency rises and the amplitude is getting smaller. This is a highly undesirable phenomenon, because it causes serious wear and tear on the actuator components. Therefore, a few methods to eliminate chattering have been proposed. The most popular of them uses function

$$\text{sat}(s) = \begin{cases} -1 & \text{for } s < -\rho \\ \frac{1}{\rho}s & \text{for } |s| \leq \rho \\ 1 & \text{for } s > \rho \end{cases} \quad (11)$$

instead of  $\text{sgn}(s)$  in the definition of the discontinuous control term. With this modification the term becomes continuous and the switching variable does not converge to zero but to the closed interval  $[-\rho, \rho]$ . Consequently, the system RP after the reaching phase termination, belongs to a layer around the switching hyperplane and therefore this strategy is called boundary layer controller (Slotine & Li, 1991).

Other approaches to the chattering elimination include:

- introduction of other nonlinear approximations of the discontinuous control term, for example the so called fractional approximation defined as

$$\text{approx}(s) = \frac{s}{\varepsilon + |s|} \quad (12)$$

where  $\varepsilon$  is a small positive constant (Ambrosino et al., 1984; Xu et al., 1996);

- replacing the boundary layer with a sliding sector (Shyu et al., 1992; Xu et al., 1996);
- using dynamic sliding mode controllers (Sira-Ramirez, 1993a; Sira-Ramirez, 1993b; Zlateva, 1996);
- using fuzzy sliding mode controllers (Palm, 1994; Palm et al., 1997);
- using second (or higher) order sliding mode controllers (Bartolini et al., 1998; Levant, 1993).

The phenomenon of chattering has been extensively analyzed in many papers using describing function method and various stability criteria (Shtessel & Lee, 1996).

As it has already been mentioned, the switching surface completely determines the plant dynamics in the sliding mode. Therefore, selecting this surface is one of the two major tasks in the process of the SMC system design. In order to stress this issue let us point out that the same controller which has been considered in the last example may result in a very different system performance, if the sliding line slope  $c$  is selected in another way. This can be easily noticed if one takes into account any negative  $c$ . Then, controller (10) still ensures stability of the sliding motion on line (9), i.e. the system RP still converges to the line, however the system is unstable since both state variables  $x_1$  and  $x_2$  tend to (either plus or minus) infinity while the system RP slides away from the origin of the phase plane along line (9).

Since sliding mode control is well known to be a robust and computationally efficient regulation technique which may be applied to nonlinear and possibly time-varying plants, then the proper design of the sliding mode controllers has recently become one of the most



extensively studied research topics within the field of control engineering. This design process usually breaks into two distinct parts: in the first part the switching surface is selected, and in the second one the control signal which always makes the system representative point approach the surface is chosen. Once the representative point hits the surface, then under the same control signal, the point remains on the surface. Thus, the switching surface fully determines the system dynamics in the sliding mode and should be carefully selected by the system designer.

In this chapter we consider the second order, nonlinear, time-varying system subject to the acceleration and velocity constraints. We introduce a continuously time-varying switching line adaptable to the initial conditions of the system which guarantees the existence of a sliding mode on this line. At the time  $t = t_0$  the line passes through the representative point, specified by the initial conditions of the system, in the error state space. Afterwards, the line moves smoothly, with a constant deceleration and a constant angle of inclination, to the origin of the space and having reached the origin the line remains fixed. Thus the proposed control algorithm eliminates the reaching phase and forces the representative point of the system to always stay on the switching line. Consequently, our control is robust with respect to the external disturbance and model uncertainty from the very beginning of the control action. Furthermore, in order to obtain good dynamic performance of the considered system, the switching line is designed in such a way that the integral absolute error (IAE) over the whole period of the control action is minimised and state constraints are satisfied at the same time. The presented method is verified by the simulation example.

The control algorithm proposed in this chapter may be regarded as an alternative solution to the elegant and currently widely accepted integral sliding mode control technique (Utkin & Shi, 1996). The main advantage of our approach is explicit consideration of state constraints in the controller design process. Furthermore, the novelty of our work demonstrates itself also in the IAE optimal performance and error convergence without oscillations or overshoots.

## 2. Problem formulation

In this chapter we consider the time-varying and nonlinear, second order system described by the following equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x, t) + \Delta f(x, t) + b(x, t)u + d(t) \end{aligned} \quad (13)$$

where  $x_1, x_2$  are the state variables of the system and  $x(t) = [x_1(t) \ x_2(t)]^T$  is the state vector,  $t$  denotes time,  $u$  is the input signal,  $b, f$  – are a priori known, bounded functions of time and the system state,  $\Delta f$  and  $d$  are functions representing the system uncertainty and external disturbances, respectively. Further in this chapter, it is assumed that there exists a strictly positive constant  $\delta$  which is the lower bound of the absolute value of  $b(x, t)$ , i.e.  $0 < \delta = \inf\{|b(x, t)|\}$ . Furthermore, functions  $\Delta f$  and  $d$  are unknown and bounded. Therefore, there exists a constant  $\mu$  which for every pair  $(x, t)$  satisfies the following inequality  $|\Delta f(x, t) + d(t)| \leq \mu$ . The initial conditions of the system are denoted as  $x_{10}, x_{20}$  where  $x_{10} = x_1(t_0), x_{20} = x_2(t_0)$ . System (13) is supposed to track the desired trajectory given as a function of time  $x_d(t) = [x_{1d}(t) \ x_{2d}(t)]^T$  where  $x_{2d}(t) = \dot{x}_{1d}(t)$  and  $x_{1d}(t)$  is a differentiable function of time. The trajectory tracking error is defined by the following vector

$$e(t) = [e_1(t) \ e_2(t)]^T = x(t) - x_d(t) \quad (14)$$

Hence, we have  $e_1(t) = x_1(t) - x_{1d}(t)$  and  $e_2(t) = x_2(t) - x_{2d}(t)$ .

In this chapter it is assumed that at the initial time  $t = t_0$ , the tracking error and the error derivative can be expressed as

$$e_1(t_0) = e_0 \neq 0, \ e_2(t_0) = 0 \quad (15)$$

where  $e_0$  is an arbitrary real number different from zero. This condition is indeed satisfied in many practical applications such as position control or set point change of second order systems. An example of these applications is point to point (PTP) control of robot manipulators, that is moving the manipulator arm from its initial location where it is originally at a halt, to another predefined position at which the arm stops and again is expected to remain at rest.

Further in this chapter, we present a detailed description of the sliding mode control strategy which ensures optimal performance of the system and its robustness with respect to both the system uncertainty  $\Delta f(x, t)$  and external disturbance  $d(t)$ .

### 3. Sliding mode controller

In order to effectively control system (13), i.e. to eliminate the reaching phase and to obtain system insensitivity with respect to both external disturbance  $d(t)$  and the model uncertainty  $\Delta f(x, t)$  from the very beginning of the system motion, we introduce a time-varying switching line. The line slope does not change during the control process, which implies that the line moves on the phase plane without rotating. In other words, the line is shifted in the state space with a constant angle of inclination. At the beginning the line moves with a constant deceleration in the state space and then it stops at a time instant  $t_f > t_0$ . Consequently, the switching line can be described by the following equation

$$s(e, t) = 0 \quad \text{where} \quad s(e, t) = e_2(t) + ce_1(t) + (Ct^2 + Bt + A)\delta \quad (16)$$

where

$$\delta = \begin{cases} 1 & \text{for } t \in [0, t_f) \\ 0 & \text{for } t \in [t_f, \infty) \end{cases} \quad (17)$$

and  $c$ ,  $A$  and  $B$  are constants. The selection of these constants will be considered further in this chapter.

In order to ensure system (13) stability in the sliding motion on the line described by equations (16) parameter  $c$  in this equation must be strictly positive, i.e.  $c > 0$ . Furthermore, in order to actually eliminate the reaching phase, and consequently to ensure insensitivity of the considered system from the very beginning of its motion, constants  $A$ ,  $B$ ,  $C$  and  $c$  should be chosen in such a way that the representative point of the system at the initial time  $t = t_0$  belongs to the switching line. For that purpose, the following condition must be satisfied

$$s[e(t_0), t_0] = e_2(t_0) + ce_1(t_0) + Ct_0^2 + Bt_0 + A = 0 \quad (18)$$

Notice that the input signal

$$u = \left\{ -f(x, t) - ce_2(t) + \dot{x}_{2d}(t) - (2Ct + B)\delta - \gamma \operatorname{sgn}[s(e, t)] \right\} / b(x, t) \quad (19)$$

where  $\gamma = \eta + \mu$  and  $\eta$  is a strictly positive constant, ensures the stability of the sliding motion on the switching line (16). In order to verify this property we consider the product  $s(e, t)\dot{s}(e, t) = s(e, t)[\dot{e}_2(t) + ce_2(t) + (2Ct + B)\delta]$ . Taking into account (13) and (19), we obtain

$$\begin{aligned} s(e, t)\dot{s}(e, t) &= s(e, t)[f(x, t) + \Delta f(x, t) + b(x, t)u + d(t) - \dot{x}_2(t) + ce_2(t) + (2Ct + B)\delta] = \\ &= s(e, t)\{\Delta f(x, t) - \gamma \operatorname{sgn}[s(e, t)] + d(t)\} \leq -\eta |s(e, t)| \end{aligned} \quad (20)$$

which proves the stability of the sliding motion on the switching line (16). In order to find the system tracking error we solve equation (16). First we consider the following equation

$$e_2(t) + ce_1(t) + Ct^2 + Bt + A = 0 \quad (21)$$

which determines the considered switching line for any time  $t \leq t_f$ , i.e. when the line moves and  $\delta=1$ . Solving equation (21) with initial condition (15) and assuming for the sake of clarity that  $t_0 = 0$ , we can calculate the tracking error and its derivative for the time  $t \in (0, t_f)$ . Furthermore, taking into account condition (18) and the assumption that  $t_0 = 0$  we obtain

$$A = -ce_0 \quad (22)$$

Then, the tracking error and its derivative can be written as

$$e_1(t) = \left( -\frac{B}{c^2} + \frac{2C}{c^3} \right) e^{-ct} - \frac{C}{c} t^2 + \frac{2C - cB}{c^2} t + e_0 + \frac{B}{c^2} - \frac{2C}{c^3} \quad (23)$$

$$e_2(t) = \left( \frac{B}{c} - \frac{2C}{c^2} \right) e^{-ct} - \frac{2C}{c} t + \frac{2C - cB}{c^2} \quad (24)$$

Now we solve equation

$$e_2(t) + ce_1(t) = 0 \quad (25)$$

which determines the considered switching line for any time  $t > t_f$  i.e. for the time when the line does not move which is equivalent to the case  $\delta=0$ . For this purpose we calculate values of (23) and its derivative (24) for  $t = t_f$

$$e_1(t_f) = \left( -\frac{B}{c^2} + \frac{2C}{c^3} \right) e^{-ct_f} - \frac{C}{c} t_f^2 + \frac{2C - cB}{c^2} t_f + e_0 + \frac{B}{c^2} - \frac{2C}{c^3} \quad (26)$$

$$e_2(t_f) = \left( \frac{B}{c} - \frac{2C}{c^2} \right) e^{-ct_f} - \frac{2C}{c} t_f + \frac{2C - cB}{c^2} \quad (27)$$

Then, after some calculations, we obtain the evolution of the tracking error

$$e_1(t) = \left[ -\frac{B}{c^2} + \frac{2C}{c^3} + \left( -\frac{C}{c} t_f^2 + \frac{2C - cB}{c^2} t_f + e_0 + \frac{B}{c^2} - \frac{2C}{c^3} \right) e^{ct_f} \right] e^{-ct} \quad (28)$$

$$e_2(t) = \left[ \frac{B}{c} - \frac{2C}{c^2} + \left( Ct_f^2 - \frac{2C - cB}{c} t_f - ce_0 - \frac{B}{c} + \frac{2C}{c^2} \right) e^{ct_f} \right] e^{-ct} \quad (29)$$

Notice that the error described by (23) and (28) converges to zero monotonically. Next, we show the procedure for finding the optimal switching line.

#### 4. Switching line design

Now we present how to choose the optimal switching line under the assumption that the line moves with a constant deceleration to the origin of the error state space. It means that we consider the line defined by (16) where  $C \neq 0$ . Notice that for the time  $t > t_f$ , switching line (16) is fixed and passes through the origin of the error state space. This leads to the condition

$$Ct_f^2 + Bt_f + A = 0 \quad (30)$$

Furthermore, in order to avoid rapid input changes, the velocity of the introduced line should change smoothly. Thus, the following condition should hold

$$2Ct_f + B = 0 \quad (31)$$

Using relations (30), (31) and (22), we obtain the formula expressing the time when the line stops moving

$$t_f = \frac{2e_0c}{B} \quad (32)$$

In order to choose the switching line parameters, the integral of the absolute error (IAE)

$$J = \int_0^\infty |e_1(t)| dt \quad (33)$$

is minimised subject to the system velocity

$$|e_2(t)| \leq v_{\max} \quad (34)$$

and the system acceleration

$$|\dot{e}_2(t)| \leq a_{\max} \quad (35)$$

constraints, where  $v_{\max}$ ,  $a_{\max}$  represent the maximum admissible velocity and maximum admissible acceleration of the considered system, respectively. In order to facilitate further minimisation procedure, we define the following positive constant

$$k = \frac{e_0c^2}{B} \quad (36)$$

From (36), we get

$$c = \sqrt{\frac{Bk}{e_0}} \quad (37)$$

We begin the procedure for finding optimal switching line parameters with calculating the IAE criterion. Substituting equations (23) and (28) into (33), calculating appropriate integrals and considering relation (37), we obtain

$$J(k, B) = \frac{|e_0|^{3/2}}{\sqrt{|B|}} \left( \frac{1}{\sqrt{k}} + \frac{2}{3} \sqrt{k} \right) \quad (38)$$

This criterion will be minimised with constraints (34) and (35). Since the considered criterion decreases with increasing value of  $B$ , the minimisation procedure of two variable function  $J(k, B)$  can be replaced by the minimisation of a single variable function. This remark will be very useful further in the chapter. Considering constraints, firstly we take into account each of the two constraints separately, and then we require both of them to be satisfied simultaneously.

#### 4.1 Velocity constraint

In this section we will consider system (13) subject to velocity constraint (34). For any time  $t \leq t_f$  the system velocity is described by equation (24) and for the time  $t \geq t_f$  by relation (29). Calculating the maximum value of  $|e_2(t)|$  we get

$$\max |e_2(t)| = \left| \frac{B}{c} \left[ \frac{\ln(1+2k)}{2k} - 1 \right] \right| \quad (39)$$

Then using relations (34), (39) and taking into account condition (37), we obtain the following inequality

$$|B| \leq \frac{v_{\max}^2 k}{|e_0|} \left[ \frac{\ln(1+2k)}{2k} - 1 \right]^{-2} \quad (40)$$

As it was mentioned, because criterion (38) decreases with increasing value of  $|B|$  the minimisation of criterion  $J$  as a function of two variables  $(k, B)$  with the velocity constraint may be replaced by the minimisation of the following single variable function

$$J_v(k) = \frac{e_0^2}{v_{\max}} \left| \frac{\ln(1+2k)}{2k} - 1 \right| \left( \frac{1}{k} + \frac{2}{3} \right) \quad (41)$$

This function, for any fixed  $k$  expresses the minimum value of criterion  $J(k, B)$  which can be achieved when the velocity constraint is satisfied. Closer analysis of this criterion as a single variable function shows that (41) reaches its minimum for numerically found argument  $k_{v \text{ opt}} \approx 13.467$ . Then, the optimal parameter  $B$  can be calculated from

$$B = \frac{v_{\max}^2 k}{|e_0|} \left[ \frac{\ln(1+2k)}{2k} - 1 \right]^{-2} \text{sgn}(e_0) \quad (42)$$

Substituting  $k_{v \text{ opt}}$  into (42), we obtain

$$B_{v \text{ opt}} = \frac{v_{\max}^2 k_{v \text{ opt}}}{|e_0|} \left[ \frac{\ln(1+2k_{v \text{ opt}})}{2k_{v \text{ opt}}} - 1 \right]^{-2} \text{sgn}(e_0) \quad (43)$$

The other switching line parameters can be derived from (22), (31), (32) and (37), and they are given below

$$A_{v \text{ opt}} = v_{\max} k_{v \text{ opt}} \left| \frac{\ln(1 + 2k_{v \text{ opt}})}{2k_{v \text{ opt}}} - 1 \right|^{-1} \text{sgn}(e_0) \quad (44)$$

$$t_{f v \text{ opt}} = \frac{2|e_0|}{v_{\max}} \left| \frac{\ln(1 + 2k_{v \text{ opt}})}{2k_{v \text{ opt}}} - 1 \right| \quad (45)$$

$$C_{v \text{ opt}} = -\frac{v_{\max}^3 k_{v \text{ opt}}}{4e_0^2} \left| \frac{\ln(1 + 2k_{v \text{ opt}})}{2k_{v \text{ opt}}} - 1 \right|^{-3} \text{sgn}(e_0) \quad (46)$$

$$c_{v \text{ opt}} = \frac{v_{\max} k_{v \text{ opt}}}{|e_0|} \left| \frac{\ln(1 + 2k_{v \text{ opt}})}{2k_{v \text{ opt}}} - 1 \right| \quad (47)$$

That concludes the analysis of the velocity constraint taken into account separately.

#### 4.2 Acceleration constraint

Now we consider the system acceleration constraint given by (35). Let us calculate the greatest value of  $|\dot{e}_2(t)|$ . The maximum absolute value of this signal, achieved at the initial time  $t_0 = 0$  is equal to  $|\dot{e}_2(0)| = |B|$ . Then, the acceleration constraint can be expressed as follows

$$|B| \leq a_{\max} \quad (48)$$

Now we will analyse the criterion J minimisation task. Notice that for any given value of k, the minimum of criterion (38) is obtained for the greatest value of  $|B|$  satisfying constraint (48). Therefore, the solution of the considered minimisation task can be found as a minimum of the following single variable function J

$$J_a(k) = \frac{|e_0|^{3/2}}{\sqrt{a_{\max}}} \left( \frac{1}{\sqrt{k}} + \frac{2}{3} \sqrt{k} \right) \quad (49)$$

In order to analyse the minimisation task we calculate the derivative of expression (49) with respect to k. Then, we conclude that function (49) reaches its minimum for  $k_{a \text{ opt}} = 1.5$  and the optimal parameter B can be calculated from

$$B = a_{\max} \text{sgn}(e_0) \quad (50)$$

The other optimal switching line parameters can be calculated from relations

$$A_{a \text{ opt}} = -\sqrt{\frac{3a_{\max}|e_0|}{2}} \text{sgn}(e_0) \quad (51)$$

$$t_{f a \text{ opt}} = \sqrt{\frac{6|e_0|}{a_{\max}}} \quad (52)$$

$$C_{a \text{ opt}} = -\frac{a_{\text{max}}^{3/2}}{2\sqrt{6|e_0|}} \text{sgn}(e_0) \quad (53)$$

$$c_{a \text{ opt}} = \sqrt{\frac{3a_{\text{max}}}{2|e_0|}} \quad (54)$$

That ends our presentation of the algorithm for switching line design with the acceleration constraint.

### 4.3 Velocity and acceleration constraint

Finally, we consider both of constraints, i.e. the system velocity and the system acceleration and we require that they are satisfied at the same time. In order to minimise the considered criterion with constraints (34) and (35), we will minimise the following function of a single variable  $k$

$$J_{v a}(k) = \max[J_v(k), J_a(k)] \quad (55)$$

This minimisation task can be solved by considering three cases (which are illustrated in Figs. 8-10):

1.  $J_v(k_{a \text{ opt}}) \leq J_a(k_{a \text{ opt}})$
2.  $J_v(k_{v \text{ opt}}) \geq J_a(k_{v \text{ opt}})$
3.  $J_v(k_{a \text{ opt}}) > J_a(k_{a \text{ opt}})$  and  $J_v(k_{v \text{ opt}}) < J_a(k_{v \text{ opt}})$

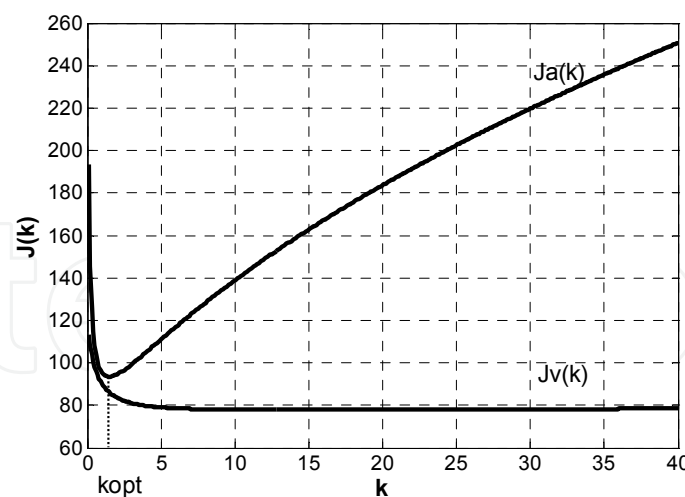


Fig. 8. Criteria  $J_v(k)$  and  $J_a(k)$  - case 1.

In the first case, the optimal value of  $k$  is given by  $k_{\text{opt}} = k_{a \text{ opt}} = 1.5$ , and then parameter  $B_{\text{opt}}$  is given by formula (50). In the second case we obtain that  $k_{\text{opt}} = k_{v \text{ opt}} \approx 13.467$  and  $B_{\text{opt}}$  can be calculated from equation (42). In the last case, in order to find the optimal solution, we solve (numerically) equation  $J_v(k) - J_a(k) = 0$  in the interval  $(k_{a \text{ opt}}, k_{v \text{ opt}})$ . Substituting numerically found value  $k_{\text{opt}}$  into either (42) or (50), we get the optimal value of  $B$ . The other



optimal switching line parameters can be derived from (22), (31), (32) and (37). In this way we design the switching line which is optimal in the sense of the IAE criterion and guarantees that the state constraints are satisfied.

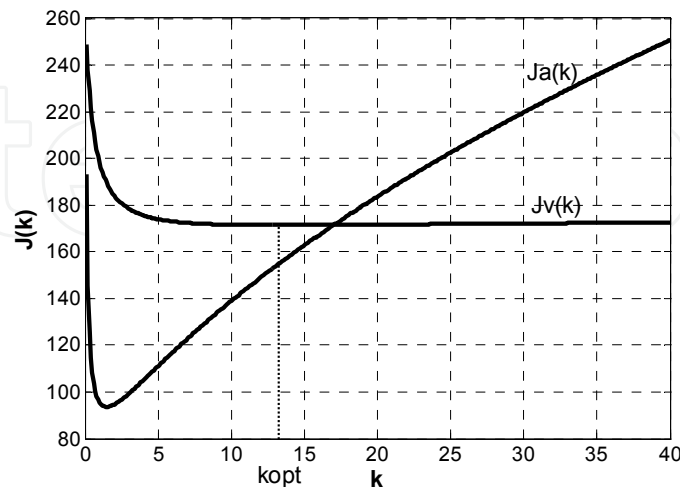


Fig. 9. Criteria  $J_v(k)$  and  $J_a(k)$  - case 2.

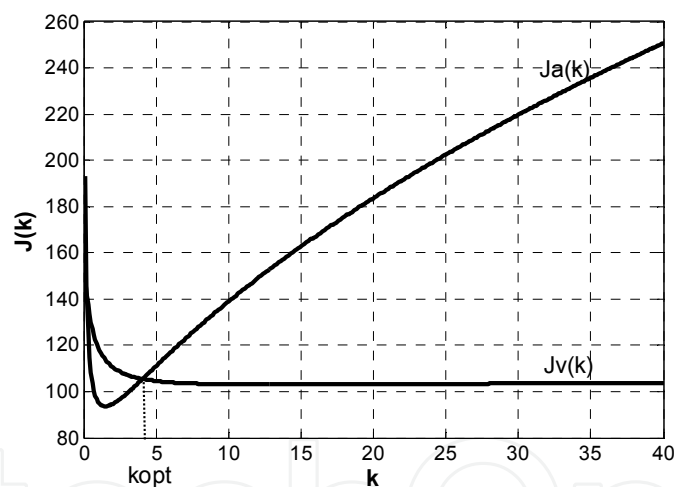


Fig. 10. Criteria  $J_v(k)$  and  $J_a(k)$  - case 3.

## 5. Simulation examples

In order to illustrate and verify the proposed method of the switching line design, we consider a suspended load described as follows

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = [-0.15x_2 + F - f(x_1, x_2)]/m \quad (56)$$

where  $m = 1$  kg and  $f(x_1, x_2) = 0.1\text{sgn}(x_2) + 0.049x_2 / (\pi|x_2| + 0.1)$  represents model uncertainty, i.e. unknown friction in the system. Consequently,  $\gamma = 0.15$ . The initial condition  $x_0 = 0.1$  m. The demand position of system (56) is  $x_d = 7$  m. We require that  $v_{\max} = 0.3$  m/s and  $a_{\max} = 0.1$  m/s<sup>2</sup>. Then, using the presented algorithm, we obtain that



$J_v(k_{a\text{opt}}) > J_a(k_{a\text{opt}})$  and  $J_v(k_{v\text{opt}}) < J_a(k_{v\text{opt}})$ , and the optimal value of  $k$  can be found numerically. In the considered example it is equal to  $k_{\text{opt}} \approx 4.0612$ . Consequently, we obtain the following set of the optimal parameters  $A_{\text{opt}} \approx 1.674$  m/s,  $B_{\text{opt}} = -0.1$  m/s<sup>2</sup>,  $c_{\text{opt}} \approx 0.2426$  1/s and  $C_{\text{opt}} \approx 0.0015$  m/s<sup>3</sup>. The line stops moving at the time instant  $t_{f\text{opt}}$  equal to 33.48s.

Simulation results for the system with this line are shown in Figures 11 – 14. From Figure 11 it can be seen that the load reaches its demand position without oscillations or overshoots. Figure 12 presents the system velocity. The system acceleration is illustrated in Figure 13. The plots confirm that the required constraints are always satisfied. Furthermore, the system is insensitive from the very beginning of the control process. Figure 14 illustrates the phase trajectory of the controlled plant.

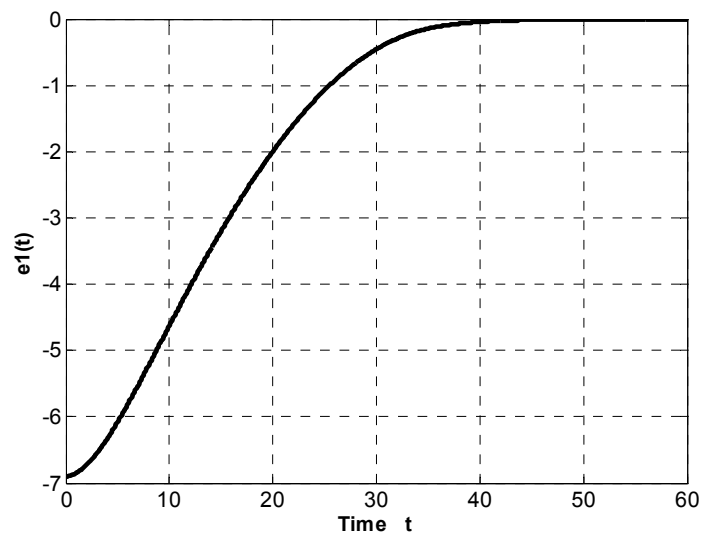


Fig. 11. System error evolution.

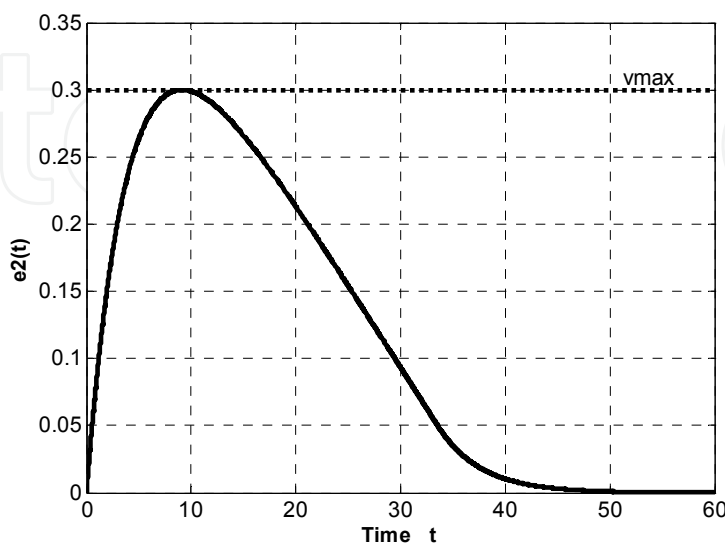


Fig. 12. System velocity.

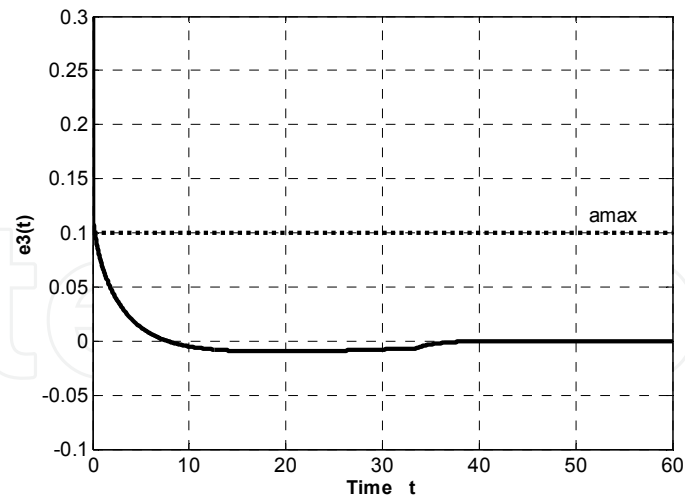


Fig. 13. System acceleration.

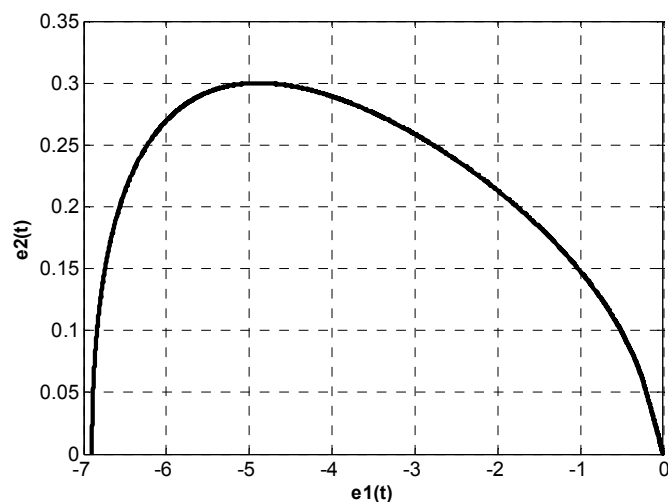


Fig. 14. Phase trajectory.

## 6. Conclusion

In this chapter, we proposed a method of sliding mode control. This method employs the time-varying switching line which moves with a decreasing velocity and a constant angle of inclination to the origin of the error state space. Parameters of this line are selected in such a way that integral the absolute error (IAE) is minimised with the system acceleration and the system velocity constraints. Furthermore, the tracking error converges to zero monotonically and the system is insensitive with respect to external disturbance and the model uncertainty from the very beginning of the control action.

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## **Sliding Mode Control**

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The main objective of this monograph is to present a broad range of well worked out, recent application studies as well as theoretical contributions in the field of sliding mode control system analysis and design. The contributions presented here include new theoretical developments as well as successful applications of variable structure controllers primarily in the field of power electronics, electric drives and motion steering systems. They enrich the current state of the art, and motivate and encourage new ideas and solutions in the sliding mode control area.

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