We are IntechOpen, the world’s leading publisher of Open Access books
Built by scientists, for scientists

3,700
Open access books available

108,500
International authors and editors

1.7 M
Downloads

154
Countries delivered to

TOP 1%
Our authors are among the most cited scientists

12.2%
Contributors from top 500 universities

WEB OF SCIENCE™
Selection of our books indexed in the Book Citation Index in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com
Exponential Stability of Uncertain Switched System with Time-Varying Delay

Eakkapong Duangdai¹ and Piyapong Niamsup¹,²,³

¹²Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200

²Center of Excellence in Mathematics CHE, Si Ayutthaya Rd.,Bangkok 10400

³Materials Science Research Center, Faculty of Science, Chiang Mai University, Chiang Mai 50200 Thailand

1. Introduction

During the past decades, many researchers have investigated stability of switched systems; due to its potential for real world application such as transportation systems, computer systems, communication systems, control of mechanical systems, etc. A switched system is composed of a family of continuous time (Alan & Lib, 2008; Alan & Lib, 2009, Alan et al., 2008; Hien et al., 2009; Hien & Phat, 2009; Kim et al., 2006; Li et al., 2009; Li et al., 2009; Lien et al., 2009; Lib et al., 2008) or discrete time systems (Wu et al., 2004) and a switching condition determining at any time instant which subsystem is activated.

In recent years, the stability of systems with time delay has received considerable attention. Switched system in which all subsystems are stable was studied in (Lien et al., 2009) and switched system in which subsystems are both stable and unstable was studied in (Alan & Lib, 2008; Alan & Lib, 2009, Alan et al., 2008). The commonly used approach to stability analysis of switched systems is Lyapunov theory and some important preliminaries results have been applied to obtain sufficient conditions for stability of switched systems. A single Lyapunov function approach is used in (Alan & Lib, 2008) and a multiple Lyapunov functions approach is used in (Hien et al., 2009; Kim et al., 2006; Li et al., 2009; Lien et al., 2009; Lib et al., 2008) and the references therein. The asymptotical stability of the linear with time delay and uncertainties has been considered in (Lien et al., 2009). In (L.V.Hien et al., 2009), the authors investigated the exponential stability and stabilization of switched linear systems with time varying delay and uncertainties by using the strictly complete systems of matrices approach. The strictly complete of the matrices has been also used for the switching condition, see (Hien et al., 2009; Huang et al., 2005; Niamsup, 2008; Lib et al., 2008, Wu et al., 2004). In this paper, stability analysis for switched linear and nonlinear systems with uncertainties and time-varying delay are studied. We obtain the new conditions for exponential stability of switched system in which subsystems consist of stable and unstable subsystems. The stability conditions are derived in terms of linear matrix inequality (LMI) by using a new Lyapunov

*Corresponding author (Email:scipnmsp@chiangmai.ac.th)
function. The free weighting matrices and Newton-Leibniz formula are applied. As a result, the obtained stability conditions are less conservative comparing to some previous existing results in the literatures. In particular, comparing to (Alan & Lib, 2008), our results give a much less conservative results, namely, for stable subsystems, the condition that state matrices are Hurwitz stable is not required. Moreover, advantages of the paper are that the delay is time-varying and switched system may have uncertainties. The paper is organized as follows. In section 1, problem formulation and introduction is addressed. In section 2, we give some notations, definitions and the preliminary results that will be used in this paper. Switching design for the exponential stability of the switched system is presented in Section 3. In section 4, numerical examples are given to illustrate the theoretical results. The paper ends with conclusions and cited references.

2. Preliminaries

The following notations will be used throughout this paper. \( \mathbb{R}^n \) denotes the n-dimensional Euclidean space. \( \mathbb{R}^{n \times n} \) denotes the space of all matrices of \( n \times n \)-dimensions. \( A^T \) denotes the transpose of \( A \). \( I \) denotes the identity matrix. \( \lambda(A), \lambda_M(A), \lambda_m(A) \) denote the set of all eigenvalues of \( A \), the maximum eigenvalue of \( A \), and the minimum eigenvalue of \( A \), respectively. For all real symmetric matrix \( X \), the notation \( X > 0 \) means that \( X \) is positive definite (positive semidefinite, negative definite, negative semidefinite, respectively.) For a vector \( x \), \( \|x\| = \sup_{s \in [-h_M,0]} \|x(t+s)\| \) with \( ||x|| \) being the Euclidean norm of vector \( x \).

The switched system under the consideration is described by

\[
\dot{x}(t) = [A_{\sigma} + \Delta A_{\sigma}(t)]x(t) + [B_{\sigma} + \Delta B_{\sigma}(t)]x(t-h(t)) + f_{\sigma}(t,x(t),x(t-h(t))), \quad t > 0,
\]

\[
x(t) = \phi(t), \quad t \in [-h_M,0],
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector. \( \sigma(\cdot) : \mathbb{R}^n \rightarrow S = \{1,2,...,N\} \) is the switching function. Let \( i \in S = S_u \cup S_s \) such that \( S_u = \{1,2,...,r\} \) and \( S_s = \{r+1,r+2,...,N\} \) be the set of the unstable and stable modes, respectively. \( N \) denotes the number of subsystems. \( A_i, B_i \in \mathbb{R}^{n \times n} \) are given constant matrices. \( \Delta A_i(t), \Delta B_i(t) \) are uncertain matrices satisfying the following conditions:

\[
\Delta A_i(t) = E_{1i}F_{1i}(t)H_{1i}, \quad \Delta B_i(t) = E_{2i}F_{2i}(t)H_{2i},
\]

where \( E_{ji}, H_{ji}, j = 1,2, i = 1,2,...,N \) are given constant matrices with appropriate dimensions. \( F_{ji}(t) \) are unknown, real matrices satisfying:

\[
F_{ji}^T(t)F_{ji}(t) \leq 1, \quad j = 1,2, i = 1,2,...,N, \quad \forall t \geq 0,
\]

where \( I \) is the identity matrix of appropriate dimension. The nonlinear perturbation \( f_{\sigma}(t,x(t),x(t-h(t))) \), \( i = 1,2,...,N \) satisfies the following condition:

\[
\|f_i(t,x(t),x(t-h(t)))\| \leq \gamma_i \|x(t)\| + \delta_i \|x(t-h(t))\|
\]

for some \( \gamma_i, \delta_i > 0 \). The time-varying delay function \( h(t) \) is assumed to satisfy one of the following conditions:

(i) when \( \Delta A_i(t) = 0 \) and \( \Delta B_i(t) = 0 \) and \( f_i(t,x(t),x(t-h(t))) = 0 \)
Exponential Stability of Uncertain Switched System with Time-Varying Delay

\begin{equation}
0 \leq h_m \leq h(t) \leq h_M, \quad \dot{h}(t) \leq \mu, \quad t \geq 0,
\end{equation}

\( (ii) \) when \( \Delta A_i(t) \neq 0 \) or \( \Delta B_i(t) \neq 0 \) or \( f_i(t, x(t), x(t-h(t))) \neq 0 \)

\begin{equation}
0 \leq h_m \leq h(t) \leq h_M, \quad \dot{h}(t) \leq \mu < 1, \quad t \geq 0,
\end{equation}

where \( h_m, h_M \) and \( \mu \) are given constants.

**Definition 2.1** (Hien et al., 2009) Given \( \beta > 0 \). The system (1) is \( \beta \)-exponentially stable if there exists a switching function \( \sigma(\cdot) \) and positive number \( \gamma \) such that any solution \( x(t, \phi) \) of the system satisfies

\begin{equation}
\| x(t, \phi) \| \leq \gamma e^{-\beta t} \| \phi \|, \quad \forall t \in \mathbb{R}^+,
\end{equation}

for all the uncertainties.

**Lemma 2.1** (Hien et al., 2009) For any \( x, y \in \mathbb{R}^n \), matrices \( W, E, F, H \) with \( W > 0, F^T F \leq I \), and scalar \( \varepsilon > 0 \), one has

\( (1.) \) \( E F H + H^T F^T E \leq \varepsilon^{-1} E E^T + \varepsilon H^T H \),

\( (2.) \) \( 2x^T y \leq x^T W^{-1} x + y^T W y \).

**Lemma 2.2** (Alan & Lib, 2008) Let \( u : [t_0, \infty) \rightarrow \mathbb{R} \) satisfy the following delay differential inequality:

\begin{equation}
\dot{u}(t) \leq a u(t) + \beta \sup_{\theta \in [t-\tau, t]} u(\theta), \quad t \geq t_0.
\end{equation}

Assume that \( a + \beta > 0 \). Then, there exist positive constant \( \zeta \) and \( k \) such that

\begin{equation}
\dot{u}(t) \leq k e^{\zeta(t-t_0)}, \quad t \geq t_0,
\end{equation}

where \( \zeta = a + \beta \) and \( k = \sup_{\theta \in [t_0-\tau, t_0]} u(\theta) \).

**Lemma 2.3** (Alan & Lib, 2008) Let the following differential inequality:

\begin{equation}
\dot{u} \leq -a u(t) + \beta \sup_{\theta \in [t-\tau, t]} u(\theta), \quad t \geq t_0,
\end{equation}

hold. If \( a > \beta > 0 \), then there exist positive \( k \) and \( \zeta \) such that

\begin{equation}
\dot{u}(t) \leq k e^{-\zeta(t-t_0)}, \quad t \geq t_0,
\end{equation}

where \( \zeta = a - \beta \) and \( k = \sup_{\theta \in [t_0-\tau, t_0]} u(\theta) \).

**Lemma 2.4** (Schur Complement Lemma) (Boyd et al., 1985) Given constant symmetric \( Q, S \) and \( R \in \mathbb{R}^{n \times n} \) where \( R > 0, Q = Q^T \) and \( R = R^T \) we have

\begin{equation}
\begin{bmatrix}
Q & S \\
S^T & -R
\end{bmatrix} < 0 \Leftrightarrow Q + SR^{-1}S^T < 0.
\end{equation}

3. Main results

In this section, we establish exponential stability of uncertain switched system with time-varying delay. For simplicity of later presentation, we use the following notations:

\( \lambda^+ = \max \{ \zeta_i, \forall i \in S_u \}, \) \( \zeta_i \) denotes the growth rates of the unstable modes.

\( \lambda^- = \min \{ \zeta_i, \forall i \in S_s \}, \) \( \zeta_i \) denotes the decay rates of the stable modes.
\( T^+ (t_0, t) \) denotes the total activation times of the unstable modes over \([t_0, t)\).
\( T^- (t_0, t) \) denotes the total activation times of the stable modes over \([t_0, t)\).
\( N(t) \) denotes the number of times the system is switched on \([t_0, t)\).
\( l(t) \) denotes the number of times the unstable subsystems are activated on \([t_0, t)\).
\( N(t) - l(t) \) denotes the number of times the stable subsystems are activated on \([t_0, t)\).

\[
\Psi = \max_i \{ \lambda_M (P_i) \},
\]
\[
a_1 = \min_i \{ \lambda_M (P_i) \},
\]
\[
a_2 = \max_i \{ \lambda_M (P_i) \} + h_M \max_i \{ \lambda_M (Q_i) \} + \frac{h_M^2}{2} \max_i \{ \lambda_M (R_i) \}
+ h_M^2 \max_i \{ \lambda_M \left( \begin{array} {c} S_{11,i} \\ S_{12,i} \\ S_{22,i} \end{array} \right) \} 
+ 2h_M^2 \max_i \{ \lambda_M (A_i^T T_i A_i), \lambda_M (A_i^T T_i B_i), \lambda_M (B_i^T T_i A_i), \lambda_M (B_i^T T_i B_i) \}.
\]
\[
a_3 = \max_i \{ \lambda_M (P_i) \} + h_M \max_i \{ \lambda_M (Q_i) \} + \frac{h_M^2}{2} \max_i \{ \lambda_M (R_i) \}
+ h_M^2 \max_i \{ \lambda_M \left( \begin{array} {c} S_{11,i} \\ S_{12,i} \\ S_{22,i} \end{array} \right) \}.
\]
\[
\Omega_{1,i} = \left[ \Phi_{11,i} \Phi_{12,i} \right].
\]
\[
\Phi_{11,i} = A_i^T P_i + P_i A_i + \gamma_i R_i + h_M S_{11,i} + h_M A_i^T T_i A_i,
\]
\[
\Phi_{12,i} = B_i^T P_i + h_M S_{12,i} + h_M A_i^T T_i B_i,
\]
\[
\Phi_{13,i} = - (1 - \mu)e^{-2h_M \lambda_i} Q_i + h_M S_{22,i} + h_M B_i^T T_i B_i.
\]
\[
\Omega_{2,i} = \left[ \Phi_{21,i} \Phi_{22,i} \right].
\]
\[
\Phi_{21,i} = A_i^T P_i + P_i A_i + \gamma_i R_i + h_M S_{11,i} + h_M A_i^T T_i A_i + h_M X_{11,i} + Y_i + Y_i^T,
\]
\[
\Phi_{22,i} = B_i^T P_i + h_M S_{12,i} + h_M A_i^T T_i B_i + h_M X_{12,i} - Y_i + Z_i^T,
\]
\[
\Phi_{23,i} = - (1 - \mu)e^{-2h_M \lambda_i} Q_i + h_M S_{22,i} + h_M B_i^T T_i B_i + h_M X_{22,i} - Z_i - Z_i^T.
\]
\[
\Omega_{3,i} = \left[ \Phi_{31,i} \Phi_{32,i} \right].
\]
\[
\Phi_{31,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \gamma_i^2 H_i H_i + \epsilon_i P_i E_{i1}^T E_{i1} P_i + \epsilon_i P_i E_{i2}^T E_{i2} P_i,
\]
\[
\Phi_{32,i} = B_i^T P_i + h_M S_{12,i},
\]
\[
\Phi_{33,i} = - (1 - \mu)e^{-2h_M \lambda_i} Q_i + h_M S_{22,i} + \epsilon_i^2 H_i H_i.
\]
\[
\Theta_{1,i} = \left[ \Phi_{41,i} \Phi_{42,i} \right].
\]
\[
\Phi_{41,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \epsilon_i^3 \gamma_i I + \epsilon_i S_i P_i + \epsilon_i^4 H_i H_i
+ \epsilon_i P_i E_{i4}^T E_{i4} P_i + \epsilon_i P_i E_{i5}^T E_{i5} P_i,
\]
\[
\Phi_{42,i} = B_i^T P_i + h_M S_{12,i}.
\]
\[ \Phi_{4,3,1} = -(1 - \mu)e^{-2\beta_{M}Q_{1}} + h_{M}S_{2,2,1} + \varepsilon_{3}^{-1}M + \varepsilon_{3}^{-1}H_{5}^{T}H_{5}. \]

### 3.1 Exponential stability of linear switched system with time-varying delay

In this section, we deal with the problem for exponential stability of the zero solution of system (1) without the uncertainties and nonlinear perturbation (\(\Delta A_{i}(t) = \Delta B_{i}(t) = 0, f_{i}(t,x(t),x(t - h(t))) = 0\)).

**Theorem 3.1** The zero solution of system (1) with \(\Delta A_{i}(t) = \Delta B_{i}(t) = 0\) and \(f_{i}(t,x(t),x(t - h(t))) = 0\) is exponentially stable if there exist symmetric positive definite matrices \(P_{i}, Q_{i}, R_{i}\),

\[
\begin{bmatrix}
S_{1,1,1} & S_{1,1,2} \\
S_{1,2,1}^{T} & S_{1,2,2}
\end{bmatrix}, T_{i}
\]

and appropriate dimension matrices \(Y_{i}, Z_{i}\) such that the following conditions hold:

**A1.** (i) For \(i \in S_{u}\),

\[
\Omega_{1,i} > 0.
\]

(ii) For \(i \in S_{s}\),

\[
\Omega_{2,i} < 0 \text{ and } \Omega_{3,i} \geq 0.
\]

**A2.** Assume that, for any \(t_{0}\) the switching law guarantees that

\[
\inf_{t_{0} \leq t} \frac{T^{-}(t_{0}, t)}{T^{+}(t_{0}, t)} = \frac{\lambda^{+} + \lambda^{*}}{\lambda^{-} - \lambda^{*}}
\]

where \(\lambda^{*} \in (0, \lambda^{-})\). Furthermore, there exists \(0 < \nu < \lambda^{*}\) such that

(i) If the subsystem \(i \in S_{u}\) is activated in time intervals \([t_{k-1}, t_{k})\), \(k = 1, 2, ...,\) then

\[
\ln \psi - v(t_{k} - t_{k-1}) \leq 0, \; k = 1, 2, ..., N(t) - 1.
\]

(ii) If the subsystem \(j \in S_{s}\) is activated in time intervals \([t_{j-1}, t_{j})\), \(k = 1, 2, ...,\) then

\[
\ln \psi + \xi_{M} - v(t_{j} - t_{j-1}) \leq 0, \; k = 1, 2, ..., N(t) - 1.
\]

**Proof.** Consider the following Lyapunov functional:

\[
V_{1}(x_{t}) = V_{1,1}(x(t)) + V_{2,1}(x_{t}) + V_{3,1}(x_{t}) + V_{4,1}(x_{t}) + V_{5,1}(x_{t})
\]

where \(x_{t} \in C([-h_{M}, 0], \mathbb{R}^{n})\), \(x_{t}(s) = x(t + s), s \in [-h_{M}, 0]\) and

\[
V_{1,1}(x(t)) = x^{T}(t)P_{1}x(t),
\]

\[
V_{2,1}(x_{t}) = \int_{t-h(t)}^{t} e^{2\beta(t-s)}x^{T}(s)Q_{1}x(s)ds,
\]

\[
V_{3,1}(x_{t}) = \int_{t-h(t)}^{t} \int_{t-s}^{t} e^{2\beta(s-t)}x^{T}(s)K_{2}x(s)ds,
\]

\[
V_{4,1}(x_{t}) = \int_{t-h(t)}^{t} \int_{t-s}^{t} e^{2\beta(t-s)}x^{T}(s)K_{3}x(s)ds,
\]

It is easy to verify that

\[
\alpha_{1} \| x(t) \|^{2} \leq V_{1}(x_{t}) \leq \alpha_{2} \| x_{t} \|^{2}, \; t \geq 0.
\]

www.intechopen.com
We have
\[ V_{1,i}(x(t)) \leq \max_i \{ \lambda_M(P_i) \} \| x(t) \|^2 \]
\[ = \frac{\max_i \{ \lambda_M(P_i) \}}{\min_j \{ \lambda_m(P_j) \}} \min_j \{ \lambda_m(P_j) \} x^T(t)x(t) \]
\[ \leq \frac{\max_i \{ \lambda_M(P_i) \}}{\min_j \{ \lambda_m(P_j) \}} x^T(t)P_j x(t) \]
\[ = \frac{\max_i \{ \lambda_M(P_i) \}}{\min_j \{ \lambda_m(P_j) \}} V_{1,j}(x(t)). \]

Let \( \psi = \frac{\max_i \{ \lambda_M(P_i) \}}{\min_j \{ \lambda_m(P_j) \}} \). Obviously \( \psi \geq 1 \) and we get
\[ V_i(x(t)) \leq \psi V_j(x(t)), \forall i, j \in S. \] (11)

Taking derivative of \( V_{1,i}(x(t)) \) along trajectories of any subsystem \( i \)th we have
\[ \dot{V}_{1,i}(x(t)) = x^T(t)P_i x(t) + x^T(t)P_i \dot{x}(t) \]
\[ = \sum_{i=1}^{N} \lambda_i(t)[x^T(t)A_i^TP_i x(t) + x^T(t-h(t))B_i^TP_i x(t) + x^T(t)P_i A_i x(t) + x^T(t)P_i B_i x(t-h(t))]. \]

Next, by taking derivative of \( V_2,i(x(t)), V_3,i(x(t)), V_4,i(x(t)) \) and \( V_5,i(x(t)) \), respectively, along the system trajectories yields
\[ \dot{V}_2,i(x(t)) = x^T(t)Q_i x(t) - (1 - h(t))e^{-2\beta h(t)}x^T(t-h(t))Q_i x(t-h(t)) - 2\beta V_2,i(x(t)) \]
\[ \leq x^T(t)Q_i x(t) - (1 - \mu)e^{-2\beta h(t)}x^T(t-h(t))Q_i x(t-h(t)) - 2\beta V_2,i(x(t)), \]
\[ \dot{V}_3,i(x(t)) = \int_{-h(t)}^{0} x^T(t)R_i x(t) - e^{2\beta h(t)}x^T(t+s)R_i x(t+s)ds - 2\beta V_3,i(x(t)) \]
\[ \leq h_M x^T(t)R_i x(t) - \int_{-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_i x(s)ds - 2\beta V_3,i(x(t)). \]
\[ V_{4,i}(x_t) = \int_{-h(t)}^{0} \left[ \begin{array}{c} x(\xi) \\ x(\xi - h(\xi)) \end{array} \right]^T \left[ \begin{array}{cc} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{array} \right] \left[ \begin{array}{c} x(\xi) \\ x(\xi - h(\xi)) \end{array} \right] \, ds \\
-2\beta_2 \int_{-h(t)}^{0} \left[ \begin{array}{c} x(t+s) \\ x(t+s-h(t+s)) \end{array} \right]^T \left[ \begin{array}{cc} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{array} \right] \left[ \begin{array}{c} x(t+s) \\ x(t+s-h(t+s)) \end{array} \right] \, ds \]

\[ -2\beta_2 V_{4,i}(x_t) \leq h_M \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right]^T \left[ \begin{array}{cc} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{array} \right] \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right] \]

\[ - h_M \int_{1-h(t)}^{t} e^{\beta(s-t)} \left[ \begin{array}{c} x(s) \\ x(s-h(s)) \end{array} \right]^T \left[ \begin{array}{cc} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{array} \right] \left[ \begin{array}{c} x(s) \\ x(s-h(s)) \end{array} \right] \, ds \]

Then, the derivative of \[ V_i(x_t) \] along any trajectory of solution of (I) is estimated by

\[ \dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right]^T \Omega^{*}_{i,j} \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right] - 2\beta_2 V_{4,i}(x_t) \]

\[ - \int_{1-h(t)}^{t} e^{\beta(s-t)} x^T(s) R_i x(s) \, ds - 2\beta_2 V_{4,i}(x_t) \]

\[ - \int_{1-h(t)}^{t} e^{\beta(s-t)} \left[ \begin{array}{c} x(s) \\ x(s-h(s)) \end{array} \right]^T \left[ \begin{array}{cc} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{array} \right] \left[ \begin{array}{c} x(s) \\ x(s-h(s)) \end{array} \right] \, ds \]

\[ -2\beta_4 V_{4,i}(x_t) + h_M \dot{x}(t)^T T_i \dot{x}(t) - \frac{1}{2} \int_{1-h(t)}^{t} x^T(s) T_i x(s) \, ds \]

\[ - \frac{1}{2} \int_{1-h(t)}^{t} \dot{x}(t)^T T_i \dot{x}(s) \, ds \]
where
\[
\Omega^j_{i,j} = \left[ A^T_i P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,j} \ast (1 - \mu e^{-2h_M Q_i + h_M S_{22,j}}) \right]
\]

Since
\[
\int_{-h(t)}^t \int_{t+h(t)}^t e^{2\beta(\xi - t)} x^T(\xi) R_i x(\xi) d\xi \, ds \leq \int_{-h(t)}^t \int_{t+h(t)}^t e^{2\beta(\xi - t)} x^T(\xi) R_i x(\xi) d\xi \, ds
\]
we have
\[
-\int_{t-h(t)}^t e^{2\beta(s-t)} x^T(s) R_i x(s) \, ds \leq -\frac{1}{h_M} \int_{-h(t)}^t \int_{t+h(t)}^t e^{2\beta(s-t)} x^T(s) R_i x(s) d\xi \, ds
\]
\[
= -\frac{1}{h_M} V_{3,i}(x_t).
\]

Similarly, we have
\[
-\int_{t-h(t)}^t e^{2\beta(s-t)} \left[ \begin{array}{c} x(s) \\ x(s-h(s)) \end{array} \right] ^T \left[ \begin{array}{cc} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{array} \right] \left[ \begin{array}{c} x(s) \\ x(s-h(s)) \end{array} \right] ds \leq -\frac{1}{h_M} V_{4,i}(x_t),
\]
and
\[
-\frac{1}{2} \int_{t-h(t)}^t \dot{x}(s) T_x \dot{x}(s) ds \leq -\frac{1}{2h_M} V_{5,i}(x_t).
\]

From (12), (13), (14) and (15), we obtain
\[
V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right] ^T \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right] -2\beta V_{2,i}(x_t)
\]
\[
-2\beta + \frac{1}{h_M} (V_{3,i}(x_t) + V_{4,i}(x_t)) \leq 2 \frac{1}{h_M} V_{5,i}(x_t)
\]
\[
\frac{1}{2} \int_{t-h(t)}^t x(s) T_x x(s) ds.
\]

For \( i \in S_u \), we have
\[
V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right] ^T \left[ \begin{array}{c} x(t) \\ x(t-h(t)) \end{array} \right].
\]

By (5), (16) and Lemma 2.2, there exists \( \xi_i > 0 \) such that
\[
V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \parallel V_i(x_t) \parallel \phi^i(t-h), \; t \geq t_0.
\]
where $\xi_i = \frac{2\max\{\lambda_M(\Omega_{1,i})\}}{\min\{\lambda_m(P_i)\}}$.

For $i \in S_s$, we have that when $X_i = \begin{bmatrix} X_{11,i} & X_{12,i} \\ X_{21,i} & X_{22,i} \end{bmatrix} \geq 0$, the following holds:

$$h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} X_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - \int_{t-h(t)}^{t} e^{2\beta(t-s)} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} X_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} ds \geq 0. \quad (18)$$

Using the Newton-Leibniz formula, (Wu et al., 2004), we can write

$$x(t-h(t)) = x(t) - \int_{t-h(t)}^{t} \dot{x}(s)ds.$$  

Then, for any appropriate dimension matrices $Y_i$ and $Z_i$, we have

$$2[x^T(t)Y_i + x^T(t-h(t))Z_i][x(t) - \int_{t-h(t)}^{t} \dot{x}(s)ds - x(t-h(t))] = 0. \quad (19)$$

It follows that

$$2x^T(t)Y_i x(t) - 2x^T(t)Y_i \int_{t-h(t)}^{t} \dot{x}(s)ds - 2x^T(t)Y_i x(t-h(t)) + 2x^T(t-h(t))Z_i x(t)$$

$$- 2x^T(t-h(t))Z_i \int_{t-h(t)}^{t} \dot{x}(s)ds + 2x^T(t-h(t))Z_i x(t-h(t)) = 0. \quad (19)$$

From (16) with (18) and (19), we have

$$V_i(x_i) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Omega_{2,i} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_i)$$

$$- (2\beta + \frac{1}{h_M})(V_{3,i}(x_i) + V_{4,i}(x_i)) - \frac{1}{2h_M} V_5(x_i)$$

$$- \int_{t-h(t)}^{t} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} ds. \quad (20)$$

By (6), (20) and Lemma 2.3, there exist $\zeta_i > 0$ such that

$$V_i(x_i) \leq \sum_{i=1}^{N} \lambda_i(t) \| V_i(x_i) \| e^{-\zeta_i(t-t_0)}, \quad t \geq t_0. \quad (21)$$

where $\zeta_i = \min\{\frac{\min\{\lambda_m(-\Omega_{2,i})\}}{\max\{\lambda_m(P_i)\}}, 2\beta, \frac{1}{2h_M}\}$.

Let $N(t)$ denotes the number of times the system is switched on $[t_0, t)$ such that $\lim_{t \to +\infty} N(t) = +\infty$. Suppose that $\sigma(t_0) = i_0, \sigma(t_1) = i_1, \ldots$ and $\sigma(t) = i$.  

www.intechopen.com
Let \( l(t) \) denotes the number of times the unstable subsystems are activated on \([t_0, t)\) and \( N(t) - l(t) \) denotes the number of times the stable subsystems are activated on \([t_0, t)\). Suppose that \( t_0 < t_1 < t_2 < \ldots \) and \( \lim_{n \to +\infty} t_n = +\infty \).

From (11), (17) and (21), suppose that the \( j \) th subsystem of unstable mode is activated on the interval \([t_j, t_{j+1})\),
- if the \( j \) th subsystem of unstable mode is activated on the interval \([t_{j-1}, t_j)\), then
  \[
  V_j(x_t) \leq \psi \| V_i(x_{t_{j-1}}) \| e^{\xi(t_{j-1}-t_j)}e^{\delta(t_{j-1})}, \quad t \in [t_j, t_{j+1})
  \]
- if the \( i \) th subsystem of stable mode is activated on the interval \([t_{j-1}, t_j)\), then
  \[
  V_j(x_t) \leq \psi \| V_i(x_{t_{j-1}}) \| e^{\omega(t_{j-1}-t_j)}e^{\delta(t_{j-1})}, \quad t \in [t_j, t_{j+1})
  \]
Suppose that the \( j \) th subsystem of stable mode is activated on the interval \([t_j, t_{j+1})\),
- if the \( i \) th subsystem of unstable mode is activated on the interval \([t_{j-1}, t_j)\), then
  \[
  V_j(x_t) \leq \psi \| V_i(x_{t_{j-1}}) \| e^{\xi(t_{j-1}-t_j)}e^{\delta(t_{j-1})}, \quad t \in [t_j, t_{j+1})
  \]
- if the \( i \) th subsystem of stable mode is activated on the interval \([t_{j-1}, t_j)\), then
  \[
  V_j(x_t) \leq \psi \| V_i(x_{t_{j-1}}) \| e^{-\omega(t_{j-1}-t_j)}e^{-\delta(t_{j-1})}, \quad t \in [t_j, t_{j+1})
  \]

In general, we get
\[
V_j(x_t) \leq \prod_{m=1}^{l(t)} \psi e^{\xi_0(t_{n-1}-t)} \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\xi_0 M_n} e^{-\xi(t_{n-1})} \times \| V_i(x_{t_0}) \| e^{\xi(t_{n-1}-t)}
\]
\[
\leq \prod_{m=1}^{l(t)} \psi e^{\lambda(t_{n-1}-t)} \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\xi_0 M_n} e^{-\lambda(t_{n-1})} \times \| V_i(x_{t_0}) \| e^{-\lambda(t_{n-1}-t)}
\]
\[
t \geq t_0. \quad \text{Using (7), we have}
\]
\[
V_j(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\xi_0 M_n} \| V_i(x_{t_0}) \| e^{-\lambda(t_{n-1})}, \quad t \geq t_0.
\]
By (8) and (9), we get
\[
V_j(x_t) \leq \| V_i(x_{t_0}) \| e^{-(\lambda t-(t_{n-1})}, \quad t \geq t_0.
\]
Thus, by (10), we have
\[
\| x(t) \| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \| x_{t_0} \| e^{-(\lambda t-(t_{n-1})}, \quad t \geq t_0,
\]
which concludes the proof of the Theorem 3.1.

3.2 Robust exponential stability of linear switched system with time-varying delay

In this section, we give conditions for robust exponential stability of the zero solution of system (1) without nonlinear perturbation, namely \( f_i(t, x(t), x(t-h(t))) = 0 \). The following is the main result.

Theorem 3.2 The zero solution of system (1) with \( f_i(t, x(t), x(t-h(t))) = 0 \) is robustly exponentially stable if there exist positive real numbers \( \varepsilon_1, \varepsilon_2, \) positive definite matrices \( P_i, Q_i, R_i \) and
\[
\begin{bmatrix}
S_{11,i} & S_{12,i} \\
S_{21,i} & S_{22,i}
\end{bmatrix}
\]
such that the following conditions hold:
A1. (i) For \( i \in S_u \),
\[
\Xi_i > 0.
\]
(ii) For $i \in S_s$, $\Xi_i < 0$. 

A2. Assume that, for any $t_0$ the switching law guarantees that

$$
\inf_{t \geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} > \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}
$$

(24)

where $\lambda^* \in (0, \lambda^-)$. Furthermore, there exists $0 < \nu < A^*$ such that

(i) If the subsystem $i \in S_s$ is activated in time intervals $[t_{ik-1}, t_{ik})$, $k = 1, 2, ...,$
then

$$
\ln \psi - \nu(t_{ik} - t_{ik-1}) \leq 0, \ k = 1, 2, ..., l(t).
$$

(25)

(ii) If the subsystem $j \in S_s$ is activated in time intervals $[t_{jk-1}, t_{jk})$, $k = 1, 2, ...$
then

$$
\ln \psi + \xi_j h_M - \nu(t_{jk} - t_{jk-1}) \leq 0, \ k = 1, 2, ..., N(t) - 1.
$$

(26)

Proof. Consider the following Lyapunov functional:

$$
V_i(x_t) = V_{1j}(x(t)) + V_{2j}(x_t) + V_{3j}(x_t) + V_{4j}(x_t)
$$

where $x_t \in C([-h_M, 0], \mathbb{R}^n)$, $x_j(s) = x(t + s), s \in [-h_M, 0]$, and $V_{1j}(x(t)) = x^T(t)P_{x_1}x(t)$,

$$
V_{2j}(x_t) = \int_{-h(t)}^{0} \int_{t+s}^{t} \alpha_1 \| x(s) \|^2 \, ds,
$$

$$
V_{3j}(x_t) = \int_{-h(t)}^{0} \int_{t+s}^{t} \beta \| x(t) \|^2 \, ds,
$$

$$
V_{4j}(x_t) = \int_{-h(t)}^{0} \int_{t+s}^{t} \gamma \| x(t) \|^2 \, ds.
$$

It is easy to verify that

$$
\alpha_1 \| x(t) \|^2 \leq V_i(x_t) \leq \alpha_3 \| x(t) \|^2, \ t \geq 0.
$$

(27)

Similar to (11), we have

$$
V_i(x_t) \leq \psi V_i(x_t), \ \forall i, j \in S.
$$

(28)

Taking derivative of $V_{1j}(x(t))$ along trajectories of any subsystem $i$th, we have

$$
\dot{V}_{1j}(x(t)) = x^T(t)P_{x_1}x(t) + x^T(t)P_{x_1}x(t) + x^T(t)A_{1j}^T(t)P_{x_1}x(t) + x^T(t)(A_{1j}^T(t)P_{x_1}x(t) + x^T(t)P_{x_1}x(t))
$$

$$
+ x^T(t)(-h(t))B_{1j}^T(t)P_{x_1}x(t) + x^T(t)P_{x_1}x(t) + x^T(t)P_{x_1}x(t) + x^T(t)P_{x_1}x(t) + x^T(t)P_{x_1}x(t) + x^T(t)P_{x_1}x(t).
$$

Applying Lemma 2.1 and from (2) and (3), we get

$$
2x^T(t)A_{1j}^T(t)P_{x_1}x(t) \leq \epsilon_{1j}^{-1}x^T(t)H_{1j}^T(t)H_{1j}x(t) + \epsilon_{1j}x^T(t)P_{x_1}E_{1j}^T(t)P_{x_1}x(t),
$$

$$
2x^T(t)(-h(t))B_{1j}^T(t)P_{x_1}x(t) \leq \epsilon_{2j}^{-1}x^T(t)H_{2j}^T(t)H_{2j}x(t) + \epsilon_{2j}x^T(t)P_{x_1}E_{2j}^T(t)P_{x_1}x(t).
$$

(24)
Next, by taking derivative of \( V_{2,i}(x_t) \), \( V_{3,i}(x_t) \) and \( V_{4,i}(x_t) \), respectively, along the system trajectories yields

\[
V_{2,i}(x_t) \leq x^T(t)Q_i x(t) - (1 - \mu)e^{-2\beta h(t)}x^T(t - h(t))Q_i x(t - h(t)) - 2\beta V_{2,i}(x_t),
\]

\[
V_{3,i}(x_t) \leq h_M x^T(t)R_i x(t) - \int_{t-h(t)}^t e^{2\beta (s-t)}x^T(s)R_i x(s)ds - 2\beta V_{3,i}(x_t),
\]

\[
V_{4,i}(x_t) \leq h_M \left[ x(t) \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ x(s-h(s)) \end{bmatrix} \right] - 2\beta V_{4,i}(x_t).
\]

Therefore, the estimation of derivative of \( V_i(x_t) \) along any trajectory of solution of (1) is given by

\[
V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \left[ \frac{x(t)}{x(t-h(t))} \right]^T E_i \left[ \frac{x(t)}{x(t-h(t))} \right] - 2\beta V_{2,i}(x_t)
\]

\[
- \int_{t-h(t)}^t e^{2\beta (s-t)}x^T(s)R_i x(s)ds - 2\beta V_{3,i}(x_t)
\]

\[
- \int_{t-h(t)}^t e^{2\beta (s-t)} \left[ \frac{x(s)}{x(s-h(s))} \right]^T \left[ S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \right] \left[ \frac{x(s)}{x(s-h(s))} \right] ds
\]

\[-2\beta V_{4,i}(x_t).
\]

For \( i \in S_u \), we have

\[
V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \left[ \frac{x(t)}{x(t-h(t))} \right]^T E_i \left[ \frac{x(t)}{x(t-h(t))} \right].
\]

Similar to Theorem 3.1, from (22) and (29), we get

\[
V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \| V_i(x_{t_0}) \| e^{\xi_i(t-t_0)}, \ t \geq t_0,
\]

where \( \xi_i = \frac{2 \max \{ \lambda_M(E_i) \}}{\min \{ \lambda_m(P_i) \}} \).

For \( i \in S_s \), from (13), (14) and (29), we have

\[
V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \left[ \frac{x(t)}{x(t-h(t))} \right]^T E_i \left[ \frac{x(t)}{x(t-h(t))} \right] - 2\beta V_{2,i}(x_t)
\]

\[-(2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t))
\]

\[
(31)
\]
Similar to Theorem 3.1, from (23) and (31), we get
\[ V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \| V_i(x_{t_0}) \| e^{-\zeta_i(t-t_0)}, \quad t \geq t_0. \] (32)

where \( \zeta_i = \min \{ \lambda_i(-z_i) \} \).

In general, from (28), (30) and (32), with the same argument as in the proof of Theorem 3.1, we get
\[ V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi e^{\lambda^+(t_n-t_{n-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{h_n} e^{\lambda^-(t_n-t_{n-1})} \times \| V_0(x_{t_0}) \| e^{-\lambda^-(t-l(t)-1)}, \]
\[ t \geq t_0. \]

Using (24), we have
\[ V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{h_n} \times \| V_0(x_{t_0}) \| e^{-\lambda^-(t-t_0)}, \quad t \geq t_0. \]

By (25) and (26), we get
\[ V_i(x_t) \leq \| V_0(x_{t_0}) \| e^{-(\lambda^+-\nu)(t-t_0)}, \quad t \geq t_0. \]

Thus, by (27), we have
\[ \| x(t) \| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \| x_{t_0} \| e^{-\frac{1}{2}(\lambda^+-\nu)(t-t_0)}, \quad t \geq t_0, \]
which concludes the proof of the Theorem 3.2. \( \square \)

### 3.3 Robust exponential stability of switched system with time-varying delay and nonlinear perturbation

In this section, we deal with the problem for robust exponential stability of the zero solution of system (1).

**Theorem 3.3** The zero solution of system (1) is robust exponentially stable if there exist positive real numbers \( \epsilon_{3i}, \epsilon_{4i}, \epsilon_{5i} \), positive definite matrices \( P_i, Q_i, R_i \) and \( S_{1i}, S_{12i}, S_{22i} \) such that the following conditions hold:

1. (i) For \( i \in S_u \),
   \[ \Theta_i > 0. \] (33)
2. (ii) For \( i \in S_s \),
   \[ \Theta_i < 0. \] (34)

A2. Assume that, for any \( t_0 \) the switching law guarantees that
\[ \inf_{t \geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \] (35)

www.intechopen.com
where $\lambda^* \in (0, \lambda^-)$. Furthermore, there exists $0 < \nu < \lambda^*$ such that

(i) If the subsystem $i \in S_\nu$ is activated in time intervals $[t_{ik-1}, t_{ik})$, $k = 1, 2, ..., \ell(t)$,

\[ \ln \psi - \nu(t_{ik} - t_{ik-1}) \leq 0, \quad k = 1, 2, ..., \ell(t). \]  

(36)

(ii) If the subsystem $j \notin S_\nu$ is activated in time intervals $[t_{ik-1}, t_{ik})$, $k = 1, 2, ..., \ell(t)$,

\[ \ln \psi + \xi_t h_M - \nu(t_{ik} - t_{ik-1}) \leq 0, \quad k = 1, 2, ..., N(t) - 1. \]  

(37)

**Proof.** Consider the following Lyapunov functional:

\[ V_i(x_t) = V_{1,i}(x(t)) + V_{2,i}(x_t) + V_{3,i}(x_t) + V_{4,i}(x_t) \]

where $x_t \in C([-h_M, 0], \mathbb{R}^n)$, $x_t(s) = x(t + s), s \in [-h_M, 0]$ and $V_{1,i}(x(t)) = x^T(t)P_ix(t)$,

\[ V_{2,i}(x_t) = \int_{t-h(t)}^{t} \int_{t+s}^{t} 2\beta(t-s) x^T(s)Q(x(s))ds, \]

\[ V_{3,i}(x_t) = \int_{t-h(t)}^{t} \int_{t+s}^{t} 2\beta(t-s) x^T(s)R(x(s))d\xi ds, \]

\[ V_{4,i}(x_t) = \int_{t-h(t)}^{t} \int_{t+s}^{t} 2\beta(t-s) \left[ x(\xi) \right]^T \left[ \begin{array}{c} S_{11,i} \, S_{12,i} \\ S_{12,i} \, S_{22,i} \end{array} \right] \left[ \begin{array}{c} x(\xi - h(\xi)) \\ x(\xi - h(\xi)) \end{array} \right] d\xi ds. \]

It is easy to verify that

\[ \alpha_1 \| x(t) \|^2 \leq V_i(x_t) \leq \alpha_3 \| x_t \|^2, \quad t \geq 0. \]  

(38)

Similar to (11), we have

\[ V_i(x_t) \leq \psi V_j(x_t), \quad \forall i, j \in S. \]  

(39)

Taking derivative of $V_{1,i}(x(t))$ along trajectories of any subsystem $i$th we have

\[ V_{1,i}(x(t)) = x^T(t)P_ix(t) + x^T(t)P_ix(t) \]

\[ = \sum_{i=1}^{N} \lambda_i(t) x_i^T(t)A_i^T(t)P_ix(t) + x^T(t)A_i^T(t)P_ix(t) + x^T(t - h(t))B_i^T(t)P_ix(t) + x^T(t - h(t))B_i^T(t)P_ix(t) + \]

\[ + x^T(t - h(t))B_i^T(t)P_ix(t) + x^T(t)P_i\Delta A_i(t)x(t) + x^T(t)P_i\Delta B_i(t)x(t - h(t)) + x^T(t)P_i\Delta A_i(t)x(t) + x^T(t)P_i\Delta B_i(t)x(t - h(t)) + \]

\[ + x^T(t)P_if_i(t, x(t), x(t - h(t))). \]

From lemma 2.1, we have

\[ 2f_i^T(t, x(t), x(t - h(t)))P_ix(t) \leq f_i^T(t, x(t), x(t - h(t)))W_i^{-1}f_i(t, x(t), x(t - h(t))) \]

\[ + x^T(t)P_iW_iP_ix(t). \]

By choosing $W_i = \varepsilon_{3,i}I_i$ and from (4), we have

\[ 2f_i^T(t, x(t), x(t - h(t)))P_ix(t) \leq \varepsilon_{3,i}^{-1}f_i^T(t, x(t), x(t - h(t)))f_i(t, x(t), x(t - h(t))) \]

\[ + \varepsilon_{3,i}x^T(t)P_iP_ix(t) \]

\[ \leq \varepsilon_{3,i}^{-1}[\gamma x^T(t)x(t) + \delta x^T(t - h(t))x(t - h(t))] \]

\[ + \varepsilon_{3,i}x^T(t)P_iP_ix(t). \]
Applying Lemma 2.1 and from (2) and (3), we get
\[2x^T(t) \Delta A^T_i(t) P_i(t) x(t) \leq \varepsilon_i \| x(t) \|^2 \| H_i \|^2 + \varepsilon_i x^T(t) P_i E_i^T E_i P_i x(t),\]
\[2x^T(t - h(t)) \Delta B^T_i(t) P_i(t) x(t) \leq \varepsilon_i \| x(t) \|^2 \| H_i \|^2 + \varepsilon_i x^T(t) P_i E_i^T E_i P_i x(t).\]

Next, by taking derivative of \( V_{2,1}(x_i), V_{3,1}(x_i) \) and \( V_{4,1}(x_i) \), respectively, along the system trajectories yields
\[V_{2,1}(x_i) \leq x^T(t) Q_i x(t) - (1 - \mu) e^{-2\beta h(t)} x^T(t - h(t)) Q_i x(t - h(t)) - 2\beta V_{2,1}(x_i),\]
\[V_{3,1}(x_i) \leq h_M x^T(t) R_i x(t) - \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) R_i x(s) ds - 2\beta V_{3,1}(x_i),\]
\[V_{4,1}(x_i) \leq \dot{h}_M \left[ \begin{array}{ll} x(t) \\ x(t - h(t)) \end{array} \right]^T \left[ \begin{array}{cc} \Sigma_{11,1} & \Sigma_{12,1} \\ \Sigma_{21,1} & \Sigma_{22,1} \end{array} \right] \left[ \begin{array}{ll} x(t) \\ x(t - h(t)) \end{array} \right] \]
\[- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \left[ \begin{array}{ll} x(s) \\ x(s - h(s)) \end{array} \right]^T \left[ \begin{array}{cc} \Sigma_{11,1} & \Sigma_{12,1} \\ \Sigma_{21,1} & \Sigma_{22,1} \end{array} \right] \left[ \begin{array}{ll} x(s) \\ x(s - h(s)) \end{array} \right] ds \]
\[-2\beta V_{4,1}(x_i).\]

Then, the derivative of \( V_i(x_i) \) along any trajectory of solution of (1) is estimated by
\[\dot{V}_i(x_i) \leq \sum_{i=1}^{N} \lambda_i(t) \left[ \begin{array}{ll} x(t) \\ x(t - h(t)) \end{array} \right]^T \Theta_i \left[ \begin{array}{ll} x(t) \\ x(t - h(t)) \end{array} \right] - 2\beta V_{2,1}(x_i)\]
\[- \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) R_i x(s) ds - 2\beta V_{3,1}(x_i)\]
\[- \int_{t-h(t)}^{t} e^{2\beta(s-t)} \left[ \begin{array}{ll} x(s) \\ x(s - h(s)) \end{array} \right]^T \left[ \begin{array}{cc} \Sigma_{11,1} & \Sigma_{12,1} \\ \Sigma_{21,1} & \Sigma_{22,1} \end{array} \right] \left[ \begin{array}{ll} x(s) \\ x(s - h(s)) \end{array} \right] ds \]
\[-2\beta V_{4,1}(x_i).\]

For \( i \in S_{ur} \) it follows from (40) that
\[\dot{V}_i(x_i) \leq \sum_{i=1}^{N} \lambda_i(t) \left[ \begin{array}{ll} x(t) \\ x(t - h(t)) \end{array} \right]^T \Theta_i \left[ \begin{array}{ll} x(t) \\ x(t - h(t)) \end{array} \right],\]
\[\dot{V}_i(x_i) \leq \sum_{i=1}^{N} \lambda_i(t) \| V_i(x_i) \| \| \Phi_i(t - t_0) \|, t \geq t_0.\]

Similar to Theorem 3.1, from (33) and (41), we get
\[\dot{V}_i(x_i) \leq \sum_{i=1}^{N} \lambda_i(t) \| V_i(x_i) \| \| \Phi_i(t - t_0) \|, t \geq t_0,\]
where \( \xi_i = \frac{2 \max \{ \lambda_M(\Theta_i) \}}{\min \{ \lambda_m(P_i) \}}. \)
For $i \in S$, from (13), (14) and (40), we have
\[
\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \left[ x(t) \right] \left[ x(t - h(t)) \right]^T \Theta_i \left[ x(t) \right] \left[ x(t - h(t)) \right] - 2\beta V_{2,i}(x_t)
\]
\[\quad - (2\beta + \frac{1}{r_M})(V_{3,i}(x_t) + V_{4,i}(x_t)). \tag{43}\]

Similar to Theorem 3.1, from (34) and (43), we get
\[
V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \| x_{i}(x_0) \| e^{-\zeta(t-t_0)}, t \geq t_0. \tag{44}\]

where $\zeta_i = \min_i \left\{ \frac{\lambda_i(-\Theta_i)}{\max_i \{\lambda_M(P_i)\}} \right\}$.

In general, from (39), (42) and (44), with the same argument as in the proof of Theorem 3.1, we get
\[
V_i(x_t) \leq \prod_{n=1}^{l(t)} \psi e^{\lambda^*(-t_n-t_n-1)} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\lambda^*(t_n-t_n-1)} \times \| x_{i}(x_0) \| e^{-\lambda^*(t-t_0)}, t \geq t_0.
\]

By (36) and (37), we get
\[
V_i(x_t) \leq \| x_{i}(x_0) \| e^{-(\lambda^*-\nu)(t-t_0)}, t \geq t_0.
\]

Thus, by (38), we have
\[
\| x(t) \| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \| x_{i}(x_0) \| e^{-\frac{1}{2}(\lambda^*-\nu)(t-t_0)}, t \geq t_0
\]
which concludes the proof of the Theorem 3.3. \qed

4. Numerical examples

Example 4.1 Consider linear switched system (1) with time-varying delay but without matrix uncertainties and without nonlinear perturbations. Let $N = 2$, $S_a = \{1\}$, $S_b = \{2\}$. Let the delay function be $h(t) = 0.51 \sin^2 t$. We have $h_{\max} = 0.51$, $\mu = 1.02, \lambda(A_1 + B_1) = 0.0046, -0.0399, \lambda(A_2) = -0.2156, 0.0007$. Let $\beta = 0.5$.

Since one of the eigenvalues of $A_1 + B_1$ is negative and one of eigenvalues of $A_2$ is positive, we can’t use results in (Alan & Lib, 2008) to consider stability of switched system (1). By using the LMI toolbox in Matlab, we have matrix solutions of (5) for unstable subsystems and (6) for stable subsystems as the following:

For unstable subsystems, we get
\[ P_1 = \begin{bmatrix} 41.6819 & 0.0001 \\ 0.0001 & 41.5691 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 24.7813 & -0.0002 \\ -0.0002 & 24.7848 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 33.1027 & -0.0001 \\ -0.0001 & 33.1044 \end{bmatrix}, \]
\[ S_{11,1} = \begin{bmatrix} 33.1027 & -0.0001 \\ -0.0001 & 33.1044 \end{bmatrix}, \quad S_{12,1} = \begin{bmatrix} -0.0372 & -0.0023 \\ -0.0023 & 0.7075 \end{bmatrix}, \quad S_{22,1} = \begin{bmatrix} 50.0115 \end{bmatrix}. \]

For stable subsystems, we get

\[ P_2 = \begin{bmatrix} 71.8776 & 2.3932 \\ 2.3932 & 110.8889 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 7.2590 & -0.3265 \\ -0.3265 & 0.8745 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 10.4001 & -0.4667 \\ -0.4667 & 1.2806 \end{bmatrix}, \]
\[ S_{11,2} = \begin{bmatrix} 12.7990 & -0.4854 \\ -0.4854 & 3.5031 \end{bmatrix}, \quad S_{12,2} = \begin{bmatrix} -3.1787 & 0.0240 \\ 0.0240 & -2.8307 \end{bmatrix}, \quad S_{22,2} = \begin{bmatrix} 4.6346 & -0.0289 \\ -0.0289 & 4.0835 \end{bmatrix}. \]

By straightforward calculation, the growth rate is \( \lambda^+ = \xi = 2.8291 \), the decay rate is \( \lambda^- = \zeta = 0.0063 \), \( \lambda(\Omega_{1,1}) = 25.8187, 25.8188, 58.7463, 58.8011, \lambda(\Omega_{2,2}) = -10.1108, -3.7678, -2.0403, -0.7032 \) and \( \lambda(\Omega_{3,2}) = 1.4217, 4.2448, 5.4006, 9.1514, 29.3526, 30.0607 \). Thus, we may take \( \lambda^+ = 0.0001 \) and \( \nu = 0.00001 \). Thus, from inequality (7), we have \( T^- \geq 456.3226 \). By choosing \( T^+ = 0.1 \), we get \( T^- \geq 45.63226 \). We choose the following switching rules:

(i) for \( t \in [0,0.1) \cup [50,50.1) \cup [100,100.1) \cup [150,150.1) \cup \ldots \), subsystem \( i = 1 \) is activated.

(ii) for \( t \in [0.1,50] \cup [50.1,100) \cup [100.1,150) \cup [150.1,200) \cup \ldots \), subsystem \( i = 2 \) is activated.

Then, by Theorem 3.1, the switching system (1) is exponentially stable. Moreover, the solution \( x(t) \) of the system satisfies

\[ \| x(t) \| \leq 11.8915e^{-0.00045t}, \quad t \in [0,\infty). \]

The trajectories of solution of switched system switching between the subsystems \( i = 1 \) and \( i = 2 \) are shown in Figure 1, Figure 2 and Figure 3, respectively.

Fig. 1. The trajectories of solution of linear switched system.

www.intechopen.com
Example 4.2 Consider uncertain switched system (1) with time-varying delay and nonlinear perturbation. Let $N = 2$, $S_i = \{1\}, S_i = \{2\}$ where

$$
A_1 = \begin{bmatrix}
0.1130 & 0.00013 \\
0.00015 & -0.0033 \\
\end{bmatrix},
B_1 = \begin{bmatrix}
0.0002 & 0.0012 \\
0.0014 & -0.5002 \\
\end{bmatrix},
$$

$$
A_2 = \begin{bmatrix}
-5.5200 & 1.0002 \\
1.0003 & -6.5500 \\
\end{bmatrix},
B_2 = \begin{bmatrix}
0.0245 & 0.0001 \\
0.0001 & 0.0237 \\
\end{bmatrix},
$$

$$
E_{1i} = E_{2i} = \begin{bmatrix}
0.2000 & 0.0000 \\
0.0000 & 0.2000 \\
\end{bmatrix},
H_{1i} = H_{2i} = \begin{bmatrix}
0.1000 & 0.0000 \\
0.0000 & 0.1000 \\
\end{bmatrix}, i = 1, 2,
$$

$$
F_{1i} = F_{2i} = \begin{bmatrix}
\sin t & 0 \\
0 & \sin t \\
\end{bmatrix}, i = 1, 2,
$$
Exponential Stability of Uncertain Switched System with Time-Varying Delay

\[ f_1(t, x(t), x(t-h(t))) = \begin{bmatrix} 0.1x_1(t) \sin(x_1(t)) \\ 0.1x_2(t-h(t)) \cos(x_2(t)) \end{bmatrix}, \]
\[ f_2(t, x(t), x(t-h(t))) = \begin{bmatrix} 0.5x_1(t) \sin(x_1(t)) \\ 0.5x_2(t-h(t)) \cos(x_2(t)) \end{bmatrix}. \]

From
\[ \| f_1(t, x(t), x(t-h(t))) \|^2 = |0.1x_1(t) \sin(x_1(t))|^2 + |0.1x_2(t-h(t)) \cos(x_2(t))|^2 \leq 0.01x_1^2(t) + 0.01x_2^2(t-h(t)) \leq 0.01 \| x(t) \|^2 + 0.01 \| x(t-h(t)) \|^2, \]

we obtain
\[ \| f_1(t, x(t), x(t-h(t))) \| \leq 0.1 \| x(t) \| + 0.1 \| x(t-h(t)) \|. \]

The delay function is chosen as \( h(t) = 0.25 \sin^2 t \). From
\[ \| f_2(t, x(t), x(t-h(t))) \|^2 = |0.5x_1(t) \sin(x_1(t))|^2 + |0.5x_2(t-h(t)) \cos(x_2(t))|^2 \leq 0.25x_1^2(t) + 0.25x_2^2(t-h(t)) \leq 0.25 \| x(t) \|^2 + 0.25 \| x(t-h(t)) \|^2, \]

we obtain
\[ \| f_2(t, x(t), x(t-h(t))) \| \leq 0.5 \| x(t) \| + 0.5 \| x(t-h(t)) \|. \]

We may take \( h_M = 0.25 \), and from (4), we take \( \gamma_1 = 0.1, \delta_1 = 0.1, \gamma_2 = 0.5, \delta_2 = 0.5 \). Note that \( \lambda(A_1) = 0.11300016, -0.00330016 \). Let \( \beta = 0.5, \mu = 0.5 \). Since one of the eigenvalues of \( A_1 \) is negative, we can’t use results in (Alan & Lib, 2008) to consider stability of switched system (1). From Lemma 2.4, we have the matrix solutions of (33) for unstable subsystems and of (34) for stable subsystems by using the LMI toolbox in Matlab as the following:

For unstable subsystems, we get
\[ \epsilon_{31} = 0.8901, \epsilon_{41} = 0.8901, \epsilon_{51} = 0.8901, \]
\[ P_1 = \begin{bmatrix} 0.2745 & -0.0000 \\ -0.0000 & 0.2818 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.4818 & -0.0000 \\ -0.0000 & 0.5097 \end{bmatrix}, R_1 = \begin{bmatrix} 0.8649 & -0.0000 \\ -0.0000 & 0.8729 \end{bmatrix}, \]
\[ S_{11,1} = \begin{bmatrix} 0.8649 & -0.0000 \\ -0.0000 & 0.8729 \end{bmatrix}, S_{12,1} = 10^{-4} \times \begin{bmatrix} -0.1291 & -0.8517 \\ -0.8517 & 0.1326 \end{bmatrix}, \]
\[ S_{22,1} = \begin{bmatrix} 1.0877 & -0.0000 \\ -0.0000 & 1.9092 \end{bmatrix}. \]

For stable subsystems, we get
\[ \epsilon_{32} = 2.0180, \epsilon_{42} = 2.0180, \epsilon_{52} = 2.0180, \]
\[ P_2 = \begin{bmatrix} 0.2741 & 0.0407 \\ 0.0407 & 0.2323 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.3330 & -0.0069 \\ -0.0069 & 1.3330 \end{bmatrix}, R_2 = \begin{bmatrix} 1.0210 & -0.0002 \\ -0.0002 & 1.0210 \end{bmatrix}, \]
\[ S_{11,2} = \begin{bmatrix} 1.0210 & -0.0002 \\ -0.0002 & 1.0210 \end{bmatrix}, S_{12,2} = \begin{bmatrix} -0.0016 & -0.0002 \\ -0.0002 & -0.0016 \end{bmatrix}, \]
\[ S_{22,2} = \begin{bmatrix} 0.8236 & -0.0006 \\ -0.0006 & 0.8236 \end{bmatrix}. \]

By straightforward calculation, the growth rate is \( \lambda^+ = \zeta = 8.5413 \), the decay
rate is $\dot{\lambda} = \zeta = 0.1967$, $\lambda(\Theta_1) = 0.1976, 0.2079, 1.1443, 1.1723$ and $\lambda(\Theta_2) = -0.7682, -0.6494, -0.0646, -0.0588$. Thus, we may take $\lambda^* = 0.0001$ and $\nu = 0.00001$.

Thus, from inequality (35), we have $T^- \geq 43.4456 T^+$. By choosing $T^+ = 0.1$, we get $T^- \geq 4.34456$. We choose the following switching rules:

(i) for $t \in [0, 0.1) \cup [5.0, 5.1) \cup [10.0, 10.1) \cup [15.0, 15.1) \cup \ldots$, system $i = 1$ is activated.

(ii) for $t \in [0.1, 5.0) \cup [5.1, 10.0) \cup [10.1, 15.0) \cup [15.1, 20.0) \cup \ldots$, system $i = 2$ is activated.

Then, by theorem 3.3.1, the switched system (1) is exponentially stable. Moreover, the solution $x(t)$ of the system satisfies

$$\|x(t)\| \leq 1.8770 e^{-0.000045 t}, t \in [0, \infty).$$

The trajectories of solution of switched system switching between the subsystems $i = 1$ and $i = 2$ are shown in Figure 4, Figure 5 and Figure 6, respectively.

5. Conclusion

In this paper, we have studied the exponential stability of uncertain switched system with time varying delay and nonlinear perturbations. We allow switched system to contain stable and unstable subsystems. By using a new Lyapunov functional, we obtain the conditions for robust exponential stability for switched system in terms of linear matrix inequalities (LMIs) which may be solved by various algorithms. Numerical examples are given to illustrate the effectiveness of our theoretical results.

6. Acknowledgments

This work is supported by Center of Excellence in Mathematics and the Commission on Higher Education, Thailand.
We also wish to thank the National Research University Project under Thailand’s Office of the Higher Education Commission for financial support.
Fig. 5. The trajectories of solution of system $i = 1$

Fig. 6. The trajectories of solution of system $i = 2$

7. References


Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, robotics, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of controllability, observability, robustness, optimization, adaptive control, pole placement and particularly stability and robustness stabilization for this class of systems, has been one of the main interests for many scientists and researchers during the last five decades.

How to reference
In order to correctly reference this scholarly work, feel free to copy and paste the following:
