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Finite difference solutions of MFM square duct flow with heat transfer using MatLab program

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1. Introduction

Many researchers are interested in magneto-hydro-dynamics MHD since the last century due to its important applications. For example, MHD steam plants and MHD generators are used in the modern power plants. The basic concept of the MHD generator is to generate electrical energy from the motion of conductive liquid that is crossing a perpendicular magnetic field. Carnot efficiency is improved by the presence of MHD unit. Another example is the MHD pumps and flow meters. In this type of pumps, the electrical energy is converted directly to a force which is applied on the working fluid. MHD separation in metal casting with superconducting coils is another important application.

A very useful proposed application which involves MHD is the lithium cooling blanket in a nuclear fusion reactor. The high-temperature plasma is maintained in the reactor by means of a toroidal magnetic field. The liquid-lithium circulation loops, which will be located between the plasma and magnetic windings, are called lithium blankets. The lithium performs two functions: it absorbs the thermal energy released by the reaction (and subsequently used for power generation) and it participates in nuclear reactions in which tritium is produced. The lithium blanket is thus a very important reactor component. On other hand, the blanket will be acted upon by an extremely strong magnetic field. Consequently, to calculate the flow of liquid metal in channels or pipes situated at different angles to the magnetic field, and to determine the required pressure drop, heat transfer, etc., a knowledge of the appropriate MHD relationships will be necessary.

Magnetohydrodynamics (MHD) has been studied since the 19th century, but extensive investigations in this field accelerated only at the beginning of the 20th century. The first theoretical and laboratory studies of MHD flows in pipes and ducts were carried out in the 1930s. Williams published results of experiments with electrolytes flowing in insulated tubes. The tubes were placed between the poles of a magnet, and the potential difference across the flow was measured using wires passed through the walls. Hartmann and Lazarus made some very comprehensive theoretical and experimental studies of this subject. They performed their experiments with mercury which has an electrical conductivity 100,000 times greater than that of an electrolyte. This made it possible to observe a wider range of phenomena than in the experiments by Williams. In particular, Hartmann and Lazarus were...
able to investigate the change in drag (friction) and, indirectly, the suppression of turbulence caused by magnetic field. Hartmann obtained the exact solution of the flow between two parallel, non-conducting walls with the applied magnetic field normal to the walls. Shercliff, in 1956, has solved the problem of rectangular duct, from which he noticed that for high Hartmann numbers $M$ the velocity distribution consists of a uniform core with a boundary layer near the walls. This result enabled him to solve the problem for a circular pipe in an approximate manner (a first approximation which gives rise to errors of order $M^{-1}$) for large $M$ assuming walls of zero conductivity and, subsequently, walls with small conductivity. In 1962, Gold and Lykoudis has obtained an analytical solution for the MFM flow in a circular tube with zero wall conductivity while in 1968, Gardner and Lykoudis have acquired experimentally some results for circular tube with and without heat transfer. The MFM flow is also examined numerically by Al-Khawaja et al. for the case of circular tube with heat transfer and for the case of uniform wall heat flux with and without free convection. The solution for MFM square duct flow is obtained using spectral method by Al-Khawaja and Selmi for the case of uniform wall temperature. Also, the MFM combined free-and-forced convection duct flow was considered by many researchers. Chang & Lundgren considered the effect of wall conductivity for this problem. Gold analytically solved the MHD problem in a circular pipe with zero wall conductivity. His solution was an infinite series of Bessel functions, which was approximated for large $M$ with the first few terms. For the same problem, Shercliff used the second approximation (which gives rise to errors of order $M^{-2}$) to get the solution for large $M$. Gardner used Gold's solution to evaluate the exact solution for temperature profile, which turned out to be very complex. Then, he approximated velocity profile for small to moderate $M$ with a polynomial form from which he calculated the Nusselt number $Nu$. For large $M$, he used Gold's approximation to determine $Nu$. Gardner and Lykoudis experimentally studied MFM turbulent pipe flow in a transverse magnetic field with and without heat transfer. Gardner and Lo tried to solve the problem of a circular pipe flow with combined forced-and-free convection analytically using a perturbation technique in which the solutions were generated in inverse powers of the Lykoudis number, $Ly$. They obtained only the distribution of stream function and azimuthal velocity for some small Hartmann numbers $M$. Weiss studied a nonlinear two-dimensional magnetoconvection flow in a Boussinesq fluid with a series of numerical experiments. Tabeling and Chabrierie analyzed the secondary laminar flows in annular ducts of rectangular cross-section subjected to a constant axial magnetic field. They considered the cases for high $M$ and treated the equations of flow by a perturbation method involving an infinite series expansion. In addition, some researchers investigated the case of non-uniform magnetic field. Petrykowski and Walker examined the liquid-metal flows in rectangular ducts having electrically insulating top and bottom walls and perfectly conducting sides and in the presence of strong, polar, non-uniform, transverse magnetic field. They presented solutions for the boundary layers adjacent to the sides that are parallel to the magnetic field. Singh and Lal have calculated numerically the temperature distribution for steady MHD axial flow through a rectangular pipe with discontinuity in wall temperature. Mittal, Nataraja and Naidu obtained a numerical solution of the equations governing the flow of an electrically conducting, viscous, compressible gas with variable fluid properties in the presence of a uniform magnetic field. They analyzed the velocity and temperature distributions for subsonic and supersonic flows as these occur in the duct of an MHD generator. Setayesh
and Sahai studied numerically the effect of temperature-dependent transport properties on the developing magnetohydrodynamic flow and heat transfer in a parallel-plate channel whose walls are held at constant and equal temperatures.

In addition, the problem of the combined free-and-forced convection in horizontal tubes in the absence of magnetic field was investigated considerably in the 1960s. Morton solved the problem of laminar convection in uniformly heated horizontal pipes at low Rayleigh numbers Ra using a perturbation method to obtain a formula for Nusselt number Nu which is valid only for $\text{Re}Ra = 3,000$. Here, Re and Ra are Reynolds and Rayleigh numbers based on diameter, respectively. Mori, Futagami, Tokuda and Nakamura analyzed the same problem experimentally for air but for high Ra, and they noticed that Nusselt numbers would be about twice as large as those calculated by neglecting the effect of the secondary flow caused by buoyancy at $\text{Re}Ra = 4 \times 10^5$. They concluded that buoyancy has little effect on the velocity and temperature fields in turbulent flow. The critical Reynolds number (laminar-turbulent transition) was, however, affected by the secondary flow. Later, Mori and Futagami, investigated this problem theoretically on a fully developed laminar flow. On the assumption of a boundary layer (by making the velocity and temperature distributions are affected only by viscosity and thermal conductivity) along the tube wall and by use of the boundary-layer integral method, they obtained (after assuming the velocity and temperature fields are affected only by the secondary flow in the core region) the relations between Nusselt number and $\text{Re}Ra(= 10^9)$ for Prandtl number $Pr$ not far from unity. Faris and Viskanta examined this problem analytically using a perturbation method. They presented approximate analytical solutions as well as average Nusselt numbers graphically for a range of Prandtl and Grashof numbers of the physical interest. Eckert and Peterson measured the temperature profile along the vertical diameter and calculated Nusselt number as a function of Peclet number $Pe$ for the problem of the heat transfer to mercury in laminar flow through a horizontal tube with a constant heat flux. Siegwarth and Hanratty, measured the fully developed temperature field and axial velocity profile for Prandtl number $Pr = 80$ at the outlet of a long horizontal tube which is heated electrically. They also solved this problem by finite difference techniques to obtain the secondary flow pattern as well as the temperature field and axial velocity field. Newell and Bergles, formulated a numerical investigation of the effects of free convection on fully developed laminar flow in horizontal circular tubes with uniform heat flux. They obtained solutions for heat transfer and pressure drop, with both heating and cooling, for water with two limiting tube-wall conditions: low thermal conductivity (glass tube) and infinite thermal conductivity. They found that the infinite-conductivity tube exhibits higher Nu and friction factor than the glass tube, with Nu being over five times the Poiseuille value at Grashof number (based on the difference of wall and bulk mean temperatures) $\sim 10^6$. Yousef and Tarasuk, investigated experimentally the influence of free convection due to buoyancy on forced laminar flow of air in the entrance region of a horizontal isothermal tube for a narrow range of Grashof numbers (based on logarithmic mean temperature difference) from $0.8 \times 10^4$ to $8.7 \times 10^4$. That same year, Hishida, Nagano and Montesclaros, published numerical solutions without the aid of a large Prandtl number assumption for combined free-and-forced laminar convection in the entrance region of a horizontal pipe with uniform wall temperature. Chou and Hwang, studied numerically, without the aid of the large Prandtl number assumption, the Graetz problem with the effect of natural convection in a uniformly heated horizontal tube by a relatively novel vorticity-velocity method. They showed the variations in local
friction factor and Nusselt number with Rayleigh number for Prandtl numbers $Pr = 5, 2$ and $0.7$. Rustum and Soliman, investigated numerically the steady, fully-developed, laminar, mixed convection in horizontal internally-finned tubes for the case of uniform axial heat input and circumferentially uniform wall temperature. At $Pr = 7$ and for modified Grashof number varies from 0 to $2 \times 10^6$, they obtained numerical results which include the secondary flow (velocity) components, axial velocity and temperature distributions, wall-heat flux, friction factor and average Nusselt number for different fin geometries. Finally, Al-Khawaja, Agarwal, and Gardner considered numerically the problem of MFM combined-free-and-forced convection pipe flow using modified third-order-accurate upwind scheme to handle the problem of high Grashof number. However, for high Hartmann number, they refined the mesh near the boundary.

2. Background

The problem considered herein is one of the forced convection in a horizontal, circular pipe of radius $a$ in a uniform, vertical, transverse magnetic field $B_0$. A homogeneous, incompressible, viscous, electrically-conducting fluid flows through a horizontal circular pipe and is subjected to a uniform surface temperature and a uniform surface heat flux. In conjunction with defining this problem, the following assumptions are made:

a) All fluid properties are constant (the fluid considered is incompressible) and independent of the temperature.

b) The pipe is sufficiently long that it can be assumed the flow and heat transfer are fully developed and entrance or exit effects can be neglected. Further, it can be deduced that none of the variables except pressure and temperature vary linearly with axial direction.

c) The contributions of viscous and Joulean dissipation in the energy equation are small and can be neglected. This assumption has been shown to be applicable to a similar problem when no external electric field is imposed on the flow.

d) The induced magnetic field produced as a result of interaction of applied field, $B_0$, with either main or secondary flow, will be assumed negligibly small compared to $B_0$. This assumption follows from the fact that the magnetic Reynolds number based on the flow is much smaller than unity under conditions found in typical applications.

3. Basic Conservation Equations

For incompressible newtonian liquid metal fluid and steady-state conditions, the modified Navier-Stokes equations under the effect of magnetic field body force including induction and energy equations in vector forms are, respectively,

\[ \rho(V \cdot \nabla)V + \nabla(p + \mu \frac{|H|^2}{2}) = \mu I V^2 + \mu(H \cdot V)H \]  \hspace{1cm} (1)

\[ \nabla^2 H + \mu \sigma [(H \cdot V) V - (V \cdot V) H] = 0 \]  \hspace{1cm} (2)
Finite difference solutions of MFM square duct flow with heat transfer using MatLab program

\[ \rho c (V \cdot V) T = k \nabla^2 T + \mu_f \phi + \left| \frac{\mu}{\sigma} \right|^2 \]  

(3)

in addition to solenoidal conditions on the two vectors

\[ \nabla \cdot V = 0 \quad \text{and} \quad \nabla \cdot H = 0 \]  

(4)

For very small magnetic Reynolds number \( R_M \) (i.e. the induced magnetic field produced as a result of interaction of applied field, \( B_0 \), will be assumed negligibly small compared to \( B_0 \)),

the induction equation, Eq. (2), can be derived from Maxwell's equations along with the two solenoidal conditions, Eqs. (4). The last two terms in the right hand of the energy equation, Eq. (3), represent the viscous and Joulean dissipations, respectively. Those terms can be neglected compared to the other ones in the equation.

After many simplifications by assuming fully developed flow, i.e. 2-D problem (See Fig. 1), and since the flow is laminar due to damping of the fluctuations of turbulence in the presence of magnetic field, the dimensionless governing equations for this flow become

\[ \nabla^2 w^* - M \frac{\partial H^*}{\partial x} = 1 \]  

(5)

\[ \nabla^2 H^* - M \frac{\partial w^*}{\partial x} = 0 \]  

(6)

Fig. 1. MFM flow geometry
4. Numerical Investigations

4.1 Finite Difference Schemes

The partial differential equations considered here are of the elliptic type because of the steady state behavior of those equations. Some important schemes must be introduced in order to discretize those equations and get a system of linear algebraic equations with reasonable accuracy and stability if they are solved using one of the iterative methods that will be discussed below.

4.1.1 Central Difference Scheme

Central difference schemes are very well known and have been used extensively for elliptic equations, particularly, Laplace’s and Poisson’s equations. This scheme, for 2-D, can be represented by five-points formula, diagonal five formula, or nine-point formula, etc.. Unfortunately, these schemes do not work with all types of the elliptic equations since, for example, the numerical solutions of steady Navier-Stokes equations by relaxation methods using central difference scheme may become unstable and fail to converge if the grid Reynolds number exceeds the value 2. This instability occurs when both the convective and diffusion terms in the Navier-Stokes equations are central-differenced. The standard central-difference of the convective terms destroys the ellipticity of the difference equations at high Reynolds numbers because of loss of diagonal dominance in the resulting matrix.

\[ \nabla^2 \theta + 4\text{Nu} w \theta = 0 \quad (7) \]

and

\[ \nabla^2 \theta - 4w = 0 \quad (8) \]

Where the negative dimensionless pressure gradient \( \gamma \) is related to \( w^* \) by

\[ \gamma = \frac{1}{\int \int_0^1 \int \int_0^1 w \ dx \ dy} \quad (9) \]

From the force and energy balances one can show, respectively, that \( \text{fRe} = -2\gamma \) and \( \text{Nu} = -1/\theta_m \). Where the mean dimensionless temperature is given by

\[ \theta_m = \frac{1}{\int \int_0^1 \int \int_0^1 \theta w \ dx \ dy} \quad (10) \]

Definitions of other dimensionless variables are described in the notation section. The boundary conditions are \( w^* = 0 \) (from no-slip condition), \( H^* = 0 \) (from electrically insulated surface), and \( \theta = 0 \) (for isothermal surface and constant surface heat flux).
4.1.2 First-Order and Second-Order-Accurate Upwind Schemes

The first-order-accurate upwind scheme has been used by many researchers to handle stability problem in the convective terms for high Reynolds number (in the present case is the square root of Grashof number). The diffusion operator and the source terms could be left as central-differenced.

There are two main disadvantages of employing these difference operators. First, the introduction of large artificial diffusion in the direction of the bias, thereby resulting in considerable loss of accuracy. Second, the overall accuracy of the algorithm being $O(|Gr|^{1/2}h)$, at high Grashof number, even with a reasonably fine mesh, the error of $O(|Gr|^{1/2}h)$ may become so dominant as to obscure the effects of physical diffusivity on the flow. Here, $Gr$ is the Grashof number and $h$ is the mesh size. Although considerable grid refinement, in principle, can alleviate the problem, the necessary degree of refinement is often impractical because of computer-time and storage limitations. For flow problem in a two-dimensional driven-square cavity (this flow structure has become a standard test case for evaluating the accuracy, stability and efficiency of various Navier-Stokes algorithm), only few investigators have computed the flowfield beyond Reynolds numbers of 1000.

Atias, Wolfshein and Israeli employed a second-order-accurate upwind scheme to discretize the convective terms in the vorticity transport equation. The overall accuracy of this scheme is $O(Re h^2)$, based on Reynolds number. However, Atias, Wolfshein and Israeli on the basis of Von Neumann type stability analysis for the linearized vorticity equation, find that a Gauss-Seidel solution of the second-order upwind scheme is stable if the mesh Reynolds number is less than $2 + (8)^{1/2}$ (compared with value 2 for a central difference scheme).

4.1.3 Third-Order-Accurate Upwind Scheme

This scheme was first introduced by Agarwal for computing Navier-Stokes solutions at high Reynolds numbers. He used this scheme to solve for example, flow in a 2-D driven square cavity, 2-D flow in a channel with sudden symmetric expansion, 2-D flow in a channel with a symmetrical placed blunt base, the flowfield of a 2-D impinging jet and 3-D flow in a driven cubic box. For all cases, he obtained a good agreement with computations of other investigators as well as with available experimental data. He obtained a highly accurate solution for Reynolds numbers up to 10,000 for flow in a 2-D driven square cavity. However, this scheme suffers from two main disadvantages. First, the numerical treatment of the boundary conditions requires extra care because the algorithm uses a five-point difference formula instead of the standard three-point formula for the first derivatives. Second, the solution by line relaxation requires a pentadiagonal matrix inversion.

4.1.4 Modified Third-Order-Accurate Upwind Scheme

A modification to the third-order upwind scheme also was presented by Agarwal which makes the scheme second-order-accurate, but frees it from the disadvantages discussed above. The new algorithm has low artificial diffusion compared to the second order upwind scheme.
4.2 Iterative Methods for Solving Systems of Linear Algebraic Equations

Methods for solving systems of linear algebraic equations are classified as either direct or iterative. Direct methods are those which give the solution (exactly, if round-off error does not exist) in a finite and predeterminable number of operations using an algorithm which is often quite complicated. Iterative methods consist of a repeated application of an algorithm which is usually quite simple. They yield the exact answer only as a limit of a sequence, but, if the iterative procedure converges, one can come within $\epsilon$ (small value) of the answer in a finite but usually not predeterminable number of operations. Thus, the iterative methods will be used. This class of methods is sometimes referred to by relaxation methods. Those methods are broken into point- (or explicit-) iterative methods and block- (or implicit-) iterative methods. In brief, for point-iterative methods, the same simple algorithm is applied to each point where the unknown function is to be determined in successive iterative sweeps whereas in block iterative methods, subgroups of points are singled out for solution by elimination (direct) schemes in an overall iterative procedure.

4.2.1 Point-Gauss-Seidel Iteration

Overall this method is explicit and the steps which summarize the application of the point Gauss-Seidel iteration on a general system of algebraic equations would be as following,

(a) Make initial guesses for all unknowns.
(b) Solve each equation for the unknown whose coefficient is largest in magnitude (to satisfy stability criteria as it will be seen later), using the guessed values initially and the most recently computed values thereafter for the other unknowns in each equation.
(c) Repeat iteratively the solution of the equations in this manner until changes in the unknown become small.

4.2.2 Sufficient Condition for Convergence of The Gauss-Seidel Procedure

The point-Gauss-Seidel iterative method is simple but only converges under certain conditions related to diagonal dominance of the matrix of coefficients. Fortunately, the differencing of many steady-state conservation statements provides this diagonal dominance. Then, the sufficient condition for convergence of this method which is applied on a system of algebraic equations can be if the system is irreducible (cannot be arranged so that some of the unknowns can be determined by solving less than m equations) and if the resulting matrix of coefficient from the difference equation has a property of diagonal dominance. This is a sufficient condition which means that the convergence may sometimes be observed when the above condition is not met.

Now, the above iterative convergence criteria can be related to the system of algebraic equations, which results from differencing the elliptic equations. By inspection, it can be shown that the coefficient largest in magnitude belongs to $s_{ij}$, where $s$ is dependent variable. Then, those equations would establish a sparse matrix which has a property of the diagonal dominance, and hence, the Gauss-Seidel iteration would converge.

4.2.3 Successive Over-Relaxation (SOR)

Successive over-relaxation is a technique which can be used to accelerate any iterative procedure but it is proposed here primarily as a refinement to the Gauss-Seidel method.
Successive under-relaxation (SUR) appears to be most appropriate when the convergence at a point is taking on an oscillatory pattern and tending to overshoot the final solution. Over-relaxation is usually appropriate for numerical solutions to Laplace’s equation with Dirichlet boundary conditions. Under-relaxation is sometimes called for in elliptic problems when the equations are nonlinear. Occasionally, for nonlinear problems, under-relaxation is even observed to be necessary for convergence.

In general, there is no specific formula which determines the optimum value of relaxation factor. Sometimes the determination of optimum factor could be obtained by numerical experiments.

### 4.2.4 Line-Iterative Relaxation Method

Line-iterative relaxation algorithm sometimes is referred to as block-iterative method and since this method has an implicit nature, then it is known as implicit-iterative method. Although this procedure is workable with almost any iterative algorithm, it makes sense to work within the framework of the Gauss-Seidel method with SOR or SUR.

Again, over-relaxation or under-relaxation can be used here. There are many alternative ways in applying SOR or SUR.

### 5. Solution

In this paper, the MFM problem for two heat transfer limits; constant temperature and constant heat flux boundary conditions, is investigated numerically for square duct (See Fig. 1). The modified dimensionless Navier-Stokes equations with uniform-temperature-condition having energy equation (Eq. 7), and uniform-heat-flux-condition having energy equation (Eq. 8) are transferred into finite-difference equations (using the central-difference scheme) and given as

\[
w_{i-1, j}^* + w_{i+1, j}^* + w_{i, j-1}^* + w_{i, j+1}^* - 4w_{i, j}^* - \frac{1}{2} M \Delta x (H_{i+1, j}^* - H_{i-1, j}^*) = (\Delta x)^2 \quad (11)
\]

\[
H_{i-1, j}^* + H_{i+1, j}^* + H_{i, j-1}^* + H_{i, j+1}^* - 4H_{i, j}^* - \frac{1}{2} M \Delta x (w_{i+1, j}^* - w_{i-1, j}^*) = 0 \quad (12)
\]

\[
\theta_{i-1, j} + \theta_{i+1, j} + \theta_{i, j-1} + \theta_{i, j+1} - \theta_{i, j} - 4(\Delta x)^2 \text{Nu} w_{i, j} \theta_{i, j} = 0 \quad (13)
\]

and

\[
\theta_{i-1, j} + \theta_{i+1, j} + \theta_{i, j-1} + \theta_{i, j+1} - 4(\Delta x)^2 w_{i, j} = 0 \quad (14)
\]

with the following definitions of dimensionless pressure gradient and mean dimensionless temperature given, respectively, as

\[
\gamma = \frac{1}{\sum_{i=0}^{l} \sum_{j=0}^{m} w_{i,j}^* (\Delta x)^2}
\]
and

\[ \theta_m = \frac{\sum_{i=0}^{I} \sum_{j=0}^{J} \theta_{i,j} w_{i,j} (\Delta x)^2}{\sum_{i=0}^{I} \sum_{j=0}^{J} w_{i,j} (\Delta x)^2} \]  \hspace{1cm} (16)

Beside the following boundary conditions: \( H^* = 0 \) (for electrically insulated wall), \( w^* = 0 \) (from no-slip condition), and \( \theta = 0 \) (from the definition of dimensionless temperature). The last boundary condition is valid for the two heat transfer limits as given in the reference. It should be noted that the above finite-difference equations are derived by making the mesh size (either in \( x \) or \( y \) direction) to be uniform and to have the same value for both directions. The non-linear energy equation (Eq. 13) for constant temperature condition or the linear equation (Eq. 14) for constant heat flux condition is solved simultaneously with the axial momentum (Eq. 11) and induction (Eq. 12) equations using Gauss-Seidel iterative method. The program utilized to achieve this task is MatLab software. For low to moderate Hartmann number (\( M = 0 \) to 100), a uniform 101 by 101 mesh is used while for high Hartmann number (\( M = 200 \)), a uniform 201 by 201 mesh is used. The convergence of the solution is tested by using root-mean-square residual \( R \) (defined in the nomenclature section given below). The significance of this residual \( R \) is that once it reaches a very small number compared to unity, then the solution will be acceptable. As shown in Fig. 2, the convergence of normalized axial velocity residuals increases as the Hartman number increases. The residual at \( M = 200 \) reaches \( 10^{-3} \) after 500 iterations while the residual, without magnetic field, reaches \( 10^{-3} \) after 5800 iterations. The same behavior can be said for the normalized magnetic field residuals (See Fig. 3). The situation will be different for the dimensionless temperature. There are two cases. First, the residual for uniform temperature case converges more slowly, particularly, for \( M = 200 \). In this Hartmann number a value of \( 10^{-3} \) for the
residual can be reached if the number of iterations exceeds 29500. However, the number of iterations, at $M = 0$, should approach 7700 to have a residual value of $10^{-3}$ (See Fig. 4). Second, the residual for uniform heat flux condition, converges to $10^{-3}$ if number of iterations reaches 33800 for $M = 200$, while at $M = 0$, the residual will converge to same value if number of iterations exceeds 10600 (See Fig. 5).

Fig. 3. Residuals for normalized magnetic field

Fig. 4. Residuals for dimensionless temperature with uniform temperature boundary condition
6. Results

Some noticeable heat transfer results are obtained for the MFM square duct flow with uniform temperature and heat flux boundary conditions. The flow (velocity and pressure) was studied so extensively in reference for the same flow conditions. For more details, the reader should refer to reference to notice, in the provided figures, the flattening of the axial velocity (due to the presence of the magnetic field) and the increase of the friction factor with the field. The negative dimensionless temperature distributions at the mid-plane (either along or normal to the magnetic field) always decrease as the Hartmann number increases for both boundary condition limits, See Figs. 6, 7, and 8. This is because the temperature distributions are more homogenous as the magnetic field is turned on. This can be seen from the results presented in and is due to the fact that velocity profile becomes more flattened as M increases, particularly along the direction of the magnetic field. Also we notice that the temperature distributions along and normal to the field are almost identical for any Hartmann number. This is supported by the color bands shown in Figs. 9. Figures 9 (a), (b), and (c) are the color bands for the case of uniform heat flux boundary conditions, whereas Figs. 9 (d), (e), and (f) show the color bands for uniform temperature boundary condition. The uniformity of the temperature across the duct is greater for the former case. This explains the reasons why this case has higher Nusselt number for any Hartmann number as will be further explained in the next paragraph.
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Fig. 6. Negative dimensionless temperature distribution along magnetic field and at the mid plane (\( y^* = 0.5 \)) for both thermal boundary conditions with \( M = 0 \)

Fig. 7. Negative dimensionless temperature distribution along and normal to magnetic field and at the mid plane for both directions and for both thermal boundary conditions with \( M = 20 \)
Finally, as expected the Nusselt number, $Nu$, increases as $M$ increases for both cases of constant wall temperature and constant heat flux boundary conditions. Starting from conventional flow ($M = 0$), results for $Nu$ coincide well with those obtained by the analytical approach given in reference for both boundary conditions. The present work gives values for $Nu$ as 3.606 and 2.977 for constant heat flux and constant temperature boundary conditions, respectively, while the analytical $Nu$ values for the same conditions are 3.61 and 2.98. For any Hartmann number $M$, the highest Nusselt number is shown by the results to correspond always to the case of circular tube with constant heat flux taken from reference, whilst the case of square duct with uniform temperature has the lowest $Nu$ values (See Fig. 10). The solution for the present work agrees very well with the reference where spectral method was used for the case of uniform temperature boundary condition.
Finally, as expected the Nusselt number, $\text{Nu}$, increases as $M$ increases for both cases of constant wall temperature and constant heat flux boundary conditions. Starting from conventional flow ($M = 0$), results for $\text{Nu}$ coincide well with those obtained by the analytical approach given in reference for both boundary conditions. The present work gives values for $\text{Nu}$ as 3.606 and 2.977 for constant heat flux and constant temperature boundary conditions, respectively, while the analytical $\text{Nu}$ values for the same conditions are 3.61 and 2.98. For any Hartmann number $M$, the highest $\text{Nu}$ number is shown by the results to correspond always to the case of circular tube with constant heat flux taken from reference, whilst the case of square duct with uniform temperature has the lowest $\text{Nu}$ values (See Fig. 10). The solution for the present work agrees very well with the reference where spectral method was used for the case of uniform temperature boundary condition.

Fig. 9. Dimensionless temperature color bands. (a) $M = 0$, uniform heat flux; (b) $M = 20$, uniform heat flux with the same contour value shown in a; (c) $M = 200$, uniform heat flux with the same contour value shown in a; (d) $M = 0$, uniform temperature; (e) $M = 20$, uniform temperature with the same contour value shown in d; (f) $M = 200$, uniform temperature with the same contour value shown in d
7. Conclusion

The problem considered here is a square duct flow with electrically conducting fluid and with two heat transfer limits. The problem is analyzed numerically when a uniform transverse magnetic field is applied to the duct. The assumption of laminar flow is mostly valid in MFM flows since the turbulences will be damped out due the opposing force induced in the flow.

In reference, the fluid mechanic part of this problem was considered extensively and the results were shown using the spectral method. Also, the heat transfer results for only uniform temperature boundary condition were shown. In the present work, we consider two heat transfer limits (uniform heat flux and temperature boundary conditions) numerically using iterative Gauss-Seidel method, and the software package MatLab is utilized to achieve this approach. The results obtained for the case of constant temperature condition agree very well with reference.

In future, we can extend this work to include the aspect ratio (i.e. general rectangular cross section). This ratio will be added to the problem as dimensionless independent parameter beside Hartmann number M. Also, it is a good idea to include the natural convection, which makes us to be concerned with a problem of combined forced-and-free convection flow in a transverse magnetic field. Off course, it will be highly non-linear and we must employ an accurate and stable algorithm. This dilemma will add extra complexity to the problem, beside the independent Grashof number will appear in the governing equations.

![Fig. 10. Nusselt number versus Hartmann number for different flow geometries](www.intechopen.com)
8. References


9. Appendix

The four basic dimensionless equations were simplified using the finite difference scheme to get a system of algebraic equations. The central difference approximation was used since it is more accurate than the forward and backward differences. Equations 5, 6, 7 and 8 represent the four basic dimensionless equations, which are transformed equations 11, 12, 13 and 14, respectively. We used 101 by 101 mesh size for low and moderate Hartmann number M (from 0 to 100). For high Hartmann number (= 200), we used 201 by 201 mesh size. The algebraic equations were solved numerically using the MatLab software. Equations 11 and 12 were solved simultaneously by employing Gauss Seidel method. Then \( \gamma \) is determined (from Eq. 9) once \( \theta \) is obtained. The double integration was approximated by the summation in both \( x \) and \( y \) directions as given in Eq. 15. For the constant surface heat flux boundary condition, Eq. 14 (linear) is solved by employing the iterative Gauss Seidel method. Also \( \theta_m \) can be obtained once \( \theta \) is determined. \( \theta_m \) is found from Eq. 10. The double integration was approximated by the summation in both \( x \) and \( y \) directions as given in Eq. 16.

For constant surface temperature boundary conditions, an initial guess for Nusselt number Nu was assumed, then Eq. 14 (non linear) was solved using successive substitution. From Eq. 10, \( \theta_m \) is obtained and a more accurate Nu is found, then another approximation of \( \theta \) was solved using the new value of Nu. This process was repeated until the error becomes very small.
In our calculations, we used the root mean square residuals R (defined in the program) to check the convergence for each flow variable. Once R < 10^{-7}, then the iterations are stopped. From the definition of θ, the thermal boundary condition at the surfaces, for both the uniform surface heat flux and constant surface temperature, is θ = 0.

The problem was solved by many researchers and they employed different software packages to solve the resulting simultaneous algebraic equations. They used, for example, Fortran and C++ languages and spectral method. But here, the MatLab is employed and noticed that this program is so efficient and powerful for solving such problem.

The Matlab programs for uniform surface heat flux and uniform surface temperature are presented below.

### 9.1 Uniform Surface Heat Flux

% Solution for MHD flow inside square duct for const. heat flux B.C.'s
% a<x<b, c<y<d
% M = Hartmann number
% n = number of subintervals for x
% m = number of subintervals for y
% h = delta x*
% k = delta y*
% Note: In this program delta x = delta y
a=0; b=1; c=0; d=1; num_iter=20000; M=100;
n=100; m=100; h=(b-a)/n; k=(d-c)/m;
c1=1/4; c2=(h*M)/8; c3=(h^2)/4;
ws=zeros(n+1, m+1);
Hs=zeros(n+1, m+1);

% B.C.'s at the four corners for w*(no slip conditions) & H* (electrically insulated surface)
ws(1,1)=0;
ws(n+1,1)=0;
ws(1,m+1)=0;
ws(n+1,m+1)=0;
Hs(1,1)=0;
Hs(n+1,1)=0;
Hs(1,m+1)=0;
Hs(n+1,m+1)=0;

% B.C.'s at the four sides for w*(no slip conditions) & H* (electrically insulated surface)
for i=2:n
    ws(i,1)=0;
    ws(i,m+1)=0;
    Hs(i,1)=0;
    Hs(i,m+1)=0;
end
for j=2:m
    ws(1,j)=0;
    ws(n+1,j)=0;
end
Finite difference solutions of MFM square duct flow with heat transfer using MatLab program

\begin{verbatim}
Hs(1,j)=0;
Hs(n+1,j)=0;
end
for it=1:num_iter
    wsave=w;
    Hsave=H;
    Rws=0;
    RHs=0;
    % Solution for w* & H*
    for i=2:n
        for j=2:m
            ws(i,j)= c1*(ws(i,j+1)+ws(i,j-1)+ws(i+1,j)+ws(i-1,j))-c2*(Hs(i+1,j)-Hs(i-1,j))-c3;
            Hs(i,j)= c1*(Hs(i,j+1)+Hs(i,j-1)+Hs(i+1,j)+Hs(i-1,j))-c2*(ws(i+1,j)-ws(i-1,j));
            % Rws = Root mean square residuals for w*
            Rws=Rws+sqrt((w(i,j)-wsave(i,j))^2);
            RHs=RHs+sqrt((H(i,j)-Hsave(i,j))^2);
        end
    end
    Rwss(it,1)=Rws;
    RHss(it,1)=RHs;
    if (RHs<1e-8 & Rws<1e-8)
        break
    end
end
% gamma = Non-dimensional pressure gradient
% w = Dimensionless axial velocity
% H = Dimensionless induced axial magnetic field
% f = friction factor
gamma=1/sum(sum(ws*h^2));
w=ws*gamma;
H=Hs*gamma;
f=-2*gamma;
% t = Dimensionless temperature (theta)
t=zeros(n+1,m+1);
% B.C.'s at the four corners for theta (uniform surface heat flux)
t(1,1)=0;
t(1,m+1)=0;
t(n+1,1)=0;
t(n+1,m+1)=0;
% B.C.'s at the four sides for theta (uniform surface heat flux)
for i=2:n
    t(i,m+1)=0;
    t(i,1)=0;
end
for j=2:m
end
\end{verbatim}
t(1,j)=0;  
t(n+1,j)=0;  
end  
for itt=1:num_iter  
tsave=t;  
Rt=0;  
% Solution for theta  
for i=2:n  
    for j=2:m  
        \( t(i,j) = 1/4 \times (t(i-1,j) + t(i+1,j) + t(i,j-1) + t(i,j+1)) - h^2 \times w(i,j) \)  
    end  
end  
Rtt(itt,1)=Rt;  
if Rt<1e-8  
    break  
end  
end  
% thetam = Mean dimensionless temperature  
% nu = Nusselt number (Nu)  
\( \text{thetam} = \frac{\text{sum}(\text{sum}(t \times w \times h^2))}{\text{sum}(\text{sum}(w \times h^2))} \)  
nu=1/thetam;  
% Display solution with x from left to right  
ws=[ws';];  
w=[w';];  
Hs=[Hs';];  
H=[H';];  
x=a:h:b;  
y=c:k:d;  

9.2 Uniform Surface Temperature

% Solution for MHD flow inside square duct for const. temperature B.C.'s  
% a<x<b , c<y<d  
% M = Hartmann number  
% n = number of subintervals for x  
% m = number of subintervals for y  
% h = delta x*  
% k = delta y*  
% Note: In this program delta x = delta y  
a=0;  
b=1;  
c=0;  
d=1;  
n=100;  
m=100;  
h=(b-a)/n;  
k=(d-c)/m;  
c1=1/4;  
c2=(h*M)/8;  
c3=(h^2)/4;  
% ws = negative normalized axial velocity (w*)  
% Hs = normalized induced axial magnetic field (H*)  
ws=zeros(n+1, m+1);  
Hs=zeros(n+1, m+1);
Finite difference solutions of MFM square duct flow with heat transfer using MatLab program

\[ H_s = \text{zeros}(n+1, m+1); \]

% B.C.'s at the four corners for \( w^* \) (no slip conditions) & \( H^* \) (electrically insulated surface)
\[ w_s(1,1) = 0; \]
\[ w_s(n+1,1) = 0; \]
\[ w_s(1,m+1) = 0; \]
\[ w_s(n+1,m+1) = 0; \]
\[ H_s(1,1) = 0; \]
\[ H_s(n+1,1) = 0; \]
\[ H_s(1,m+1) = 0; \]
\[ H_s(n+1,m+1) = 0; \]

% B.C.'s at the four sides for \( w^* \) (no slip conditions) & \( H^* \) (electrically insulated surface)
for \( i = 2:n \)
\[ w_s(i,1) = 0; \]
\[ w_s(i,m+1) = 0; \]
\[ H_s(i,1) = 0; \]
\[ H_s(i,m+1) = 0; \]
end
for \( j = 2:m \)
\[ w_s(1,j) = 0; \]
\[ w_s(n+1,j) = 0; \]
\[ H_s(1,j) = 0; \]
\[ H_s(n+1,j) = 0; \]
end
for \( it = 1: \text{num\_iter} \)
\[ w_{\text{save}} = w; \]
\[ H_{\text{save}} = H; \]
\[ R_{ws} = 0; \]
\[ R_{Hs} = 0; \]
end

% Solution for \( w^* \) & \( H^* \)
for \( i = 2:n \)
for \( j = 2:m \)
\[ w_s(i,j) = c_1 \times (w_s(i,j+1)+w_s(i,j-1)+w_s(i+1,j)+w_s(i-1,j)) - c_2 \times (H_s(i+1,j)-H_s(i-1,j)) - c_3; \]
\[ H_s(i,j) = c_1 \times (H_s(i,j+1)+H_s(i,j-1)+H_s(i+1,j)+H_s(i-1,j)) - c_2 \times (w_s(i+1,j)-w_s(i-1,j)); \]
end
end

% \( R_{ws} \) = Root mean square residuals for \( w^* \)
% \( R_{Hs} \) = Root mean square residuals for \( H^* \)
\[ R_{ws} = R_{ws} + \sqrt{(w(i,j)-w_{\text{save}}(i,j))^2}; \]
\[ R_{Hs} = R_{Hs} + \sqrt{(H(i,j)-H_{\text{save}}(i,j))^2}; \]
end

\[ R_{ws}(it,1) = R_{ws}; \]
\[ R_{Hs}(it,1) = R_{Hs}; \]
if (\( R_{Hs} < 1 \times 10^{-8} \) & \( R_{ws} < 1 \times 10^{-8} \))
\break
end
end

% gamma = Non-dimensional pressure gradient

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9.3 Derivation of the Energy Equations

The detailed derivation of the simplified energy equation are given in the literature and the key behind this derivation is to neglect the last two terms, in Eq. 3, which represent the viscous and Joulean dissipations, respectively, and to apply the energy balance given by:

\[
\text{Heat supply to fluid from surface} = \text{Energy removed by fluid by convection}
\]

Then the simplified dimensionless energy equations, Eqs. 15 and 16 given above, can be obtained if the above assumptions are applied, the axial conductive term is omitted, and proper dimensionless variables are defined.

\[
\text{w} = \text{Dimensionless axial velocity}
\]
\[
\text{H} = \text{Dimensionless induced axial magnetic field}
\]
\[
\text{f} = \text{friction factor}
\]
\[
\gamma = \frac{1}{\text{sum(sum(}ws^2)}
\]
\[
w = ws \gamma
\]
\[
H = Hs \gamma
\]
\[
f = -2 \gamma
\]
\[
\text{t} = \text{Dimensionless temperature (theta)}
\]
\[
t = \text{zeros}(n+1,m+1)
\]
\[
\text{B.C.'s at the four corners for theta (uniform surface temperature)}
\]
\[
t(1,1) = 0;
\]
\[
t(1,m+1) = 0;
\]
\[
t(n+1,1) = 0;
\]
\[
t(n+1,m+1) = 0;
\]
\[
\text{B.C.'s at the four sides for theta (uniform surface temperature)}
\]
\[
\text{for i=2:n}
\]
\[
t(i,m+1) = 0;
\]
\[
t(i,1) = 0;
\]
\[
\text{end}
\]
\[
\text{for j=2:m}
\]
\[
t(1,j) = 0;
\]
\[
t(n+1,j) = 0;
\]
\[
\text{end}
\]
\[
\text{for itt=1:num_iter}
\]
\[
\text{tsave} = t;
\]
\[
\text{Rt} = 0;
\]
\[
\% Solution for theta
\]
\[
\text{for i=2:n}
\]
\[
\text{for j=2:m}
\]
\[
t(i,j) = (t(i-1,j)+t(i+1,j)+t(i,j-1)+t(i,j+1))/(4-4*h^2*nu*w(i,j));
\]
\[
\% Rt = Root mean square residuals for theta
\]
\[
\text{Rt} = \text{Rt+sqrt((}t(i,j)-\text{tsave(i,j)})^2)
\]
\[
\end
\]
\[
\% thetam = Mean dimensionless temperature
\]
\[
\% nu = Nusselt number (Nu)
\]
\[
\text{thetam} = (\text{sum(sum(}t*ws^2))/\text{sum(sum(ws^2)))
\]
\[
\text{nu} = -1/\text{thetam};
\]
\[
\text{Rtt(itt,1) = Rt;
\]
\[
\% if Rt<1e-8
\]
\[
\% break
\]
\[
\% Display solution with x from left to right
\]
\[
\text{ws} = [\text{ws}']
\]
\[
\text{w} = [\text{w}']
\]
9.3 Derivation of the Energy Equations

The detailed derivation of the simplified energy equation are given in the literature and the key behind this derivation is to neglect the last two terms, in Eq. 3, which represent the viscous and Joulean dissipations, respectively, and to apply the energy balance given by:

Heat supply to fluid from surface = Energy removed by fluid by convection

Then the simplified dimensionless energy equations, Eqs. 15 and 16 given above, can be obtained if the above assumptions are applied, the axial conductive term is omitted, and proper dimensionless variables are defined.
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This book is a collection of 19 excellent works presenting different applications of several MATLAB tools that can be used for educational, scientific and engineering purposes. Chapters include tips and tricks for programming and developing Graphical User Interfaces (GUIs), power system analysis, control systems design, system modelling and simulations, parallel processing, optimization, signal and image processing, finite different solutions, geosciences and portfolio insurance. Thus, readers from a range of professional fields will benefit from its content.

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